# Tetrahedron equation, 3D reflection equation and generalized quantum groups

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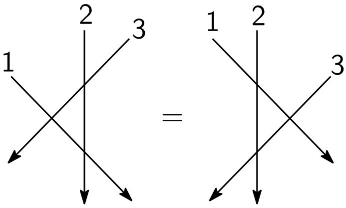
# Key to integrability in 2D

Yang-Baxter equation

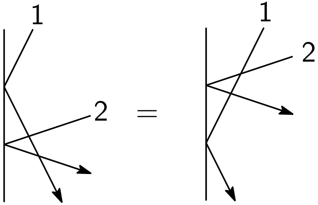
$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$




$R$ : 2 particle scattering

Reflection equation

$R_{21}K_2R_{12}K_1 = K_1R_{21}K_2R_{12}$



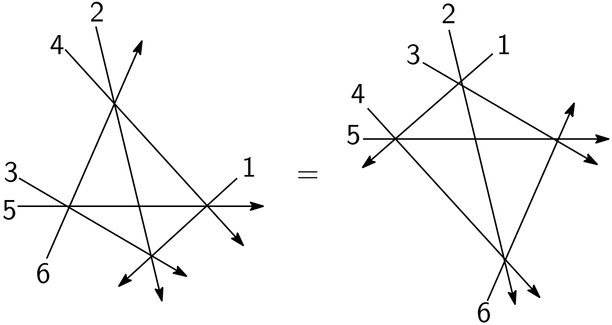

$K$ : Reflection at boundary

# What about 3D?

**Tetrahedron equation** (A.B. Zamolodchikov, 1980)

$$R : F \otimes F \otimes F \to F \otimes F \otimes F \qquad \text{(3D } R\text{)}$$

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$$




$$R = \begin{cases} \text{3 string scattering amplitude in (2+1)D} \\ \text{local Boltzmann weight of the vertex in 3D} \end{cases}$$

# Status of finding solutions and relevant maths

**2D**

- Infinitely many solutions constructed systematically by representation theory of the Drinfeld-Jimbo quantum affine algebra $U_q(\hat{g})$ ($\hat{g}$ = affine Kac-Moody algebra).

**3D**

- A few classes of solutions are known.
- Systematic framework yet to be developed.
- One such approach is by quantized algebra of functions $A_q(g)$ ($g$ = finite dimensional simple Lie algebra).
- $A_q(g)$ is the quantum group corresponding to the dual of $U_q(g)$. Studied by Drinfeld (87), Vaksman-Soibelman (89,91), Reshetikhin-Takhtajan-Faddeev (90), Noumi-Yamada-Mimachi (92), Kashiwara (93), Geiss-Leclerc-Schröer (2011-) etc.

- **Simplest example:**

$$SL_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid [t_{ij}, t_{kl}] = 0,\ t_{11}t_{22} - t_{12}t_{21} = 1 \right\}.$$

$A_q(sl_2)$ is generated by $t_{11}, t_{12}, t_{21}, t_{22}$ with the relations

$$t_{11}t_{21} = qt_{21}t_{11},\ t_{12}t_{22} = qt_{22}t_{12},\ t_{11}t_{12} = qt_{12}t_{11},\ t_{21}t_{22} = qt_{22}t_{21},$$
$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1.$$

Hopf algebra with coproduct $\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$.

- **Fock representation** $\boldsymbol{\pi_1 : \mathbf{A_q(sl_2)} \to \mathrm{End}(\mathbf{F_q})}$

$F_q = \oplus_{m \geq 0} \mathbb{C}|m\rangle$ : $q$-oscillator Fock space

$$\pi_1 : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{a}^- & \mathbf{k} \\ -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}$$
$$\mathbf{k}|m\rangle = q^m|m\rangle,\ \mathbf{a}^+|m\rangle = |m+1\rangle,\ \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle.$$

## Theorem (Classification of irreducible representations. Soibelman 1991)

1. *Irreducible reps.* $\overset{1:1}{\longleftrightarrow}$ *elements of the Weyl group* $W(g)$ *(up to a "torus degree of freedom").*

   *Set* $\pi_i :=$ *the irreducible rep. for the simple reflection* $s_i \in W(g)$ *($i$ : a vertex of the Dynkin diagram of $g$).*

2. *The irreducible rep. corresponding to the reduced expression* $s_{i_1} \cdots s_{i_r} \in W(g)$ *is realized as the tensor product* $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$.

