New solutions to the tetrahedron equation from quantized six-vertex models

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1. Introduction
2. Quantized (6V & YBE)
3. Solutions
4. Conjectures on RRRR=RRRR

1. Introduction

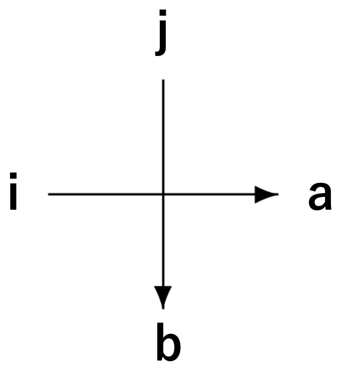
2-dimensional (2D) R-matrix

$$R : V \otimes V \to V \otimes V \quad \text{i.e. } R \in \mathrm{End}(V^{\otimes 2})$$

$$V = \oplus_n \mathbb{C}|n\rangle = \begin{cases} \text{space of 1-particle states} \\ \text{space of local spin states} \end{cases}$$

$$R(|i\rangle \otimes |j\rangle) = \sum_{ab} R_{ij}^{ab} |a\rangle \otimes |b\rangle$$

$R =$



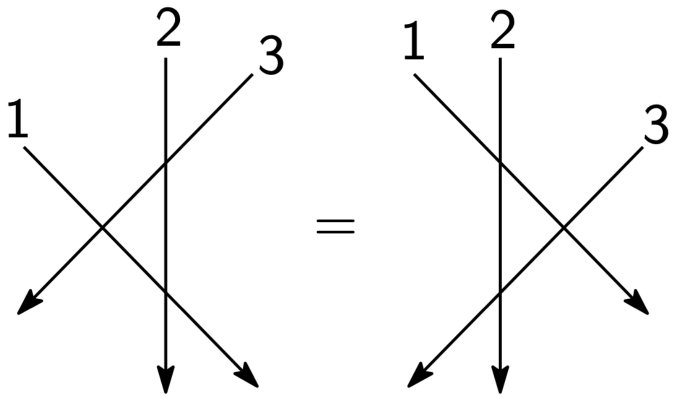
$\cdots \begin{cases} \text{2 particle scattering amplitude in (1+1)D} \\ \text{local Boltzmann weight of the vertex in 2D} \end{cases}$

Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ \in \mathrm{End}(V^{\otimes 3}),$$

where R_{ij} acts on the ith and jth components:

$$R_{12} : V \otimes V \otimes V, \quad R_{23} : V \otimes V \otimes V, \quad R_{13} : V \otimes V \otimes V$$



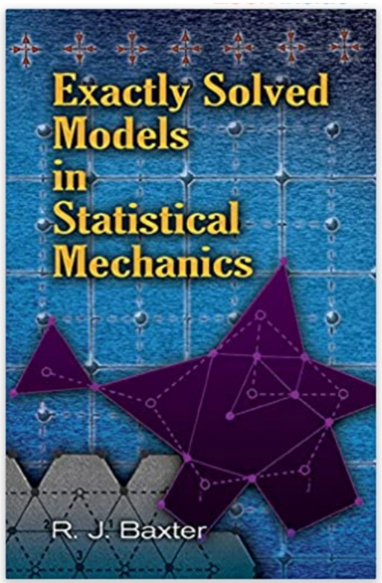
Naively, for N state model

N^4 unknows
N^6 equations

Yang-Baxter equation implies

- Factorization of 3 particle scattering amplitude into 2 body ones
- Commutativity of row transfer matrices in lattice models

Key to quantum integrability in 2D

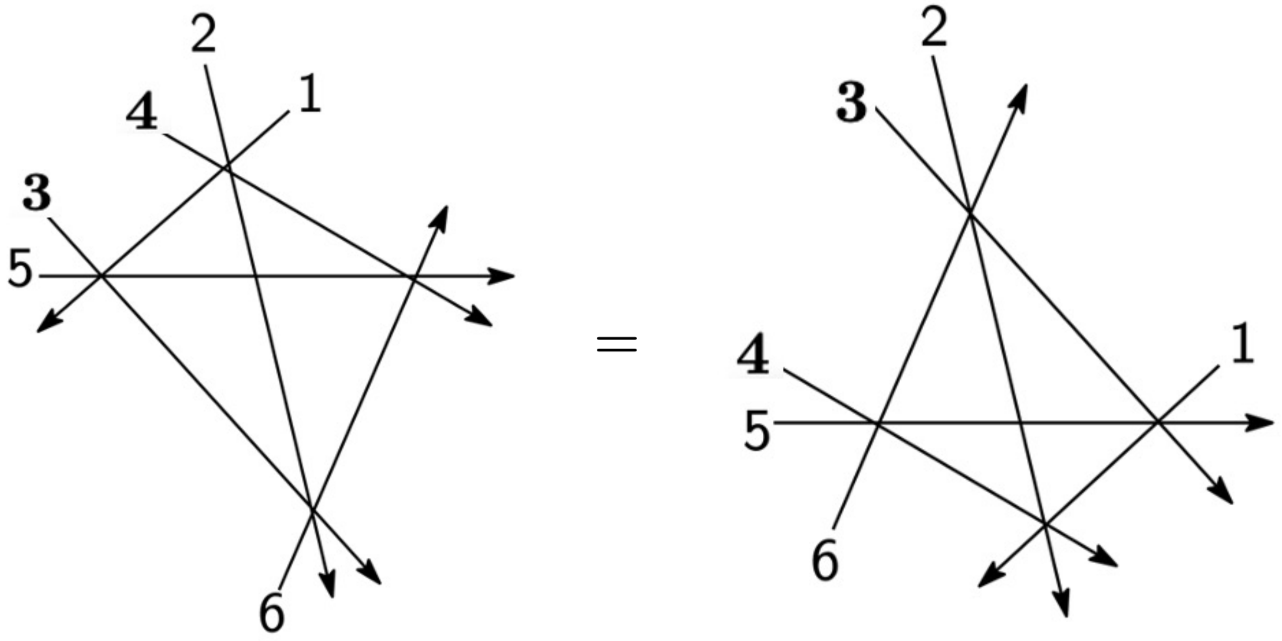


What about 3D?

Tetrahedron equation (A.B. Zamolodchikov (1980))

$$R: \ V \otimes V \otimes V \to V \otimes V \otimes V$$

$$R_{456} R_{236} R_{135} R_{124} = R_{124} R_{135} R_{236} R_{456} \in \mathrm{End}(V^{\otimes 6})$$



$$R = \begin{cases} \text{3 string scattering amplitude in (2+1)D} \\ \text{local Boltzmann weight of the vertex in 3D} \end{cases}$$

Naively, for N state model

N^6 unknows
N^{12} equations

The first non-trivial solution was established in early 80's by Zamolodchikov and Baxter.

