Tetrahedron equation, 3D reflection equation and generalized quantum groups

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Key to integrability in 2D

Yang-Baxter equation
\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \]

Reflection equation
\[ R_{21} K_2 R_{12} K_1 = K_1 R_{21} K_2 R_{12} \]

\( R \): 2 particle scattering

\( K \): Reflection at boundary
What about 3D?

**Tetrahedron equation** (A.B. Zamolodchikov, 1980)

\[ R : F \otimes F \otimes F \rightarrow F \otimes F \otimes F \quad (3D \ R) \]

\[ R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123} \]

\[ R = \begin{cases} 
3 \text{ string scattering amplitude in (2+1)D} \\
\text{local Boltzmann weight of the vertex in 3D}
\end{cases} \]
Status of finding solutions and relevant maths

2D

- Infinitely many solutions constructed systematically by representation theory of the Drinfeld-Jimbo quantum affine algebra $U_q(\hat{g})$ ($\hat{g} =$ affine Kac-Moody algebra).

3D

- A few classes of solutions are known.
- Systematic framework yet to be developed.
- One such approach is by quantized algebra of functions $A_q(g)$ ($g =$ finite dimensional simple Lie algebra).
- $A_q(g)$ is the quantum group corresponding to the dual of $U_q(g)$. Studied by Drinfeld (87), Vaksman-Soibelman (89,91), Reshetikhin-Takhtajan-Faddeev (90), Noumi-Yamada-Mimachi (92), Kashiwara (93), Geiss-Leclerc-Schröer (2011-) etc.
Simplest example:

\[ SL_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid [t_{ij}, t_{kl}] = 0, \ t_{11}t_{22} - t_{12}t_{21} = 1 \right\}. \]

\( A_q(sl_2) \) is generated by \( t_{11}, t_{12}, t_{21}, t_{22} \) with the relations

\[
\begin{align*}
t_{11}t_{21} &= qt_{21}t_{11}, \\
t_{12}t_{22} &= qt_{22}t_{12}, \\
t_{11}t_{12} &= qt_{12}t_{11}, \\
t_{21}t_{22} &= qt_{22}t_{21},
\end{align*}
\]

\[
\begin{align*}
[t_{12}, t_{21}] &= 0, \\
[t_{11}, t_{22}] &= (q - q^{-1})t_{21}t_{12}, \\
t_{11}t_{22} - qt_{12}t_{21} &= 1.
\end{align*}
\]

Hopf algebra with coproduct \( \Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj} \).

Fock representation \( \pi_1 : A_q(sl_2) \to \text{End}(F_q) \)

\( F_q = \oplus_{m \geq 0} \mathbb{C}|m\rangle : q\text{-oscillator Fock space} \)

\[
\begin{align*}
\pi_1 : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} &\mapsto \begin{pmatrix} a^- & k \\ -qk & a^+ \end{pmatrix} \\
|k\rangle \langle m | = q^m |m\rangle, &\quad a^+ \langle m | = | m + 1 \rangle, &\quad a^- \langle m | = (1 - q^{2m}) |m - 1 \rangle.
\end{align*}
\]
Theorem (Classification of irreducible representations. Soibelman 1991)

1. **Irreducible reps.** \( \longleftrightarrow \) elements of the Weyl group \( W(g) \) (up to a “torus degree of freedom”).

   Set \( \pi_i \) := the irreducible rep. for the simple reflection \( s_i \in W(g) \) (\( i \) : a vertex of the Dynkin diagram of \( g \)).

2. The irreducible rep. corresponding to the reduced expression \( s_{i_1} \cdots s_{i_r} \in W(g) \) is realized as the tensor product \( \pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \).