## Crucial Corollary

If $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}$ are 2 different reduced expressions, then

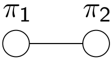
$$\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}.$$

$\Longrightarrow$ Exists the unique map $\Phi$ called intertwiner such that

$$(\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}) \circ \Phi = \Phi \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_r})$$

## Example

$$A_q(sl_3) = \langle t_{ij} \rangle_{i,j=1}^{3}$$




Fock representations

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}}$$

$W(sl_3) = \langle s_1, s_2 \rangle$. $\qquad s_2 s_1 s_2 = s_1 s_2 s_1$ (Coxeter relation)

$\Longrightarrow \quad \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_1 \otimes \pi_2 \otimes \pi_1$ as representations on $(F_q)^{\otimes 3}$

Exists the intertwiner $\Phi : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}$ such that

$$(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi = \Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1).$$

## Explicit form

$$R := \Phi P_{13}, \quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x,$$

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R^{abc}_{ijk} |a\rangle \otimes |b\rangle \otimes |c\rangle.$$

$$R^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c} \sum_{\lambda,\mu\geq 0,\lambda+\mu=b} (-1)^\lambda q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \times \begin{bmatrix} i, j, c+\mu \\ \mu, \lambda, i-\mu, j-\lambda, c \end{bmatrix}.$$

$$(q)_m = \prod_{j=1}^{m} (1-q^j), \quad \begin{bmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{bmatrix} = \frac{\prod_{m=1}^{r} (q^2)_{i_m}}{\prod_{m=1}^{s} (q^2)_{j_m}}$$

## Theorem (Kapranov-Voevodsky 1994)

*$R$ satisfies the tetrahedron eq.* $R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$.

**Essence of proof**. Consider $A_q(sl_4)$ and $W(sl_4) = \langle s_1, s_2, s_3\rangle$.

$$s_2s_1s_2 = s_1s_2s_1, \quad s_3s_2s_3 = s_2s_3s_2, \quad s_1s_3 = s_3s_1,$$

$$s_1s_2s_3s_1s_2s_1 = s_3s_2s_3s_1s_2s_3 \quad \text{(longest element)}$$

The intertwiner for the last one is constructed in 2 different ways as

| | | | |
|---|---|---|---|
| $123\underline{121}$ | $\Phi_{456}$ | $12\underline{31}21$ | $P_{34}$ |
| $1\underline{232}12$ | $\Phi_{234}$ | $\underline{121}321$ | $\Phi_{123}$ |
| $\underline{13}2\underline{31}2$ | $P_{12}P_{45}$ | $21\underline{232}1$ | $\Phi_{345}$ |
| $3\underline{121}32$ | $\Phi_{234}$ | $2\underline{13}2\underline{31}$ | $P_{23}P_{56}$ |
| $321\underline{232}$ | $\Phi_{456}$ | $23\underline{121}3$ | $\Phi_{345}$ |
| $32\underline{13}23$ | $P_{34}$ | $\underline{232}123$ | $\Phi_{123}$ |
| $323123$ | | $323123$ | |

Equate the 2 sides, substitute $\Phi_{ijk} = R_{ijk}P_{ik}$ and cancel $P_{ij}$'s. □

## Summary so far (type A case)

$$\begin{aligned} \text{Weyl group elements} &\longleftrightarrow \text{“Multi-string states”} \\ \text{Cubic Coxeter relation} &\longleftrightarrow \text{3D } R \text{ matrix} \\ \text{Reduced words for longest element} &\longleftrightarrow \text{Tetrahedron equation} \end{aligned}$$

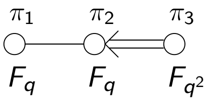
## Remark

- 3D $R$ = “Quantization” of Miquel’s theorem (1838)
  (Bazhanov-Sergeev-Mangazeev 2008)
- $q = 0$: Set-theoretical sol. to tropical (ultradiscrete) tetrahedron eq.

## Recent developments

- Type B, C, $F_4$ cases: 3D analogue of reflection equation.
- Connection to Poincaré-Birkhoff-Witt basis of $U_q^+(g)$.
- Reduction to 2D: Matrix product formula for quantum R’s of generalized quantum groups.
- Application to multispecies TASEP & TAZRP: Hidden 3D structure.