2. Quantized (6V & YBE)

YBE: $$L_{23}L_{13}L_{12} = L_{12}L_{13}L_{23}$$

Quantized YBE: $$R_{456}L_{236}L_{135}L_{124} = L_{124}L_{135}L_{236}R_{456}$$

⋯ YBE up to conjugation. Also called **RLLL relation** or local YBE, etc.

Appeared in several guises and has been studied from various viewpoints by Maillet, Nijhoff, Korepanov, Bazhanov, Kashaev, Mangazeev, Sergeev, Stroganov, Okado, Maruyama, Yoneyama, K,⋯.

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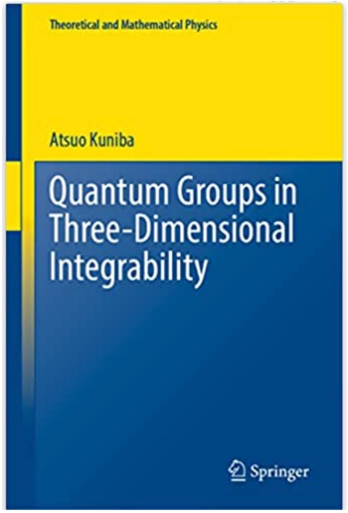
Has a connection to the quantum group theory, and generalizations to

Quantized reflection equation $\quad K(LGLG) = (GLGL)K$

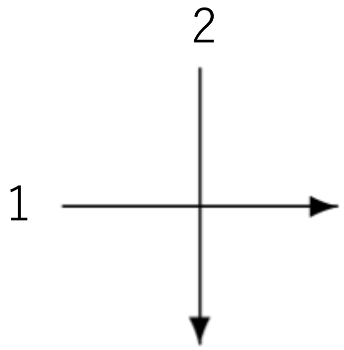
Quantized G_2 reflection equation $\quad F(LJLJLJ) = (JLJLJL)F$

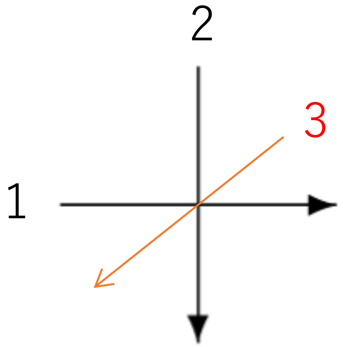
Intriguing applications to stationary states in multispecies totally asymmetric simple-exclusion/zero-range processes in 1D.



“auxiliary space”

$$L_{12} \xrightarrow{\text{“quantization”}} L_{\underbrace{12}_{\mathbb{C}^2 \otimes \mathbb{C}^2}3}$$





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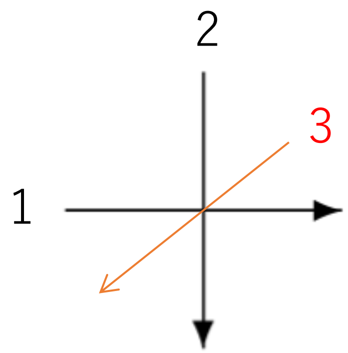

⋯ L-matrix whose elements are

$$\begin{cases} \mathcal{W}_q \ \text{(q-Weyl algebra)-valued} \\[2em] \mathcal{O}_q \ \text{(q-Oscillator algebra)-valued} \end{cases}$$

$$L_{12} \xrightarrow{\text{“quantization”}} L_{\underbrace{12}_{\mathbb{C}^2 \otimes \mathbb{C}^2}3}$$

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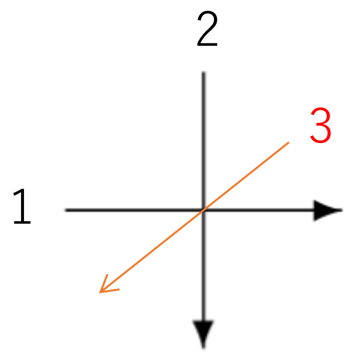


⋯ L-matrix whose elements are

$\mathcal{W}_q$ (q-Weyl algebra)-valued $\rightarrow$ L^X, L^Z — Two representations of $\mathcal{W}_q$

$\mathcal{O}_q$ (q-Oscillator algebra)-valued $\longrightarrow$ L^O

We will consider L^X, L^Z, L^O.

$\mathcal{W}_q$ q-Weyl algebra (generators $X^{\pm 1},\ Z^{\pm 1}$)

$$XZ = qZX$$

Representations $\begin{cases} \pi_X : \ X|m\rangle = q^m|m\rangle, \ \ Z|m\rangle = |m+1\rangle \\ \\ \pi_Z : \ X|m\rangle = |m-1\rangle, \ \ Z|m\rangle = q^m|m\rangle \end{cases}$

These are irreducible representations on

$$F = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}|m\rangle$$

π_X vs π_Z $\cdots$ "coordinate" vs "momentum " representations
of the q-canonical commutation relation.

$\mathcal{O}_q$ q-Oscillator algebra (generators $\mathbf{k}$, $\mathbf{a}^+, \mathbf{a}^-$)

$$\mathbf{k}\,\mathbf{a}^+ = q\mathbf{a}^+\mathbf{k}, \quad \mathbf{a}^+\mathbf{a}^- = 1 - \mathbf{k}^2$$

$$\mathbf{k}\,\mathbf{a}^- = q^{-1}\mathbf{a}^-\mathbf{k}, \quad \mathbf{a}^-\mathbf{a}^+ = 1 - q^2\mathbf{k}^2$$

$$\pi_O: \quad \mathbf{k}|m\rangle = q^m|m\rangle, \ \ \mathbf{a}^+|m\rangle = |m+1\rangle, \ \ \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle$$

π_O is an irreducible representation on

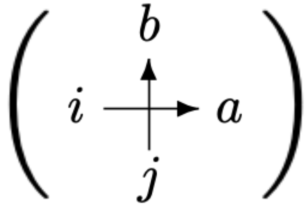
$$F_+ = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle$$

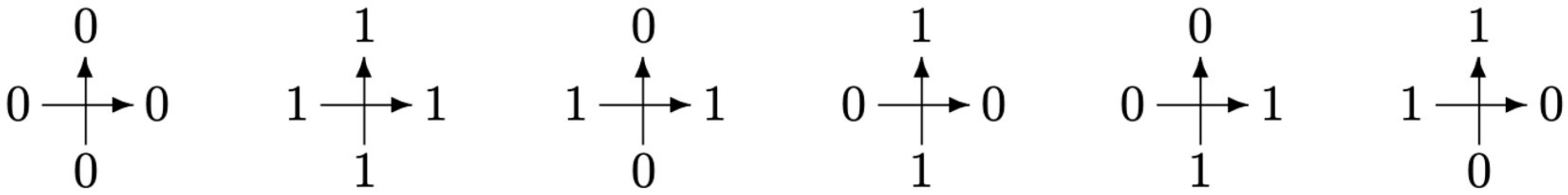
There is an embedding $\mathcal{O}_q \hookrightarrow \mathcal{W}_q$

$$\mathbf{k} \longmapsto X$$

$$\mathbf{a}^+ \longmapsto Z$$

$$\mathbf{a}^- \longmapsto Z^{-1}(1-X^2)$$

L^A $(A = X, Z, O)$ are defined by quantized 6V weights π_A 



$\left.\begin{array}{l} L^X \\ L^Z \end{array}\right\}$	r	s	twX	$-qtX$	Z	$Z^{-1}(rs - t^2wX^2)$
L^O	1	1	$\mu\mathbf{k}$	$-q\mu^{-1}\mathbf{k}$	$\mathbf{a}^+$	$\mathbf{a}^-$