**Crucial Corollary**

If \( s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r} \) are 2 different reduced expressions, then

\[
\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \cong \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}.
\]

\[\implies\text{ Exists the unique map } \Phi \text{ called } \textit{intertwiner} \text{ such that } \]

\[
(\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}) \circ \Phi = \Phi \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_r})
\]
Example

\[ A_q(sl_3) = \left\langle t_{ij} \right\rangle_{i,j=1}^3 \]

Fock representations

\[
\begin{pmatrix}
  t_{11} & t_{12} & t_{13} \\
  t_{21} & t_{22} & t_{23} \\
  t_{31} & t_{32} & t_{33}
\end{pmatrix} \quad \longrightarrow \quad
\begin{pmatrix}
  a^- & k & 0 \\
  -qk & a^+ & 0 \\
  0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & a^- & k \\
  0 & -qk & a^+
\end{pmatrix}
\]

\[ W(sl_3) = \langle s_1, s_2 \rangle. \quad s_2 s_1 s_2 = s_1 s_2 s_1 \text{ (Coxeter relation)} \]

\[ \pi_2 \otimes \pi_1 \otimes \pi_2 \cong \pi_1 \otimes \pi_2 \otimes \pi_1 \text{ as representations on } (F_q)^{\otimes 3} \]

Exists the intertwiner \( \Phi : (F_q)^{\otimes 3} \rightarrow (F_q)^{\otimes 3} \) such that

\[ (\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi = \Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1). \]
Explicit form

\[ R := \Phi P_{13}, \quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x, \]

\[ R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R_{ijk}^{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle. \]

\[ R_{ijk}^{abc} = \delta_{i+j,a+b} \delta_{j+k,b+c} \sum_{\lambda,\mu \geq 0, \lambda+\mu = b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \]

\[ \times \begin{bmatrix} i, j, c + \mu \\ \mu, \lambda, i - \mu, j - \lambda, c \end{bmatrix}. \]

\[ (q)_m = \prod_{j=1}^{m} (1 - q^j), \quad \left[ i_1, \ldots, i_r \middle| j_1, \ldots, j_s \right] = \frac{\prod_{m=1}^{r} (q^2)_{i_m}}{\prod_{m=1}^{s} (q^2)_{j_m}} \]
Theorem (Kapranov-Voevodsky 1994)

\[ R \text{ satisfies the tetrahedron eq. } R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123}. \]

**Essence of proof.** Consider \( A_q(sl_4) \) and \( W(sl_4) = \langle s_1, s_2, s_3 \rangle \).

\[
\begin{align*}
& s_2 s_1 s_2 = s_1 s_2 s_1, \quad s_3 s_2 s_3 = s_2 s_3 s_2, \quad s_1 s_3 = s_3 s_1, \\
& s_1 s_2 s_3 s_1 s_2 s_1 = s_3 s_2 s_3 s_1 s_2 s_3 \text{ (longest element)}
\end{align*}
\]

The intertwiner for the last one is constructed in 2 different ways as

\[
\begin{align*}
& 123123 \quad \Phi_{456} \quad 123123 \quad P_{34} \\
& 123212 \quad \Phi_{234} \quad 121321 \quad \Phi_{123} \\
& 132312 \quad P_{12} P_{45} \quad 212321 \quad \Phi_{345} \\
& 312132 \quad \Phi_{234} \quad 213231 \quad P_{23} P_{56} \\
& 321232 \quad \Phi_{456} \quad 231213 \quad \Phi_{345} \\
& 321323 \quad P_{34} \quad 232123 \quad \Phi_{123} \\
& 323123 \quad 323123
\end{align*}
\]

Equate the 2 sides, substitute \( \Phi_{ijk} = R_{ijk} P_{ik} \) and cancel \( P_{ij} \)'s. \( \square \)
Summary so far (type A case)

Weyl group elements $\leftrightarrow$ “Multi-string states”
Cubic Coxeter relation $\leftrightarrow$ 3D $R$ matrix
Reduced words for longest element $\leftrightarrow$ Tetrahedron equation

Remark

- 3D $R$ = “Quantization” of Miquel’s theorem (1838)
  (Bazhanov-Sergeev-Mangazeev 2008)
- $q = 0$: Set-theoretical sol. to tropical (ultradiscrete) tetrahedron eq.