$A_q(C_3) = \langle t_{ij} \rangle_{i,j=1}^{6}$: (Reshetikhin-Takhtajan-Faddeev 1990)




$\pi_k(t_{ij})$ are given as follows.

$$\pi_1 : \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} \\ 0 & 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 \\ 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} & 0 \\ 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \langle \mathbf{A}^{\pm}, \mathbf{K} \rangle = \langle \mathbf{a}^{\pm}, \mathbf{k} \rangle|_{q \to q^2}.$$

$W(C_3) = \langle s_1, s_2, s_3 \rangle$

$$s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2.$$

Write simply as $\pi_{i_1,\ldots,i_r} := \pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$. Then,

| Equivalence | Intertwiner | |
|---|---|---|
| $\pi_{13} \simeq \pi_{31}$, | $P_{12}(x \otimes y) = y \otimes x$, | (trivial) |
| $\pi_{121} \simeq \pi_{212}$, | $\Phi = R P_{13}$ | (same as type A), |
| $\pi_{2323} \simeq \pi_{3232}$, | $\Psi = K P_{14} P_{23}$ | (new). |

$$K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q), \quad R \in \mathrm{End}(F_q^{\otimes 3}).$$

## Matrix elements

$$K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K^{cmdn}_{a\,i\,b\,j} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle.$$

$$K^{cmdn}_{a\,i\,b\,j} = 0 \ \text{ unless } \ c+m+d = a+i+b, \ \ d+n-c = b+j-a.$$

**Theorem** (A more structural formula is available in K-Maruyama 2015)

$$K^{c,m,0,n}_{a,i,0,j} = \sum_{\lambda \geq 0} (-1)^{m+\lambda} \frac{(q^4)_{c+\lambda}}{(q^4)_c} q^{\phi_2} \begin{bmatrix} i, j \\ \lambda, j-\lambda, m-\lambda, i-m+\lambda \end{bmatrix},$$

$$\phi_2 = (a+c+1)(m+j-2\lambda)+m-j.$$

$$K^{cmdn}_{a\,i\,b\,j} = \frac{(q^4)_a}{(q^4)_c} \sum_{\alpha,\beta,\gamma \geq 0} \frac{(-1)^{\alpha+\gamma}}{(q^4)_{d-\beta}} q^{\phi_1} K^{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma}_{c,m+d-\alpha-\beta-\gamma,0,n+d-\alpha-\beta-\gamma}$$

$$\times \begin{bmatrix} b, d-\beta, i+b-\alpha-\beta, j+b-\alpha-\beta \\ \alpha, \beta, \gamma, m-\alpha, n-\alpha, b-\alpha-\beta, d-\beta-\gamma \end{bmatrix},$$

$$\phi_1 = \alpha(\alpha+2d-2\beta-1)+(2\beta-d)(m+n+d)+\gamma(\gamma-1)-b(i+j+b).$$

**Theorem** (K-Okado 2012)

$R$ and $K$ yield the first nontrivial solution to the **3D reflection equation** proposed by Isaev-Kulish in 1997:

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

- The proof is parallel with type $A$.
- Uses the reduced expressions of the longest element

$$s_1s_2s_3s_2s_1s_2s_3s_2s_3 = s_3s_2s_3s_2s_1s_2s_3s_2s_1 \in W(C_3).$$

- The two sides come from the 2 ways of constructing the intertwiners for

$$\pi_{123212323} \simeq \pi_{323212321}$$

out of $R$ and $K$.

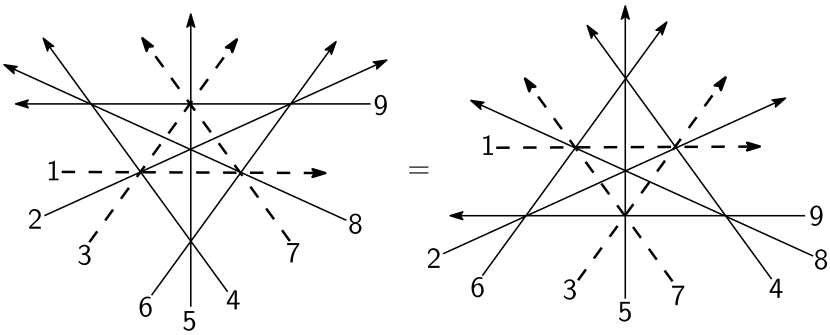
## Physical (geometric) interpretation of the 3D reflection eq.