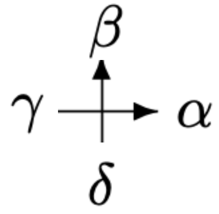
r, s, t, w, μ are parameters.

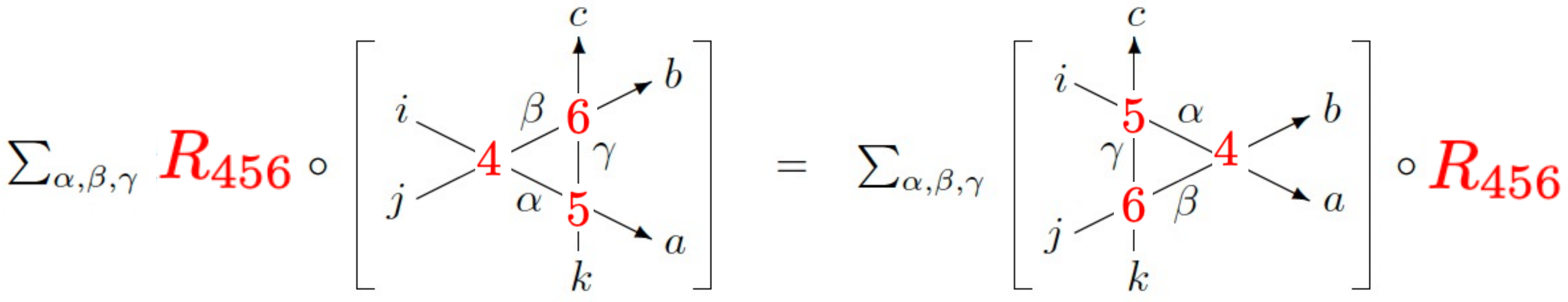
Relations of generators in $\mathcal{W}_q, \mathcal{O}_q$ may be viewed as quantizations of the "free-fermion condition" of 6V.

Quantized YBE = RLLL relation = a version of tetrahedron equation

$$R_{456}L_{236}L_{135}L_{124} = L_{124}L_{135}L_{236}R_{456}.$$

$$R\sum_{\alpha,\beta,\gamma}(\mathcal{L}^{\alpha\beta}_{ij}\otimes\mathcal{L}^{a\gamma}_{\alpha k}\otimes\mathcal{L}^{bc}_{\beta\gamma}) = \sum_{\alpha,\beta,\gamma}(\mathcal{L}^{ab}_{\alpha\beta}\otimes\mathcal{L}^{\alpha c}_{i\gamma}\otimes\mathcal{L}^{\beta\gamma}_{jk})\,R$$

$\mathcal{L}^{\alpha\beta}_{\gamma\delta} =$ 



⋯ Linear equations on R.

R depends on the choice of the three kinds of L's as

$$RL^ZL^ZL^Z = L^ZL^ZL^ZR \longrightarrow R = R^{ZZZ}$$

$$RL^ZL^OL^O = L^OL^OL^ZR \longrightarrow R = R^{OOZ}$$

In general $$RL^CL^BL^A = L^AL^BL^CR \longrightarrow R = R^{ABC}$$

RLLL relation for q-Weyl algebra case

$$Y_\alpha = Z^{-1}(r_\alpha s_\alpha - t_\alpha^2 w_\alpha X^2)$$

$$
\begin{aligned}
&R(1 \otimes X \otimes X) = (1 \otimes X \otimes X)R,\\
&R\big(r_2 t_1 X \otimes 1 \otimes Y_3 + t_3 Z \otimes Y_2 \otimes X\big) = r_1 t_2 \big(1 \otimes X \otimes Y_3\big)R,\\
&R\big(-q t_1 t_3 w_1 X \otimes Y_2 \otimes X + r_2 Y_1 \otimes 1 \otimes Y_3\big) = r_1 r_3 \big(1 \otimes Y_2 \otimes 1\big)R,\\
&r_1 t_2 R(1 \otimes X \otimes Z) = \big(r_2 t_1 X \otimes 1 \otimes Z + t_3 Y_1 \otimes Z \otimes X\big)R,\\
&R\big(q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Z \otimes Y_2 \otimes Z\big) = \big(q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Y_1 \otimes Z \otimes Y_3\big)R,\\
&R\big(t_1 w_1 X \otimes Y_2 \otimes Z + r_2 t_3 w_3 Y_1 \otimes 1 \otimes X\big) = r_3 t_2 w_2 \big(Y_1 \otimes X \otimes 1\big)R,\\
&R(X \otimes X \otimes 1) = (X \otimes X \otimes 1)R,\\
&s_3 t_2 R\big(Y_1 \otimes X \otimes 1\big) = \big(t_1 X \otimes Y_2 \otimes Z + s_2 t_3 Y_1 \otimes 1 \otimes X\big)R,\\
&s_1 s_3 R\big(1 \otimes Y_2 \otimes 1\big) = \big(-q t_1 t_3 w_3 X \otimes Y_2 \otimes X + s_2 Y_1 \otimes 1 \otimes Y_3\big)R,\\
&r_1 r_3 R(1 \otimes Z \otimes 1) = (-q t_1 t_3 w_1 X \otimes Z \otimes X + r_2 Z \otimes 1 \otimes Z)R,\\
&r_3 t_2 w_2 R(Z \otimes X \otimes 1) = \big(t_1 w_1 X \otimes Z \otimes Y_3 + r_2 t_3 w_3 Z \otimes 1 \otimes X\big)R,\\
&R(X \otimes X \otimes 1) = (X \otimes X \otimes 1)R,\\
&R(t_1 X \otimes Z \otimes Y_3 + s_2 t_3 Z \otimes 1 \otimes X) = s_3 t_2 (Z \otimes X \otimes 1)R,\\
&R\big(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Y_1 \otimes Z \otimes Y_3\big) = \big(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Z \otimes Y_2 \otimes Z\big)R,\\
&R\big(s_1 t_2 w_2 1 \otimes X \otimes Y_3 = \big(s_2 t_1 w_1 X \otimes 1 \otimes Y_3 + t_3 w_3 Z \otimes Y_2 \otimes X\big)R,\\
&R(-q t_1 t_3 w_3 X \otimes Z \otimes X + s_2 Z \otimes 1 \otimes Z) = s_1 s_3 (1 \otimes Z \otimes 1)R,\\
&R\big(t_3 w_3 Y_1 \otimes Z \otimes X + s_2 t_1 w_1 X \otimes 1 \otimes Z\big) = s_1 t_2 w_2 (1 \otimes X \otimes Z)R,\\
&R(1 \otimes X \otimes X) = (1 \otimes X \otimes X)R.
\end{aligned}
$$

