Generalized hydrodynamics in box-ball system

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(Joint with Grégoire Misguich and Vincent Pasquier)

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Commuting time evolutions, Family of conserved quantities,
Solitons obeying factorized scattering,
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Today's theme

Statistical properties of Randomised BBS in and out of equilibrium

- Limit shape problem of soliton distribution
  by thermodynamic Bethe ansatz (TBA)
- Density plateaux emerging from domain wall initial conditions
  by generalized hydrodynamics (GHD)

Box-ball system (BBS) [Takahashi-Satsuma, 1990]

$$\cdots 00000\color{red}{1111}\color{black}{0000000000000000} \cdots$$
$$\cdots 000000000\color{red}{1111}\color{black}{000000000000} \cdots$$
$$\cdots 0000000000000\color{red}{1111}\color{black}{00000000} \cdots$$

0 = empty box, 1 = box containing a ball

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$$\cdots 000001111000000000000000 \cdots$$
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- A time evolution T_∞ is defined by

(i) Move the leftmost ball to the nearest right empty box.

(ii) Do the same for the other balls until every ball is moved once.

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- soliton=consecutive balls
- velocity=amplitude (= 4 in the above example)

Scattering of 2 and 3 solitons

$\cdots$ 01111000001100000000000000000000 $\cdots$
$\cdots$ 00000111100011000000000000000000 $\cdots$
$\cdots$ 00000000011100111000000000000000 $\cdots$
$\cdots$ 00000000000011000111100000000000 $\cdots$
$\cdots$ 00000000000000110000011110000000 $\cdots$
$\cdots$ 00000000000000001100000001111000 $\cdots$

$\cdots$ 01111000110010000000000000000000 $\cdots$
$\cdots$ 00000111001101100000000000000000 $\cdots$
$\cdots$ 00000000110010011110000000000000 $\cdots$
$\cdots$ 00000000001101000001111000000000 $\cdots$
$\cdots$ 00000000000010110000000111100000 $\cdots$
$\cdots$ 00000000000001001100000000011110 $\cdots$

- **Amplitudes are individually conserved (Yang-Baxter property)**
- **Phase shift in collision of j-solion and k-soliton $= 2\min(j,k)$**

Double (classical and quantum) origin of integrability

(1) Ultra-Discretization (UD) of soliton equations

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- Key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) + \exp(\frac{b}{\varepsilon}) \right) = \max(a, b)$$

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) \times \exp(\frac{b}{\varepsilon}) \right) = a + b$$

$$(+, \times) \longrightarrow (\max, +)$$

keeps distributive law:

$$AB + AC = A(B + C) \to \max(a + b, a + c) = a + \max(b, c)$$

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- UD of a discrete KdV gives an evolution equation of BBS.

(2) Solvable vertex model at "Temperature 0"

Time evolution pattern

$$
\begin{array}{c}
\cdots 011001000000 \cdots \\
\cdots 000110100000 \cdots \\
\cdots 000001011000 \cdots \\
\cdots 000000100110 \cdots \\
\cdots 000000010001 1 \cdots
\end{array}
$$

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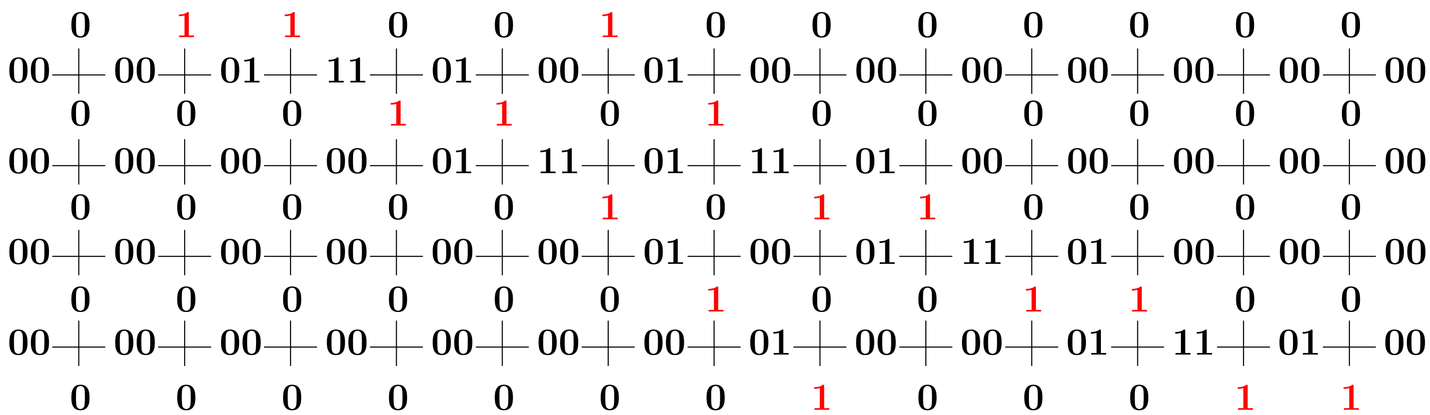
\cdots 0001101000000 \cdots \\
\cdots 0000010110000 \cdots \\
\cdots 0000001001100 \cdots \\
\cdots 0000000100011 \cdots
\end{array}
$$