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

is a "factorization" of 3 string scattering with boundary reflections.

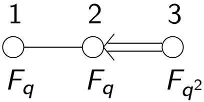
$R$ : Scattering amplitude of 3 strings.

$K$: Reflection amplitude with boundary freedom signified by spaces $1, 3, 7$.



# $B, F_4$ cases

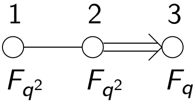
$C_3$




$R : 121 = 212$

$K : 2323 = 3232$

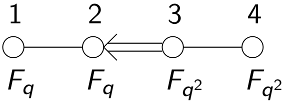
$B_3$




$S : 121 = 212$

$J : 2323 = 3232$

$F_4$




$R : 121 = 212$

$K : 2323 = 3232$

$S : 434 = 343$

$$R \in \mathrm{End}(F_q \otimes F_q \otimes F_q), \qquad K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$$

$$S = R|_{q \to q^2} \in \mathrm{End}(F_{q^2} \otimes F_{q^2} \otimes F_{q^2})$$

$$J = P_{14}P_{23}KP_{23}P_{14} \in \mathrm{End}(F_q \otimes F_{q^2} \otimes F_q \otimes F_{q^2}).$$

Both $(R, K)$ and $(S, J)$ satisfy the 3D reflection equation.

A reduced expression of the longest element of $W(F_4)$ is

$$s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1 \quad \text{(length 24)}.$$

The intertwiner for

$$\pi_{434234232123423123412321} \simeq \pi_{\text{reverse order}}$$

can be constructed by composition of $R, K, S$ in two ways, which must coincide. This leads to the $F_4$-analogue of the tetrahedron equation:

$$\begin{aligned}
& S_{14,15,16} S_{9,11,16} K_{16,10,8,7} K_{9,13,15,17} S_{4,5,16} R_{7,12,17} S_{1,2,16} R_{6,10,17} S_{9,14,18} K_{1,3,5,17} \\
\times\ & S_{11,15,18} K_{18,12,8,6} S_{1,4,18} S_{1,8,15} R_{7,13,19} K_{1,6,11,19} K_{4,12,15,19} R_{3,10,19} S_{4,8,11} K_{1,7,14,20} \\
\times\ & S_{2,5,18} R_{6,13,20} R_{3,12,20} S_{1,9,21} K_{2,10,15,20} S_{4,14,21} K_{21,13,8,3} S_{2,11,21} S_{2,8,14} R_{6,7,22} \\
\times\ & K_{2,3,4,22} S_{5,15,21} K_{11,13,14,22} R_{10,12,22} K_{2,6,9,23} R_{3,7,23} R_{19,20,22} K_{16,17,18,22} R_{10,13,23} \\
\times\ & K_{5,12,14,23} R_{3,6,24} K_{16,19,21,23} K_{4,7,9,24} R_{17,20,23} K_{5,10,11,24} R_{12,13,24} R_{17,19,24} \\
\times\ & K_{18,20,21,24} S_{5,8,9} R_{22,23,24} \ = \ \text{product in reverse order.}
\end{aligned}$$

$16R$'s, $16S$'s and $18K$'s acting on $F_{q_{i_1}} \otimes \cdots \otimes F_{q_{i_{24}}}$.

# Another aspect: Connection with PBW basis

$$U_q^+(sl_3) = \langle e_1, e_2 \rangle \text{ with Serre relation } [[e_1, e_2]_q, e_1]_q = [[e_2, e_1]_q, e_2]_q = 0.$$

$$\left( [x, y]_q := xy - qyx, \ \ [a]! = \prod_{m=1}^{a} \frac{q^m - q^{-m}}{q - q^{-1}} \right)$$

Two PBW bases: $\{E^{a,b,c}\}_{(a,b,c)\in(\mathbb{Z}_{\geq 0})^3}$, $\{E'^{a,b,c}\}_{(a,b,c)\in(\mathbb{Z}_{\geq 0})^3}$

$$E^{a,b,c} = \frac{e_1^a([e_2, e_1]_q)^b e_2^c}{[a]![b]![c]!}, \quad E'^{a,b,c} = E^{a,b,c}|_{e_1 \leftrightarrow e_2}$$