$R = R^{ZZZ}$ case

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} R^{a,b,c}_{i,j,k} |a\rangle \otimes |b\rangle \otimes |c\rangle$$

$(0,0,1,0,0,1):\quad R^{a,b+1,c+1}_{i,j,k} = R^{a,b,c}_{i,j-1,k-1},$

$(0,0,1,0,1,0):\quad q^{c} r_2 w_1 R^{a+1,b,c}_{i,j,k} + q^{-a+b} w_3 (r_1 s_1 R^{a,b,c+1}_{i,j,k} + t_1 w_1 R^{a+2,b,c+1}_{i,j,k}) = q^{k} r_1 w_2 R^{a,b,c}_{i,j-1,k},$

$(0,0,1,1,0,0):\quad q^{a+c} r_2 R^{a,b,c}_{i,j,k} + q^{b+1} t_1 w_3 R^{a+1,b,c+1}_{i,j,k} = q^{j} r_1 r_3 R^{a,b,c}_{i,j,k},$

$(0,1,0,0,0,1):\quad q^{-c} r_1 w_2 (r_3 s_3 R^{a,b+1,c}_{i,j,k} + t_3 w_3 R^{a,b+1,c+2}_{i,j,k}) = q^{-k} r_2 w_1 (r_3 s_3 R^{a,b,c}_{i-1,j,k} + q^2 t_3 w_3 R^{a,b,c}_{i-1,j,k-2}) + q^{i-j} w_3 (r_2 s_2 R^{a,b,c}_{i,j,k-1} + q^2 t_2 w_2 R^{a,b,c}_{i,j-2,k-1}),$

$(0,1,0,0,1,0):\quad q r_2 t_3 w_1 R^{a+1,b,c+1}_{i,j,k} + q^{-a+b-c} (r_3 s_3 (r_1 s_1 R^{a,b,c}_{i,j,k} + t_1 w_1 R^{a+2,b,c}_{i,j,k}) + t_3 w_3 (r_1 s_1 R^{a,b,c+2}_{i,j,k} + t_1 w_1 R^{a+2,b,c+2}_{i,j,k})) = q r_2 t_3 w_1 R^{a,b,c}_{i-1,j,k-1} + q^{i-j+k} (r_2 s_2 R^{a,b,c}_{i,j,k} + q^2 t_2 w_2 R^{a,b,c}_{i,j-2,k}),$

$(0,1,0,1,0,0):\quad q^{a} r_2 t_3 R^{a,b,c+1}_{i,j,k} + q^{b-c} t_1 (r_3 s_3 R^{a+1,b,c}_{i,j,k} + t_3 w_3 R^{a+1,b,c+2}_{i,j,k}) = q^{i} r_3 t_2 R^{a,b,c}_{i,j-1,k},$

$(0,1,1,0,1,1):\quad R^{a+1,b+1,c}_{i,j,k} = R^{a,b,c}_{i-1,j-1,k},$

$(0,1,1,1,0,1):\quad q^{a} s_3 w_2 R^{a,b+1,c}_{i,j,k} = q^{i} s_2 w_3 R^{a,b,c}_{i,j,k-1} + q^{j-k} w_1 (r_3 s_3 R^{a,b,c}_{i-1,j,k} + t_3 w_3 q^2 R^{a,b,c}_{i-1,j,k-2}),$

$(0,1,1,1,1,0):\quad q^{b} s_1 s_3 R^{a,b,c}_{i,j,k} = q^{i+k} s_2 R^{a,b,c}_{i,j,k} + q^{j+1} t_3 w_1 R^{a,b,c}_{i-1,j,k-1},$

$(1,0,0,0,0,1):\quad q^{-b} r_1 r_3 (r_2 s_2 R^{a,b,c}_{i,j,k} + t_2 w_2 R^{a,b+2,c}_{i,j,k}) = q^{-j+1} t_1 w_3 (r_2 s_2 R^{a,b,c}_{i-1,j,k-1} + q^2 t_2 w_2 R^{a,b,c}_{i-1,j-2,k-1}) + q^{-i-k} r_2 (r_1 s_1 (r_3 s_3 R^{a,b,c}_{i,j,k} + q^2 t_3 w_3 R^{a,b,c}_{i,j,k-2}) + q^2 t_1 w_1 (r_3 s_3 R^{a,b,c}_{i-2,j,k} + q^2 t_3 w_3 R^{a,b,c}_{i-2,j,k-2})),$

$(1,0,0,0,1,0):\quad q^{-a} r_3 t_2 (r_1 s_1 R^{a,b+1,c}_{i,j,k} + t_1 w_1 R^{a+2,b+1,c}_{i,j,k}) = q^{-i} r_2 t_3 (r_1 s_1 R^{a,b,c}_{i,j,k-1} + t_1 w_1 q^2 R^{a,b,c}_{i-2,j,k-1}) + q^{-j+k} t_1 (r_2 s_2 R^{a,b,c}_{i-1,j,k} + q^2 t_2 w_2 R^{a,b,c}_{i-1,j-2,k}),$

$(1,0,0,1,0,0):\quad R^{a+1,b+1,c}_{i,j,k} = R^{a,b,c}_{i-1,j-1,k},$

$(1,0,1,0,1,1):\quad q^{-a} s_2 w_3 (r_1 s_1 R^{a,b,c+1}_{i,j,k} + t_1 w_1 R^{a+2,b,c+1}_{i,j,k}) + q^{-b+c} w_1 (r_2 s_2 R^{a+1,b,c}_{i,j,k} + t_2 w_2 R^{a+1,b+2,c}_{i,j,k}) = q^{-i} s_3 w_2 (r_1 s_1 R^{a,b,c}_{i,j-1,k} + q^2 t_1 w_1 R^{a,b,c}_{i-2,j-1,k}),$