Recent developments

- Type B, C, $F_4$ cases: 3D analogue of reflection equation.
- Connection to Poincaré-Birkhoff-Witt basis of $U_q^+(g)$.
- Reduction to 2D: Matrix product formula for quantum $R$’s of generalized quantum groups.
- Application to multispecies TASEP & TAZRP: Hidden 3D structure.
\[ A_q(C_3) = \langle t_{ij} \rangle_{i,j=1}^6 : \] (Reshetikhin-Takhtajan-Faddeev 1990)

\[ \pi_k(t_{ij}) \text{ are given as follows.} \]

\[ \pi_1 : \]
\[
\begin{pmatrix}
 a^- & k & 0 & 0 & 0 & 0 \\
 -qk & a^+ & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & a^- & -k \\
 0 & 0 & 0 & 0 & qk & a^+
\end{pmatrix},
\]
\[ \pi_2 : \]
\[
\begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & a^- & k & 0 & 0 & 0 \\
 0 & -qk & a^+ & 0 & 0 & 0 \\
 0 & 0 & 0 & a^- & -k & 0 \\
 0 & 0 & 0 & qk & a^+ & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
\[ \pi_3 : \]
\[
\begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & A^- & K & 0 & 0 \\
 0 & 0 & -q^2K & A^+ & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
\[ \langle A^\pm, K \rangle = \langle a^\pm, k \rangle_{q \rightarrow q^2}. \]
\[ W(C_3) = \langle s_1, s_2, s_3 \rangle \]

\[ s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2. \]

Write simply as \[ \pi_{i_1, \ldots, i_r} := \pi_{i_1} \otimes \cdots \otimes \pi_{i_r}. \] Then,

<table>
<thead>
<tr>
<th>Equivalence</th>
<th>Intertwiner</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>( \pi_{13} \simeq \pi_{31} ),</td>
<td>( P_{12}(x \otimes y) = y \otimes x ),</td>
<td>(trivial)</td>
</tr>
<tr>
<td>( \pi_{121} \simeq \pi_{212} ),</td>
<td>( \Phi = R P_{13} )</td>
<td>(same as type A),</td>
</tr>
<tr>
<td>( \pi_{2323} \simeq \pi_{3232} ),</td>
<td>( \Psi = K P_{14} P_{23} )</td>
<td>(new).</td>
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\[ K \in \text{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q), \quad R \in \text{End}(F_{q^3}). \]
Matrix elements

\[ K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K_{a\,i\,b\,j}^{cmdn} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle. \]

\[ K_{a\,i\,b\,j}^{cmdn} = 0 \text{ unless } c + m + d = a + i + b, \quad d + n - c = b + j - a. \]

Theorem (A more structural formula is available in K-Maruyama 2015)

\[ K_{a\,i\,0\,j}^{c,m,0,n} = \sum_{\lambda \geq 0} (-1)^{m+\lambda} \frac{(q^4)^{c+\lambda}}{(q^4)^c} q^{\phi_2} \begin{bmatrix} i, j \\ \lambda, j - \lambda, m - \lambda, i - m + \lambda \end{bmatrix}, \]

\[ \phi_2 = (a + c + 1)(m + j - 2\lambda) + m - j. \]

\[ K_{a\,i\,b\,j}^{cmdn} = \frac{(q^4)^a}{(q^4)^c} \sum_{\alpha, \beta, \gamma \geq 0} (-1)^{\alpha+\gamma} \frac{(q^4)^{\alpha+\gamma}}{(q^4)^{d-\beta}} q^{\phi_1} K_{c,m+d-\alpha-\beta-\gamma,0,n+d-\alpha-\beta-\gamma}^{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma} \]

\[ \times \begin{bmatrix} b, d - \beta, i + b - \alpha - \beta, j + b - \alpha - \beta \\ \alpha, \beta, \gamma, m - \alpha, n - \alpha, b - \alpha - \beta, d - \beta - \gamma \end{bmatrix}, \]

\[ \phi_1 = \alpha(\alpha + 2d - 2\beta - 1) + (2\beta - d)(m + n + d) + \gamma(\gamma - 1) - b(i + j + b). \]
Theorem (K-Okado 2012)

$R$ and $K$ yield the first nontrivial solution to the 3D reflection equation proposed by Isaev-Kulish in 1997:

$$R_{489} K_{3579} R_{269} R_{258} K_{1678} K_{1234} R_{654} = R_{654} K_{1234} K_{1678} R_{258} R_{269} K_{3579} R_{489}.$$ 

- The proof is parallel with type $A$.
- Uses the reduced expressions of the longest element

$$s_1 s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_1 \in W(C_3).$$

- The two sides come from the 2 ways of constructing the intertwiners for

$$\pi_{12321232} \simeq \pi_{32321232}$$

out of $R$ and $K$. 
Physical (geometric) interpretation of the 3D reflection eq.

\[ R_{489} K_{3579} R_{269} R_{258} K_{1678} K_{1234} R_{654} = R_{654} K_{1234} K_{1678} R_{258} R_{269} K_{3579} R_{489}. \]

is a “factorization” of 3 string scattering with boundary reflections.