Then $E^{a,b,c} = \sum_{ijk} R^{abc}_{i,j,k} E'^{k,j,i}$ (Sergeev 2008)

3D R = transition matrix of the PBW bases of $U_q^+(sl_3)$

## Theorem (K-Okado-Yamada 2013)

For arbitrary simple Lie algebra $g$, set

$\Phi :=$ Intertwiner of Soibelman irreducible representations of $A_q(g)$,

$\Gamma :=$ Transition matrix of the PBW bases of $U_q^+(g)$.

Then $\Phi = \Gamma$.

Now we proceed to the last topic: **2D Reduction**

Tetrahedron equation $\Longrightarrow$ Yang-Baxter equation

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$
$$\Longrightarrow \quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

## Contents

- 3D $L$-operator : $RLLL = LLLR$
- Mixed $n$-product of $R$ and $L \Longrightarrow 2^n$-solutions to YBE
- Generalized quantum groups $\mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n), \mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n)$ ($\epsilon_i = 0, 1$)
- Main theorem

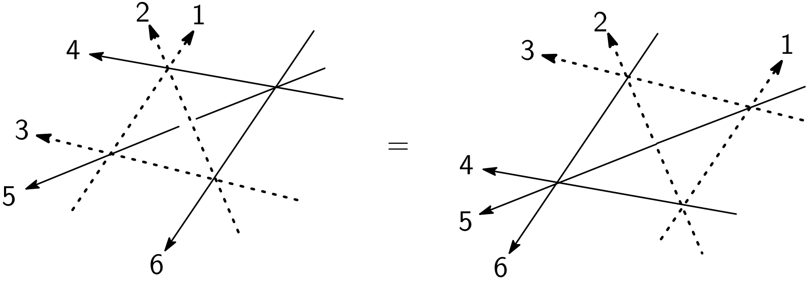
# 3D $L$-operator: $q$-oscillator valued 6-vertex model

$$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1, \quad F = F_q, \quad L = (L^{\gamma,\delta}_{\alpha,\beta}) \in \mathrm{End}(V \otimes V \otimes F)$$

$$L(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum_{\gamma,\delta} v_\gamma \otimes v_\delta \otimes L^{\gamma,\delta}_{\alpha,\beta}|m\rangle, \quad L^{\gamma,\delta}_{\alpha,\beta} \in \mathrm{End}(F)$$

$$L^{0,0}_{0,0} = L^{1,1}_{1,1} = 1, \;\; L^{0,1}_{0,1} = -q\mathbf{k}, \;\; L^{1,0}_{1,0} = \mathbf{k}, \;\; L^{0,1}_{1,0} = \mathbf{a}^-, \;\; L^{1,0}_{0,1} = \mathbf{a}^+.$$

$$L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124} \quad \text{(Bazhanov-Sergeev 2006)}$$



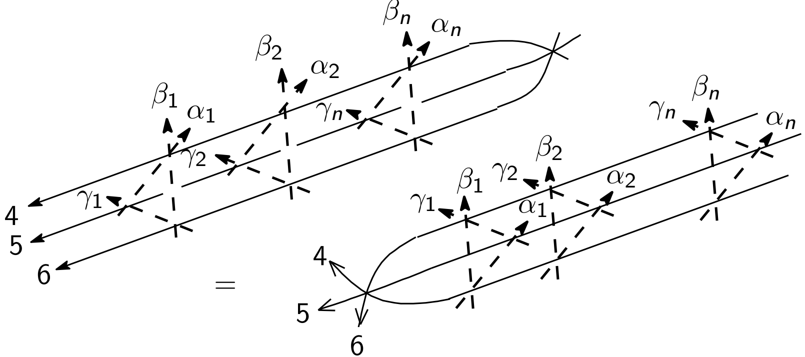
# $n$-layer tetrahedron equation

Write $RRRR = RRRR$ and $LLLR = RLLL$ as

$$\big(M^{(\epsilon)}_{\alpha\beta 4}M^{(\epsilon)}_{\alpha\gamma 5}M^{(\epsilon)}_{\beta\gamma 6}\big)R_{456} = R_{456}\big(M^{(\epsilon)}_{\beta\gamma 6}M^{(\epsilon)}_{\alpha\gamma 5}M^{(\epsilon)}_{\alpha\beta 4}\big), \quad M^{(0)} = R, \;\; M^{(1)} = L$$