$(1,0,1,1,0,1):\quad q s_2 t_1 w_3 R^{a+1,b,c+1}_{i,j,k} + q^{a-b+c} (r_2 s_2 R^{a,b,c}_{i,j,k} + t_2 w_2 R^{a,b+2,c}_{i,j,k}) = q s_2 t_1 w_3 R^{a,b,c}_{i-1,j,k-1} + q^{-i+j-k} (r_1 s_1 (r_3 s_3 R^{a,b,c}_{i,j,k} + q^2 t_3 w_3 R^{a,b,c}_{i,j,k-2}) + q^2 t_1 w_1 (r_3 s_3 R^{a,b,c}_{i-2,j,k} + q^2 t_3 w_3 R^{a,b,c}_{i-2,j,k-2})),$

$(1,0,1,1,1,0):\quad q^{c} s_1 t_2 R^{a,b+1,c}_{i,j,k} = q^{k} s_2 t_1 R^{a,b,c}_{i-1,j,k} + q^{-i+j} t_3 (r_1 s_1 R^{a,b,c}_{i,j,k-1} + q^2 t_1 w_1 R^{a,b,c}_{i-2,j,k-1}),$

$(1,1,0,0,1,1):\quad q^{-a-c} s_2 (r_1 s_1 (r_3 s_3 R^{a,b,c}_{i,j,k} + t_3 w_3 R^{a,b,c+2}_{i,j,k}) + t_1 w_1 (r_3 s_3 R^{a+2,b,c}_{i,j,k} + t_3 w_3 R^{a+2,b,c+2}_{i,j,k})) + q^{-b+1} t_3 w_1 (r_2 s_2 R^{a+1,b,c+1}_{i,j,k} + t_2 w_2 R^{a+1,b+2,c+1}_{i,j,k}) = q^{-j} s_1 s_3 (r_2 s_2 R^{a,b,c}_{i,j,k} + q^2 t_2 w_2 R^{a,b,c}_{i,j-2,k}),$

$(1,1,0,1,0,1):\quad q^{-c} s_2 t_1 (r_3 s_3 R^{a+1,b,c}_{i,j,k} + t_3 w_3 R^{a+1,b,c+2}_{i,j,k}) + q^{a-b} t_3 (r_2 s_2 R^{a,b,c+1}_{i,j,k} + t_2 w_2 R^{a,b+2,c+1}_{i,j,k}) = q^{-k} s_1 t_2 (r_3 s_3 R^{a,b,c}_{i,j-1,k} + q^2 t_3 w_3 R^{a,b,c}_{i,j-1,k-2}),$

$(1,1,0,1,1,0):\quad R^{a,b+1,c+1}_{i,j,k} = R^{a,b,c}_{i,j-1,k-1}.$

Result

(1) There always exists a unique R up to normalization in each sector specified by an appropriate parity condition.

The solution R^{ABC} is explicitly obtained for

ABC	feature	locally finiteness
ZZZ	factorized	no
OZZ	${}_2\phi_1$	no
ZZO	${}_2\phi_1$	no
ZOZ	${}_3\phi_2$-like	no
OOZ	factorized	yes
ZOO	factorized	yes
OZO	factorized	no
OOO	${}_2\phi_1$	yes
XXZ	factorized	no
ZXX	factorized	no
XZX	factorized	no

New except for OOO.

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} R^{a,b,c}_{i,j,k} |a\rangle \otimes |b\rangle \otimes |c\rangle$$

We say that R is **locally finite** if $R^{a,b,c}_{i,j,k} = 0$ for all but finitely many (a,b,c) for any given (i,j,k).

(2) Relation to quantized coordinate ring $A_q(sl_n)$

(3) Conjecture on tetrahedron equation of type RRRR = RRRR

$$(z;q)_m = \frac{(z;q)_\infty}{(zq^m;q)_\infty}, \qquad (z;q)_\infty = \prod_{n\geq 0}(1-zq^n)$$

$${}_2\phi_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};q,z\right) = \sum_{n\geq 0}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n$$

3. Solutions

R^{ZZZ}

$$R^{ZZZ} \in \mathrm{End}(F \otimes F \otimes F)$$

$$(r_1, s_1, t_1, w_1) \quad (r_2, s_2, t_2, w_2) \quad (r_3, s_3, t_3, w_3)$$

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{t_1 t_3 w_1}\right)^{\frac{d_1}{2}} \left(\frac{s_2}{t_1 t_3 w_3}\right)^{\frac{d_2}{2}} \left(\frac{t_2}{s_1 t_3}\right)^{\frac{d_3}{2}} \left(\frac{t_2 w_2}{s_3 t_1 w_1}\right)^{\frac{d_4}{2}}$$
$$\times\, q^{\varphi} \frac{\Phi_{d_2}\left(\frac{s_1 s_3}{s_2}\right) \Phi_{d_3}\left(\frac{r_3 w_2}{s_3 w_1}\right) \Phi_{d_4}\left(\frac{r_1 w_3}{s_1 w_2}\right)}{\Phi_{-d_1}\left(\frac{q^2 r_1 r_3}{r_2}\right) \Phi_{d_3+d_4}\left(\frac{r_1 r_3 w_3}{s_1 s_3 w_1}\right)},$$

$$\varphi = \frac{1}{4}\left((d_1 - d_2)(d_1 + d_2 + d_3 + d_4) + d_3 d_4\right) - d_1,$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} a + c - j \\ b - i - k \end{pmatrix}, \quad \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -a - b + c + i + j - k \\ a - b - c - i + j + k \end{pmatrix}$$

$$\Phi_m(z) = \frac{1}{(zq^m; q^2)_\infty} \quad (m \in \mathbb{Z}),$$

$a, b, c, i, j, k \in \mathbb{Z}$

Factorized
Not locally finite

R^{OZZ}

$$R^{OZZ} \in \mathrm{End}(F_+ \otimes F \otimes F)$$

$$\mu \quad (r_2, s_2, t_2, w_2) \quad (r_3, s_3, t_3, w_3)$$

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{r_3}\right)^a \left(\frac{s_3}{s_2}\right)^i \left(\frac{t_2 w_2}{\mu s_2}\right)^{-b+j} \left(-\frac{\mu t_3}{r_3}\right)^{-c+k} \frac{(z;q^2)_a}{(q^2;q^2)_a} q^{(a-b+j-1)c-(i-b+j-1)k-aj+bi} ;$$
$$\times {}_2\phi_1\left(\begin{matrix} q^{-2i}, z^{-1}q^2 \\ z^{-1}q^{-2a+2} \end{matrix}; q^2, yq^{2i+2j-2a-2b}\right).$$

$$a, i \in \mathbb{Z}_{\geq 0},\ b, c, j, k \in \mathbb{Z}$$

$$x = \frac{\mu^2 s_2}{r_2 w_2}, \qquad y = \frac{r_3 w_3}{\mu^2 s_3}, \qquad z = xq^{2k-2c+2}$$

q-hypergeometric, instead of factorization.
Not locally finite.