\( R \) : Scattering amplitude of 3 strings.
\( K \): Reflection amplitude with boundary freedom signified by spaces 1, 3, 7.
Both \((R, K)\) and \((S, J)\) satisfy the 3D reflection equation.
A reduced expression of the longest element of $W(F_4)$ is

$$s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_2 s_1$$

(length 24).

The intertwiner for

$$\pi_{434234232123423123412321} \sim \pi_{\text{reverse order}}$$

can be constructed by composition of $R$, $K$, $S$ in two ways, which must coincide. This leads to the $F_4$-analogue of the tetrahedron equation:

$$S_{14,15,16} S_{9,11,16} K_{16,10,8,7} K_{9,13,15,17} S_{4,5,16} R_{7,12,17} S_{1,2,16} R_{6,10,17} S_{9,14,18} K_{1,3,5,17}$$

$$\times S_{11,15,18} K_{18,12,8,6} S_{1,4,18} S_{1,8,15} R_{7,13,19} K_{1,6,11,19} K_{4,12,15,19} R_{3,10,19} S_{4,8,11} K_{1,7,14,20}$$

$$\times S_{2,5,18} R_{6,13,20} R_{3,12,20} S_{1,9,21} K_{2,10,15,20} S_{4,14,21} K_{21,13,8,3} S_{2,11,21} S_{2,8,14} R_{6,7,22}$$

$$\times K_{2,3,4,22} S_{5,15,21} K_{11,13,14,22} R_{10,12,22} K_{2,6,9,23} R_{3,7,23} R_{19,20,22} K_{16,17,18,22} R_{10,13,23}$$

$$\times K_{5,12,14,23} R_{3,6,24} K_{16,19,21,23} K_{4,7,9,24} R_{17,20,23} K_{5,10,11,24} R_{12,13,24} R_{17,19,24}$$

$$\times K_{18,20,21,24} S_{5,8,9} R_{22,23,24} = \text{product in reverse order}.$$

16$R$'s, 16$S$'s and 18$K$'s acting on $F_{q_1} \otimes \cdots \otimes F_{q_{24}}$. 
Another aspect: Connection with PBW basis

$U_q^+(sl_3) = \langle e_1, e_2 \rangle$ with Serre relation $[[e_1, e_2]_q, e_1]_q = [[e_2, e_1]_q, e_2]_q = 0.$

$$([x, y]_q : = xy - qyx, \; [a]! = \prod_{m=1}^a \frac{q^m - q^{-m}}{q - q^{-1}})$$

Two PBW bases: $\{E^{a,b,c}\}_{(a,b,c) \in (\mathbb{Z}_{\geq 0})^3}$, $\{E'^{a,b,c}\}_{(a,b,c) \in (\mathbb{Z}_{\geq 0})^3}$

$$E^{a,b,c} = \frac{e_1^a([e_2, e_1]_q)^b e_2^c}{[a]! [b]! [c]!}, \quad E'^{a,b,c} = E^{a,b,c} |_{e_1 \leftrightarrow e_2}$$

Then $E^{a,b,c} = \sum_{ijk} R_{i,j,k}^{abc} E'^{k,j,i}$ (Sergeev 2008)

3D $R =$ transition matrix of the PBW bases of $U_q^+(sl_3)$

Theorem (K-Okado-Yamada 2013)

For arbitrary simple Lie algebra $g$, set

$\Phi :=$ Intertwiner of Soibelman irreducible representations of $A_q(g)$,

$\Gamma :=$ Transition matrix of the PBW bases of $U_q^+(g)$.