$$M^{(\epsilon)}_{\alpha\beta 4} \in \mathrm{End}(\overset{\alpha}{W}{}^{(\epsilon)} \otimes \overset{\beta}{W}{}^{(\epsilon)} \otimes \overset{4}{F}), \text{ etc}, \quad W^{(0)} = F, \;\; W^{(1)} = V$$

$$\prod_{1\le i\le n}\big(M^{(\epsilon_i)}_{\alpha_i\beta_i 4}M^{(\epsilon_i)}_{\alpha_i\gamma_i 5}M^{(\epsilon_i)}_{\beta_i\gamma_i 6}\big)R_{456} = R_{456}\prod_{1\le i\le n}\big(M^{(\epsilon_i)}_{\beta_i\gamma_i 6}M^{(\epsilon_i)}_{\alpha_i\gamma_i 5}M^{(\epsilon_i)}_{\alpha_i\beta_i 4}\big)$$



# 2D reduction

$$\prod_i \left(M^{(\epsilon_i)}_{\alpha_i\beta_i 4} M^{(\epsilon_i)}_{\alpha_i\gamma_i 5} M^{(\epsilon_i)}_{\beta_i\gamma_i 6}\right) R_{456} = R_{456} \prod_i \left(M^{(\epsilon_i)}_{\beta_i\gamma_i 6} M^{(\epsilon_i)}_{\alpha_i\gamma_i 5} M^{(\epsilon_i)}_{\alpha_i\beta_i 4}\right) \quad \cdots (\sharp)$$

$\overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F}$ can be eliminated in two ways:

$$\text{(A)} \ \ \mathrm{Tr}_{456}\left(x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} \ (\sharp)\right), \qquad \text{(B)} \ \ {}_{456}\langle \chi | x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} \ (\sharp) | \chi \rangle_{456}$$

where $[x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6}, R_{456}] = 0$, ${}_{456}\langle \chi | R_{456} = {}_{456}\langle \chi |$, $R_{456} | \chi \rangle_{456} = | \chi \rangle_{456}$

$\mathbf{h}|m\rangle = m|m\rangle$, $|\chi\rangle_{456} = |\chi\rangle_4 \otimes |\chi\rangle_5 \otimes |\chi\rangle_6$, $|\chi\rangle = \sum_{m\geq 0} \frac{|m\rangle}{(q)_m}$.

Both lead to YBE: $S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x) S_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(xy) S_{\boldsymbol{\beta},\boldsymbol{\gamma}}(y) = S_{\boldsymbol{\beta},\boldsymbol{\gamma}}(y) S_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(xy) S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$ for

$$\text{(A)} \ \ S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = \mathrm{Tr}_3\left(z^{\mathbf{h}_3} M^{(\epsilon_1)}_{\alpha_1\beta_1 3} \cdots M^{(\epsilon_n)}_{\alpha_n\beta_n 3}\right),$$

$$\text{(B)} \ \ S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) = {}_3\langle \chi | z^{\mathbf{h}_3} M^{(\epsilon_1)}_{\alpha_1\beta_1 3} \cdots M^{(\epsilon_n)}_{\alpha_n\beta_n 3} | \chi \rangle_3.$$

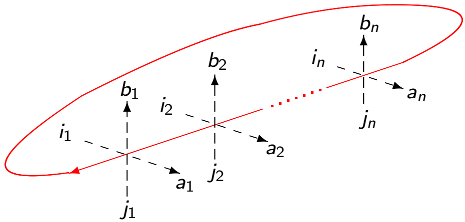
$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z) \in \mathrm{End}(W \otimes W), \quad W = W^{(\epsilon_1)} \otimes \cdots \otimes W^{(\epsilon_n)}.$$