$\mathrm{R}^{\mathrm{OOZ}}$

$$R^{OOZ} \in \mathrm{End}(\underset{\mu_1}{\underset{\uparrow}{F_+}} \otimes \underset{\mu_2}{\underset{\uparrow}{F_+}} \otimes \underset{(r_3, s_3, t_3, w_3)}{\underset{\uparrow}{F}})$$

$\exists$ unique solution iff $\frac{\mu_1}{\mu_2} = q^d$ for some $d \in \mathbb{Z}$.

$$R^{a,b,c}_{i,j,k} = s_3^i(\mu_2 t_3)^{-a}\left(\frac{\mu_2 s_3}{t_3 w_3}\right)^j\left(\frac{t_3^2 w_3}{r_3 s_3}\right)^e q^{cj-bk}\frac{(q^{2+2e-2j};q^2)_j(q^{2a+2};q^2)_{i-a}}{(q^2;q^2)_f(q^{2a-2e};q^2)_{e-a}}$$

$$e = \frac{1}{2}(a-c+j+k+d), \quad f = \frac{1}{2}(b+c+i-k-d). \qquad a,b,i,j \in \mathbb{Z}_{\geq 0},\ c,k \in \mathbb{Z}$$

Factorized. Locally finite.

R^{OOO}

$$R^{OOO} \in \mathrm{End}(\underset{\mu_1}{\underset{\uparrow}{F_+}} \otimes \underset{\mu_2}{\underset{\uparrow}{F_+}} \otimes \underset{\mu_3}{\underset{\uparrow}{F_+}})$$

$$R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i \left(-\frac{\mu_1}{\mu_3}\right)^b \left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}$$

$$\times \frac{(q^2;q^2)_{a+b}}{(q^2;q^2)_a (q^2;q^2)_b} {}_2\phi_1\left(\begin{matrix} q^{-2b}, q^{-2i} \\ q^{-2a-2b} \end{matrix}; q^2, q^{-2c}\right)$$

$$a,b,c,i,j,k \in \mathbb{Z}_{\geq 0}$$

Locally finite.
For this 3d R, representation theoretical origin
(quantized coordinate ring, PBW bases,$\cdots$) is known.

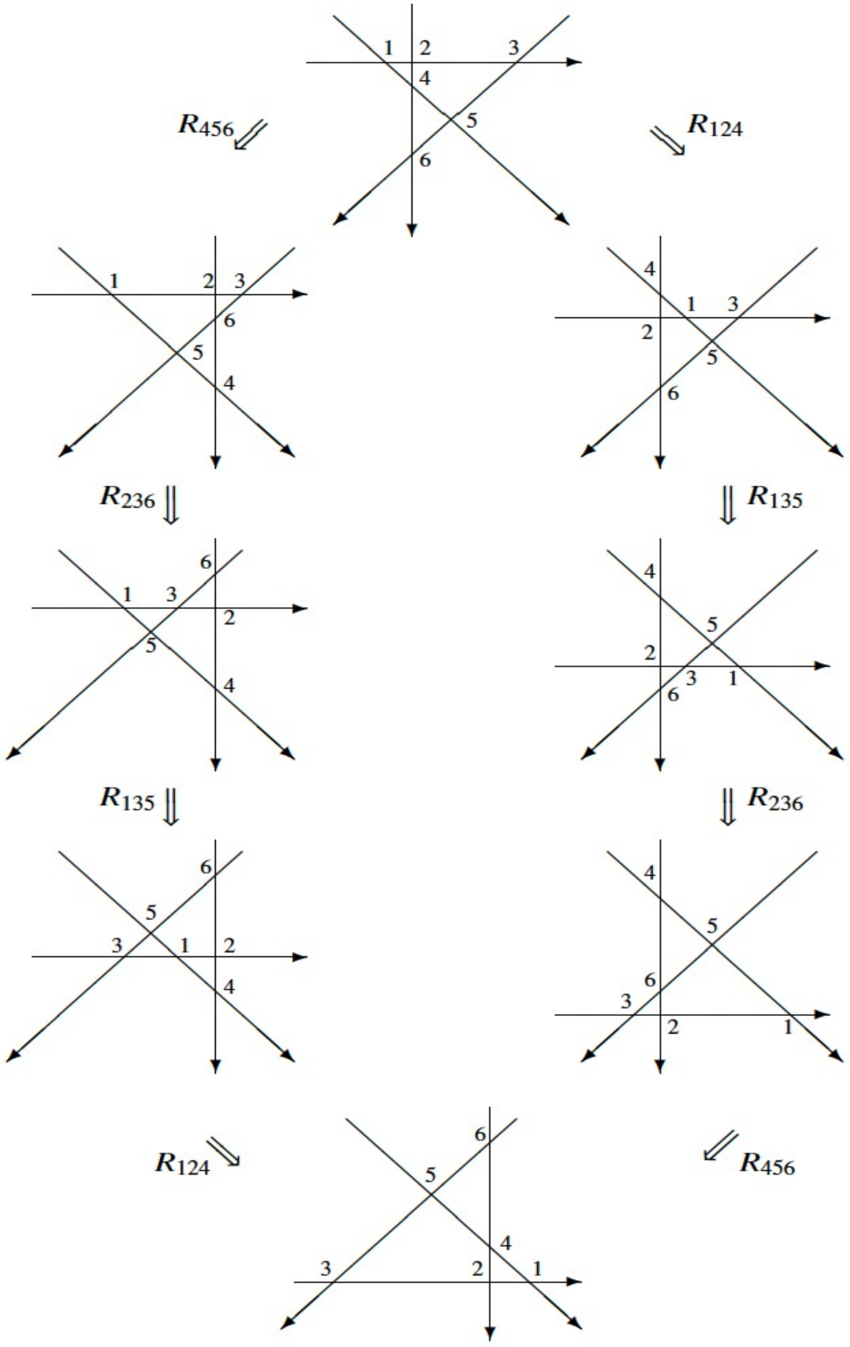
4. Conjecture on RRRR=RRRR

$$
\begin{aligned}
&R_{124}R_{135}R_{236}R_{456}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}\\
&=R_{124}R_{135}R_{236}L_{\beta\gamma4}L_{\alpha\gamma5}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\gamma\delta1}R_{456}\\
&=R_{124}R_{135}L_{\beta\gamma4}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}R_{236}R_{456}\\
&=R_{124}R_{135}L_{\beta\gamma4}L_{\beta\delta2}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\alpha\beta6}R_{236}R_{456}\\
&=R_{124}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}L_{\alpha\delta3}L_{\alpha\gamma5}L_{\alpha\beta6}R_{135}R_{236}R_{456}\\
&=L_{\gamma\delta1}L_{\beta\delta2}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},\\
&=L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},
\end{aligned}
$$

$$
\begin{aligned}
&R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\beta\delta2}L_{\gamma\delta1}\\
&=R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}L_{\alpha\delta3}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}\\
&=R_{456}R_{236}R_{135}L_{\alpha\beta6}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\beta\delta2}L_{\beta\gamma4}R_{124}\\
&=R_{456}R_{236}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}L_{\alpha\delta3}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\beta\gamma4}R_{135}R_{124}\\
&=R_{456}R_{236}L_{\gamma\delta1}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\alpha\gamma5}L_{\beta\gamma4}R_{135}R_{124}\\
&=R_{456}L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}R_{236}R_{135}R_{124}\\
&=L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{456}R_{236}R_{135}R_{124}.
\end{aligned}
$$

$(R_{124}R_{135}R_{236}R_{456})^{-1}R_{456}R_{236}R_{135}R_{124}$ commutes with $L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}$.