Then $\Phi = \Gamma$. 
Now we proceed to the last topic: **2D Reduction**

Tetrahedron equation $\Rightarrow$ Yang-Baxter equation

$$
R_{124} R_{135} R_{236} R_{456} = R_{456} R_{236} R_{135} R_{124}
\Rightarrow
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}
$$

**Contents**

- 3D $L$-operator: $RLLL = LLLR$
- Mixed $n$-product of $R$ and $L \Rightarrow 2^n$-solutions to YBE
- Generalized quantum groups $\mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n), \mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n)$ ($\epsilon_i = 0, 1$)
- Main theorem
3D $L$-operator: $q$-oscillator valued 6-vertex model

$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$, $F = F_q$, $L = (L_{\alpha,\beta}^{\gamma,\delta}) \in \text{End}(V \otimes V \otimes F)$

$L(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum_{\gamma,\delta} v_\gamma \otimes v_\delta \otimes L_{\alpha,\beta}^{\gamma,\delta}|m\rangle$, $L_{\alpha,\beta}^{\gamma,\delta} \in \text{End}(F)$

$L_{0,0}^{0,0} = L_{1,1}^{1,1} = 1$, $L_{0,1}^{0,1} = -qk$, $L_{1,0}^{1,0} = k$, $L_{0,1}^{0,1} = a^-$, $L_{0,1}^{1,0} = a^+$.

$L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124}$ (Bazhanov-Sergeev 2006)
Write $RRRR = RRRR$ and $LLLR = RLLL$ as

\[
(M^{(e)}_{\alpha \beta 4} M^{(e)}_{\alpha \gamma 5} M^{(e)}_{\beta \gamma 6}) \ R_{456} = R_{456} (M^{(e)}_{\beta \gamma 6} M^{(e)}_{\alpha \gamma 5} M^{(e)}_{\alpha \beta 4}),
\]

\[M^{(e)}_{\alpha \beta 4} \in \text{End}(W(\epsilon) \otimes W(\epsilon) \otimes F),\text{ etc.,}\]

\[
\prod_{1 \leq i \leq n} (M^{(e_i)}_{\alpha_i \beta_i 4} M^{(e_i)}_{\alpha_i \gamma_i 5} M^{(e_i)}_{\beta_i \gamma_i 6}) \ R_{456} = R_{456} \prod_{1 \leq i \leq n} (M^{(e_i)}_{\beta_i \gamma_i 6} M^{(e_i)}_{\alpha_i \gamma_i 5} M^{(e_i)}_{\alpha_i \beta_i 4})
\]

$M^{(0)} = R, \ M^{(1)} = L$

$W^{(0)} = F, \ W^{(1)} = V$
2D reduction

$$\prod_i (M^{(\epsilon_i)}_{\alpha_i \beta_i \gamma_i} M^{(\epsilon_i)}_{\alpha_i \gamma_i} M^{(\epsilon_i)}_{\beta_i \gamma_i}) R_{456} = R_{456} \prod_i (M^{(\epsilon_i)}_{\beta_i \gamma_i} M^{(\epsilon_i)}_{\alpha_i \gamma_i} M^{(\epsilon_i)}_{\alpha_i \beta_i}) \cdots (#)$$

$F \otimes F \otimes F$ can be eliminated in two ways:

(A) $\text{Tr}_{456} (x^h (xy)^h y^h)$ $(#)$,  
(B) $456 \langle \chi | x^h (xy)^h y^h (#) | \chi \rangle_{456}$

where $[x^h (xy)^h y^h, R_{456}] = 0$, $456 \langle \chi | R_{456} = 456 \langle \chi |$, $R_{456} | \chi \rangle_{456} = | \chi \rangle_{456}$

$h | m \rangle = m | m \rangle$, $| \chi \rangle_{456} = | \chi \rangle_4 \otimes | \chi \rangle_5 \otimes | \chi \rangle_6$, $| \chi \rangle = \sum_{m \geq 0} \frac{|m \rangle}{(q)_m}$.