# Matrix elements of $S(z)$

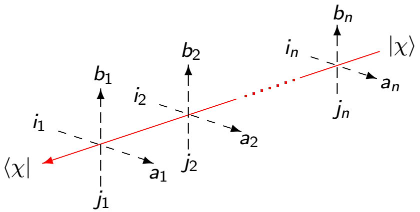
$$S(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a},\mathbf{b}} S^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}(z)|\mathbf{a}\rangle \otimes |\mathbf{b}\rangle,$$

$$|\mathbf{a}\rangle = |a_1, \ldots, a_n\rangle \in W^{(\epsilon_1)} \otimes \cdots \otimes W^{(\epsilon_n)}, \text{ etc.}$$

Matrix element $S^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}(z)$ is depicted as



(A) Trace reduction



(B) Boundary vector reduction

**Problem**:

Find a characterization of $S(z)$ obtained by (A) and (B) in the framework of the quantum group theory for each choice of $(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$.

**Example**:

$(\epsilon_1, \ldots, \epsilon_5) = (01101)$.

$$\text{(A)}\ \ S(z) = \mathrm{Tr}(RLLRL), \qquad \text{(B)}\ \ S(z) = \langle \chi | RLLRL | \chi \rangle,$$
$$S(z) \in \mathrm{End}(W \otimes W), \quad W = F \otimes V \otimes V \otimes F \otimes V$$

### Result

They are quantum $R$-matrices for some specific representations of generalized quantum groups $\mathcal{U}_A = \mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n)$ and $\mathcal{U}_B = \mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n)$ defined in the sequel.

# Def. $\mathcal{U}_A(\epsilon_1, \dots, \epsilon_n), \mathcal{U}_B(\epsilon_1, \dots, \epsilon_n)$ $(\epsilon_i = 0, 1)$

$\mathcal{U}_A$ and $\mathcal{U}_B$ are Hopf algebras over $\mathbb{C}(q^{\frac{1}{2}})$ with generators $e_i, f_i, k_i^{\pm 1}$ $(0 \le i \le \tilde{n})$ and relations ($\tilde{n} = n-1$ for $\mathcal{U}_A$, $\tilde{n} = n$ for $\mathcal{U}_B$)

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0,$$

$$k_i e_j = D_{i,j} e_j k_i, \quad k_i f_j = D_{i,j}^{-1} f_j k_i, \quad [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{r_i - r_i^{-1}}.$$

$$p = \mathrm{i} q^{-\frac{1}{2}}, \qquad q_i = q \ (\epsilon_i = 0), \quad q_i = -q^{-1} \ (\epsilon_i = 1),$$

$$r_i = q \ \text{ for } \ \mathcal{U}_A, \qquad r_i = \begin{cases} p & i = 0, n, \\ p^2 & 0 < i < n \end{cases} \ \text{ for } \ \mathcal{U}_B,$$

$$D_{i,j} = \prod_{k \in \langle i \rangle \cap \langle j \rangle} {q_k}^{2\delta_{i,j} - 1}, \qquad \langle i \rangle = \begin{cases} \{i, i+1\} & \text{for } \ \mathcal{U}_A, \\ \{i, i+1\} \cap [1, n] & \text{for } \ \mathcal{U}_B. \end{cases}$$

$$\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i$$

**Special cases**: (up to Serre-type relations)

- $\forall \epsilon_i = 0, 1$ cases = quantized Kac-Moody algebras of affine type

$$\mathcal{U}_A(0,\ldots,0) = U_q(A^{(1)}_{n-1}), \quad \mathcal{U}_A(1,\ldots,1) = U_{-q^{-1}}(A^{(1)}_{n-1}),$$
$$\mathcal{U}_B(0,\ldots,0) = U_q(D^{(2)}_{n+1}), \quad \mathcal{U}_B(1,\ldots,1) = U_{-q^{-1}}(D^{(2)}_{n+1}).$$

- $\mathcal{U}_A(\overbrace{0,\ldots,0}^{\kappa},\overbrace{1,\ldots,1}^{\kappa'}),\ \ \mathcal{U}_B(\overbrace{0,\ldots,0}^{\kappa},\overbrace{1,\ldots,1}^{\kappa'})$
  $=$ affinization of quantum superalgebras of type $A$ and $B$.

In general, $\mathcal{U}_A$ and $\mathcal{U}_B$ are examples of generalized quantum groups introduced and being developed by Heckenberger (2010), Andruskiewitsch-Schneider (2010), Angiono-Yamane (2015), Azam-Yamane-Yousofzadeh, etc.