$\longrightarrow$ $R_{456}R_{236}R_{135}R_{124}=R_{124}R_{135}R_{236}R_{456}$ if irreducible.



In our case, a similar procedure starting from $L^F_{\alpha\beta 6}L^E_{\alpha\gamma 5}L^D_{\beta\gamma 4}L^C_{\alpha\delta 3}L^B_{\beta\delta 2}L^A_{\gamma\delta 1}$ suggests

$$R^{DEF}_{456}R^{BCF}_{236}R^{ACE}_{135}R^{ABD}_{124} = R^{ABD}_{124}R^{ACE}_{135}R^{BCF}_{236}R^{DEF}_{456}$$

where, A, B, C, D, E, F are Z or O or X.

For A=···=F=O, the irreducibility is known from the representation theory of A_q. Therefore

$$R^{OOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOO}_{124} = R^{OOO}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{OOO}_{456}$$ holds. (Kapranov-Voevodsky '94)

Other cases are yet elusive, also due to convergence issue for composition of the locally non-finite 3D R's.

$$|i\rangle\otimes|j\rangle\otimes|k\rangle\otimes|l\rangle\otimes|m\rangle\otimes|n\rangle \mapsto |a\rangle\otimes|b\rangle\otimes|c\rangle\otimes|d\rangle\otimes|e\rangle\otimes|f\rangle \ \text{ component of } RRRR = RRRR:$$

$$\sum_{u,v,w,x,y,z} R^{d,e,f}_{x,y,z}R^{b,c,z}_{v,w,n}R^{a,w,y}_{u,k,m}R^{u,v,x}_{i,j,l} = \sum_{u,v,w,x,y,z} R^{a,b,d}_{u,v,x}R^{u,c,e}_{i,w,y}R^{v,w,f}_{j,k,z}R^{x,y,z}_{l,m,n}$$

We have listed up the cases (without X) in which the above sums with prescribed a,b,c,d,e,f, I,j,k,l,m,n become finite sums due to locally-finiteness. They are given in the next page.

$$
\begin{aligned}
R_{456}^{OOO}R_{236}^{OOO}R_{135}^{ZOO}R_{124}^{ZOO} &= R_{124}^{ZOO}R_{135}^{ZOO}R_{236}^{OOO}R_{456}^{OOO},\\
R_{456}^{ZOO}R_{236}^{OOO}R_{135}^{OOO}R_{124}^{OOZ} &= R_{124}^{OOZ}R_{135}^{OOO}R_{236}^{OOO}R_{456}^{ZOO},\\
R_{456}^{OOZ}R_{236}^{OOZ}R_{135}^{OOO}R_{124}^{OOO} &= R_{124}^{OOO}R_{135}^{OOO}R_{236}^{OOZ}R_{456}^{OOZ},\\
R_{456}^{OOZ}R_{236}^{OOZ}R_{135}^{ZOO}R_{124}^{ZOO} &= R_{124}^{ZOO}R_{135}^{ZOO}R_{236}^{OOZ}R_{456}^{OOZ}.
\end{aligned}
$$

$$
\begin{aligned}
R_{456}^{OOO}R_{236}^{ZOO}R_{135}^{OOO}R_{124}^{OZO} &= R_{124}^{OZO}R_{135}^{OOO}R_{236}^{ZOO}R_{456}^{OOO},\\
R_{456}^{OZO}R_{236}^{OOO}R_{135}^{OOZ}R_{124}^{OOO} &= R_{124}^{OOO}R_{135}^{OOZ}R_{236}^{OOO}R_{456}^{OZO},\\
R_{456}^{OOO}R_{236}^{ZOO}R_{135}^{ZOO}R_{124}^{ZZO} &= R_{124}^{ZZO}R_{135}^{ZOO}R_{236}^{ZOO}R_{456}^{OOO},\\
R_{456}^{ZOO}R_{236}^{OOO}R_{135}^{ZOO}R_{124}^{ZOZ} &= R_{124}^{ZOZ}R_{135}^{ZOO}R_{236}^{OOO}R_{456}^{ZOO},\\
R_{456}^{ZOO}R_{236}^{ZOO}R_{135}^{OOO}R_{124}^{OZZ} &= R_{124}^{OZZ}R_{135}^{OOO}R_{236}^{ZOO}R_{456}^{ZOO},\\
R_{456}^{ZZO}R_{236}^{OOO}R_{135}^{OOZ}R_{124}^{OOZ} &= R_{124}^{OOZ}R_{135}^{OOZ}R_{236}^{OOO}R_{456}^{ZZO},\\
R_{456}^{ZOZ}R_{236}^{OOZ}R_{135}^{OOO}R_{124}^{OOZ} &= R_{124}^{OOZ}R_{135}^{OOO}R_{236}^{OOZ}R_{456}^{ZOZ},\\
R_{456}^{OZZ}R_{236}^{OOZ}R_{135}^{OOZ}R_{124}^{OOO} &= R_{124}^{OOO}R_{135}^{OOZ}R_{236}^{OOZ}R_{456}^{OZZ},\\
R_{456}^{ZOO}R_{236}^{ZOO}R_{135}^{ZOO}R_{124}^{ZZZ} &= R_{124}^{ZZZ}R_{135}^{ZOO}R_{236}^{ZOO}R_{456}^{ZOO},\\
R_{456}^{ZZZ}R_{236}^{OOZ}R_{135}^{OOZ}R_{124}^{OOZ} &= R_{124}^{OOZ}R_{135}^{OOZ}R_{236}^{OOZ}R_{456}^{ZZZ}.
\end{aligned}
$$