Both lead to YBE: $S_{\alpha, \beta}(x) S_{\alpha, \gamma}(xy) S_{\beta, \gamma}(y) = S_{\beta, \gamma}(y) S_{\alpha, \gamma}(xy) S_{\alpha, \beta}(x)$ for

(A) $S_{\alpha, \beta}(z) = \text{Tr}_3 (z^{h_3} M^{(\epsilon_1)}_{\alpha_1 \beta_1} \cdots M^{(\epsilon_n)}_{\alpha_n \beta_n})$, 
(B) $S_{\alpha, \beta}(z) = 3 \langle \chi | z^{h_3} M^{(\epsilon_1)}_{\alpha_1 \beta_1} \cdots M^{(\epsilon_n)}_{\alpha_n \beta_n} | \chi \rangle_3$.

$S_{\alpha, \beta}(z) \in \text{End}(W \otimes W)$, $W = W^{(\varepsilon_1)} \otimes \cdots \otimes W^{(\varepsilon_n)}$. 
Matrix elements of $S(z)$

$$S(z)(|i\rangle \otimes |j\rangle) = \sum_{a,b} S_{i,j}^{a,b}(z) |a\rangle \otimes |b\rangle,$$

$$|a\rangle = |a_1, \ldots, a_n\rangle \in \mathcal{W}(\epsilon_1) \otimes \ldots \otimes \mathcal{W}(\epsilon_n), \text{ etc.}$$

Matrix element $S_{i,j}^{a,b}(z)$ is depicted as

(A) Trace reduction

(B) Boundary vector reduction
Problem:

Find a characterization of $S(z)$ obtained by (A) and (B) in the framework of the quantum group theory for each choice of $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$.

Example:

$(\epsilon_1, \ldots, \epsilon_5) = (01101)$.

\[(A) \quad S(z) = \text{Tr}(RLLRL), \quad \quad (B) \quad S(z) = \langle \chi | RLLRL | \chi \rangle,\]

\[S(z) \in \text{End}(W \otimes W), \quad W = F \otimes V \otimes V \otimes F \otimes V\]

Result

They are quantum $R$-matrices for some specific representations of generalized quantum groups $\mathcal{U}_A = \mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n)$ and $\mathcal{U}_B = \mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n)$ defined in the sequel.
Def. $\mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n), \mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n)$ ($\epsilon_i = 0, 1$)

$\mathcal{U}_A$ and $\mathcal{U}_B$ are Hopf algebras over $\mathbb{C}(q^{1/2})$ with generators $e_i, f_i, k_i^{\pm 1}$ ($0 \leq i \leq \tilde{n}$) and relations ($\tilde{n} = n - 1$ for $\mathcal{U}_A$, $\tilde{n} = n$ for $\mathcal{U}_B$)

\[
\begin{align*}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0, \\
k_i e_j &= D_{i,j} e_j k_i, \quad k_i f_j = D_{i,j}^{-1} f_j k_i, \quad [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{r_i - r_i^{-1}}.
\end{align*}
\]

\[p = i q^{-1/2}, \quad q_i = q \ (\epsilon_i = 0), \quad q_i = -q_i^{-1} \ (\epsilon_i = 1),\]

\[r_i = q \text{ for } \mathcal{U}_A, \quad r_i = \begin{cases} p & i = 0, n, \\ p^2 & 0 < i < n \end{cases} \text{ for } \mathcal{U}_B,\]

\[D_{i,j} = \prod_{k \in \langle i \rangle \cap \langle j \rangle} q_k^{2\delta_{i,j} - 1}, \quad \langle i \rangle = \begin{cases} \{i, i + 1\} & \text{for } \mathcal{U}_A, \\ \{i, i + 1\} \cap [1, n] & \text{for } \mathcal{U}_B. \end{cases}\]

\[\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i.\]
Special cases: (up to Serre-type relations)

- $\forall \epsilon_i = 0, 1$ cases = quantized Kac-Moody algebras of affine type
  \begin{align*}
  \mathcal{U}_A(0, \ldots, 0) &= U_q(A^{(1)}_{n-1}), & \mathcal{U}_A(1, \ldots, 1) &= U_{-q^{-1}}(A^{(1)}_{n-1}), \\
  \mathcal{U}_B(0, \ldots, 0) &= U_q(D^{(2)}_{n+1}), & \mathcal{U}_B(1, \ldots, 1) &= U_{-q^{-1}}(D^{(2)}_{n+1}).
  \end{align*}

- $\mathcal{U}_A(0, \ldots, 0, 1, \ldots, 1), \mathcal{U}_B(0, \ldots, 0, 1, \ldots, 1)$
  = affinization of quantum superalgebras of type $A$ and $B$.