# Relevant irreducible representation $\pi_x$ ($\mathcal{U}_B$ case)

$$|\mathbf{m}\rangle = |m_1, \ldots, m_n\rangle \in W = W^{(\epsilon_1)} \otimes \cdots \otimes W^{(\epsilon_n)} \quad (W^{(0)} = F,\ W^{(1)} = V)$$

$$\mathbf{e}_i = (0, \ldots, \overset{i}{1}, \ldots, 0),\ \ [m] = (q^m - q^{-m})/(q - q^{-1})$$

$\pi_x : \mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n) \to \mathrm{End}(W)$ is defined by

$$e_0|\mathbf{m}\rangle = x|\mathbf{m} + \mathbf{e}_1\rangle, \qquad e_n|\mathbf{m}\rangle = [m_n]|\mathbf{m} - \mathbf{e}_n\rangle,$$

$$f_0|\mathbf{m}\rangle = x^{-1}[m_1]|\mathbf{m} - \mathbf{e}_1\rangle, \qquad f_n|\mathbf{m}\rangle = |\mathbf{m} + \mathbf{e}_n\rangle,$$

$$k_0|\mathbf{m}\rangle = p^{-1}(q_1)^{m_1}|\mathbf{m}\rangle, \qquad k_n|\mathbf{m}\rangle = p(q_n)^{-m_n}|\mathbf{m}\rangle,$$

$$e_i|\mathbf{m}\rangle = [m_i]|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle \quad (0 < i < n),$$

$$f_i|\mathbf{m}\rangle = [m_{i+1}]|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle \quad (0 < i < n),$$

$$k_i|\mathbf{m}\rangle = (q_i)^{-m_i}(q_{i+1})^{m_{i+1}}|\mathbf{m}\rangle \quad (0 < i < n).$$

$\forall \epsilon_i = 0$ case: $q$-oscillator representation of $U_q(D^{(2)}_{n+1})$

$\forall \epsilon_i = 1$ case: spin representation of $U_{-q^{-1}}(D^{(2)}_{n+1})$

**Quantum $R$ matrix**

$R(z) \in \mathrm{End}(W \otimes W)$ is characterized up to an overall scalar by

$$[PR(x/y), \Delta_{x,y}(g)] = 0 \quad \forall g \in \mathcal{U}_B,$$
$$\text{where } \Delta_{x,y} := (\pi_x \otimes \pi_y)\Delta, \quad P(u \otimes v) = v \otimes u.$$

**Theorem** (K-Okado-Sergeev 2015)

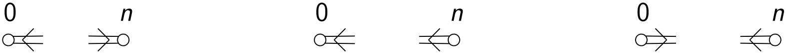
$S(z)$'s obtained by (A) trace reduction and (B) boundary vector reduction are the quantum $R$ matrices of $\mathcal{U}_A$ and $\mathcal{U}_B$, respectively.

# Ending remarks

- More is known for homogeneous case $\epsilon_1 = \cdots = \epsilon_n = 0, 1$.

  $\exists$ Two boundary vectors $|\chi_1\rangle = \sum_m \frac{|m\rangle}{(q)_m}$, $|\chi_2\rangle = \sum_m \frac{|2m\rangle}{(q^4)_m}$

  $\iff$ End shape of relevant Dynkin diagrams




| | | |
|---|---|---|
| $\langle \chi_1 \vert R \cdots R \vert \chi_1 \rangle$ | $\langle \chi_1 \vert R \cdots R \vert \chi_2 \rangle$ | $\langle \chi_2 \vert R \cdots R \vert \chi_2 \rangle$ |
| $U_q(D^{(2)}_{n+1})$ | $U_q(A^{(2)}_{2n})$ | $U_q(C^{(1)}_n)$ |

- $U_q(A^{(1)}_{n-1})$ R matrices $= \mathrm{Tr}(LL \cdots L), \mathrm{Tr}(RR \cdots R)$

  $\Longrightarrow$ Matrix product stationary states (hidden 3D structure) in 1D Totally Asymmetric Simple Exclusion Process (TASEP) and 1D Totally Asymmetric Zero Range Process (TAZRP) (K-Maruyama-Okado, 2015-2016.)

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