$$
\begin{aligned}
R_{456}^{OOO}R_{236}^{OZO}R_{135}^{OZO}R_{124}^{OOO} &= R_{124}^{OOO}R_{135}^{OZO}R_{236}^{OZO}R_{456}^{OOO},\\
R_{456}^{OOO}R_{236}^{OZO}R_{135}^{ZZO}R_{124}^{ZOO} &= R_{124}^{ZOO}R_{135}^{ZZO}R_{236}^{OZO}R_{456}^{OOO},\\
R_{456}^{OZO}R_{236}^{OOO}R_{135}^{ZOZ}R_{124}^{ZOO} &= R_{124}^{ZOO}R_{135}^{ZOZ}R_{236}^{OOO}R_{456}^{OZO},\\
R_{456}^{OOO}R_{236}^{ZZO}R_{135}^{OZO}R_{124}^{OZO} &= R_{124}^{OZO}R_{135}^{OZO}R_{236}^{ZZO}R_{456}^{OOO},\\
R_{456}^{OOZ}R_{236}^{ZOZ}R_{135}^{OOO}R_{124}^{OZO} &= R_{124}^{OZO}R_{135}^{OOO}R_{236}^{ZOZ}R_{456}^{OOZ},\\
R_{456}^{ZOO}R_{236}^{OZO}R_{135}^{OZO}R_{124}^{OOZ} &= R_{124}^{OOZ}R_{135}^{OZO}R_{236}^{OZO}R_{456}^{ZOO},\\
R_{456}^{OZO}R_{236}^{OZO}R_{135}^{OZZ}R_{124}^{OOO} &= R_{124}^{OOO}R_{135}^{OZZ}R_{236}^{OZO}R_{456}^{OZO},\\
R_{456}^{OOZ}R_{236}^{OZZ}R_{135}^{OZO}R_{124}^{OOO} &= R_{124}^{OOO}R_{135}^{OZO}R_{236}^{OZZ}R_{456}^{OOZ},\\
R_{456}^{OZO}R_{236}^{OZO}R_{135}^{ZZZ}R_{124}^{ZOO} &= R_{124}^{ZOO}R_{135}^{ZZZ}R_{236}^{OZO}R_{456}^{OZO},\\
R_{456}^{OOZ}R_{236}^{ZZZ}R_{135}^{OZO}R_{124}^{OZO} &= R_{124}^{OZO}R_{135}^{OZO}R_{236}^{ZZZ}R_{456}^{OOZ}.
\end{aligned}
$$

$$
\begin{aligned}
R^{OOO}_{456}R^{OOO}_{236}R^{ZOO}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZOO}_{135}R^{OOO}_{236}R^{OOO}_{456},\\
R^{ZOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{ZOO}_{456},\\
R^{OOZ}_{456}R^{OOZ}_{236}R^{OOO}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OOO}_{135}R^{OOZ}_{236}R^{OOZ}_{456},\\
R^{OOZ}_{456}R^{OOZ}_{236}R^{ZOO}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZOO}_{135}R^{OOZ}_{236}R^{OOZ}_{456}.
\end{aligned}
$$

$$
\begin{aligned}
R^{OOO}_{456}R^{ZOO}_{236}R^{OOO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OOO}_{135}R^{ZOO}_{236}R^{OOO}_{456},\\
R^{OZO}_{456}R^{OOO}_{236}R^{OOZ}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OOZ}_{135}R^{OOO}_{236}R^{OZO}_{456},\\
R^{OOO}_{456}R^{ZOO}_{236}R^{ZOO}_{135}R^{ZZO}_{124} &= R^{ZZO}_{124}R^{ZOO}_{135}R^{ZOO}_{236}R^{OOO}_{456},\\
R^{ZOO}_{456}R^{OOO}_{236}R^{ZOO}_{135}R^{ZOZ}_{124} &= R^{ZOZ}_{124}R^{ZOO}_{135}R^{OOO}_{236}R^{ZOO}_{456},\\
R^{ZOO}_{456}R^{ZOO}_{236}R^{OOO}_{135}R^{OZZ}_{124} &= R^{OZZ}_{124}R^{OOO}_{135}R^{ZOO}_{236}R^{ZOO}_{456},\\
R^{ZZO}_{456}R^{OOO}_{236}R^{OOZ}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOZ}_{135}R^{OOO}_{236}R^{ZZO}_{456},\\
R^{ZOZ}_{456}R^{OOZ}_{236}R^{OOO}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOO}_{135}R^{OOZ}_{236}R^{ZOZ}_{456},\\
R^{OZZ}_{456}R^{OOZ}_{236}R^{OOZ}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OOZ}_{135}R^{OOZ}_{236}R^{OZZ}_{456},\\
R^{ZOO}_{456}R^{ZOO}_{236}R^{ZOO}_{135}R^{ZZZ}_{124} &= R^{ZZZ}_{124}R^{ZOO}_{135}R^{ZOO}_{236}R^{ZOO}_{456},\\
R^{ZZZ}_{456}R^{OOZ}_{236}R^{OOZ}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOZ}_{135}R^{OOZ}_{236}R^{ZZZ}_{456}.
\end{aligned}
$$

$$
\begin{aligned}
R^{OOO}_{456}R^{OZO}_{236}R^{OZO}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OZO}_{135}R^{OZO}_{236}R^{OOO}_{456},\\
R^{OOO}_{456}R^{OZO}_{236}R^{ZZO}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZZO}_{135}R^{OZO}_{236}R^{OOO}_{456},\\
R^{OZO}_{456}R^{OOO}_{236}R^{ZOZ}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZOZ}_{135}R^{OOO}_{236}R^{OZO}_{456},\\
R^{OOO}_{456}R^{ZZO}_{236}R^{OZO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OZO}_{135}R^{ZZO}_{236}R^{OOO}_{456},\\
R^{OOZ}_{456}R^{ZOZ}_{236}R^{OOO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OOO}_{135}R^{ZOZ}_{236}R^{OOZ}_{456},\\
R^{ZOO}_{456}R^{OZO}_{236}R^{OZO}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OZO}_{135}R^{OZO}_{236}R^{ZOO}_{456},\\
R^{OZO}_{456}R^{OZO}_{236}R^{OZZ}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OZZ}_{135}R^{OZO}_{236}R^{OZO}_{456},\\
R^{OOZ}_{456}R^{OZZ}_{236}R^{OZO}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OZO}_{135}R^{OZZ}_{236}R^{OOZ}_{456},\\
R^{OZO}_{456}R^{OZO}_{236}R^{ZZZ}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZZZ}_{135}R^{OZO}_{236}R^{OZO}_{456},\\
R^{OOZ}_{456}R^{ZZZ}_{236}R^{OZO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OZO}_{135}R^{ZZZ}_{236}R^{OOZ}_{456}.
\end{aligned}
$$

Conjecture (based on computer experiments)

They are all valid.

Outlook

$$\mathrm{YBE} \underset{\textbf{Reduction via 3D}}{\overset{\text{quantization}}{\rightleftarrows}} \mathrm{QYBE}$$

generates infinite family of quantum R matrices.

$q^N = 1$: Local states can be confined to $\mathbb{Z}_N$ under $a^N + b^N = c^N$ type constraints on parameters.

Conjecture (supported by numerics)

$RRRR = RRRR$ is valid for $R = R^{ZZZ}$.

Any relation to the generalized Chiral Potts model which is connected to $R = R^{XXX}$?