In general, $\mathcal{U}_A$ and $\mathcal{U}_B$ are examples of generalized quantum groups introduced and being developed by Heckenberger (2010), Andruskiewitsch-Schneider (2010), Angiono-Yamane (2015), Azam-Yamane-Yousofzadeh, etc.
Relevant irreducible representation $\pi_x$ ($\mathcal{U}_B$ case)

$$|m\rangle = |m_1, \ldots, m_n\rangle \in \mathcal{W} = \mathcal{W}(\epsilon_1) \otimes \cdots \otimes \mathcal{W}(\epsilon_n) \quad (\mathcal{W}^{(0)} = F, \ \mathcal{W}^{(1)} = V)$$

$e_i = (0, \ldots, 1, \ldots, 0)$, $[m] = (q^m - q^{-m})/(q - q^{-1})$

$\pi_x : \mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n) \to \text{End}(\mathcal{W})$ is defined by

$$e_0|m\rangle = x|m + e_1\rangle, \quad e_n|m\rangle = [m_n]|m - e_n\rangle,$$

$$f_0|m\rangle = x^{-1}[m_1]|m - e_1\rangle, \quad f_n|m\rangle = |m + e_n\rangle,$$

$$k_0|m\rangle = p^{-1}(q_1)^{m_1}|m\rangle, \quad k_n|m\rangle = p(q_n)^{-m_n}|m\rangle,$$

$$e_i|m\rangle = [m_i]|m - e_i + e_{i+1}\rangle \quad (0 < i < n),$$

$$f_i|m\rangle = [m_{i+1}]|m + e_i - e_{i+1}\rangle \quad (0 < i < n),$$

$$k_i|m\rangle = (q_i)^{-m_i}(q_{i+1})^{m_{i+1}}|m\rangle \quad (0 < i < n).$$

$\forall \epsilon_i = 0$ case: $q$-oscillator representation of $U_q(D_{n+1}^{(2)})$

$\forall \epsilon_i = 1$ case: spin representation of $U_{-q^{-1}}(D_{n+1}^{(2)})$
Quantum $R$ matrix

$R(z) \in \text{End}(W \otimes W)$ is characterized up to an overall scalar by

$$[PR(x/y), \Delta_{x,y}(g)] = 0 \quad \forall g \in \mathcal{U}_B,$$

where $\Delta_{x,y} := (\pi_x \otimes \pi_y) \Delta, \quad P(u \otimes v) = v \otimes u$.

Theorem (K-Okado-Sergeev 2015)

$S(z)$'s obtained by (A) trace reduction and (B) boundary vector reduction are the quantum $R$ matrices of $\mathcal{U}_A$ and $\mathcal{U}_B$, respectively.
More is known for homogeneous case $\epsilon_1 = \cdots = \epsilon_n = 0, 1$.

\[ \exists \text{ Two boundary vectors } |\chi_1\rangle = \sum_m \frac{|m\rangle}{(q)_m}, \quad |\chi_2\rangle = \sum_m \frac{|2m\rangle}{(q^4)_m} \]

$\iff$ End shape of relevant Dynkin diagrams

\[
\begin{array}{ccc}
0 & \cdots & n \\
\langle \chi_1 | R \cdots R | \chi_1 \rangle & U_q(D_{n+1}^{(2)}) & \langle \chi_2 | R \cdots R | \chi_2 \rangle \\
0 & \cdots & n \\
\langle \chi_1 | R \cdots R | \chi_2 \rangle & U_q(A_{2n}^{(2)}) & \langle \chi_2 | R \cdots R | \chi_2 \rangle \\
0 & \cdots & n \\
\langle \chi_2 | R \cdots R | \chi_2 \rangle & U_q(C_{n}^{(1)}) & \end{array}
\]

$U_q(A_{n-1}^{(1)})$ R matrices $= \text{Tr}(LL\cdots L), \text{Tr}(RR\cdots R)$

$\implies$ Matrix product stationary states (hidden 3D structure)

in 1D Totally Asymmetric Simple Exclusion Process (TASEP)

and 1D Totally Asymmetric Zero Range Process (TAZRP)

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