emerges from a configuration of a 2D lattice model in statistical mechanics



by forgetting the hidden (auxiliary) variables on the horizontal edges.

- **BBS = 2D solvable vertex model associated with $U_q(\widehat{sl}_2)$ at $q = 0$**

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- **$\exists$ Integrable cellular automata with quantum group symmetry.**

Example: $\widehat{\mathrm{so}}_{10}$-automaton

$$\cdots 000\bar{2}\bar{4}211 0000\bar{1}\bar{1}\bar{4}00000000000000000000000000000000 \cdots$$
$$\cdots 00000000\bar{2}\bar{4}211 00\bar{1}\bar{1}\bar{4}00000000000000000000000000000 \cdots$$
$$\cdots 0000000000000\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4}00000000000000000000000000 \cdots$$
$$\cdots 000000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4}00000000000000000000000 \cdots$$
$$\cdots 00000000000000000000000 3\bar{0}\bar{0}\bar{3}\bar{4}\bar{4}000000000000000000 \cdots$$
$$\cdots 00000000000000000000000000 311\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}0000000000000 \cdots$$
$$\cdots 00000000000000000000000000000 31100\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}00000000 \cdots$$
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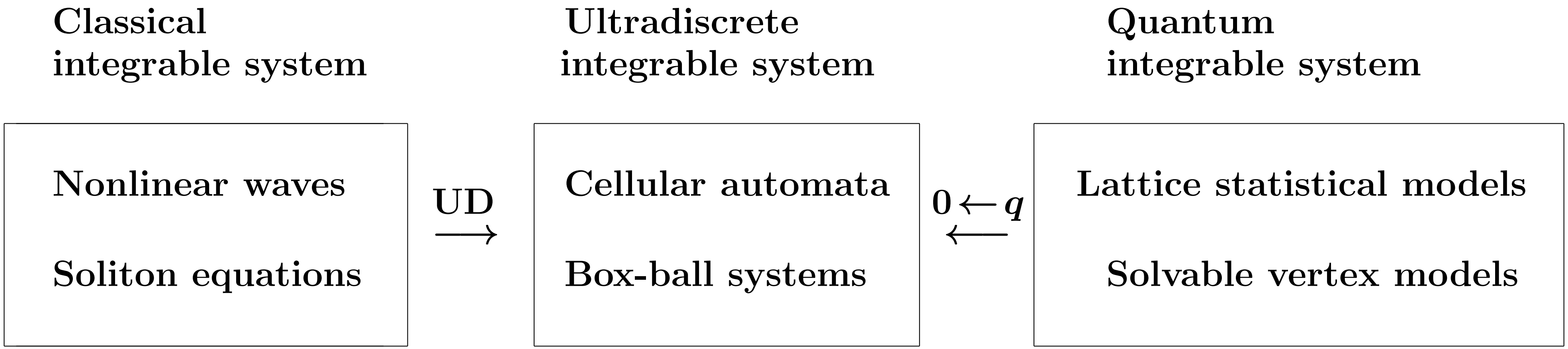
\cdots 000\bar{2}\bar{4}2110000\bar{1}\bar{1}\bar{4}00000000000000000000000000000000 \cdots \\
\cdots 00000000\bar{2}\bar{4}21100\bar{1}\bar{1}\bar{4}00000000000000000000000000000 \cdots \\
\cdots 0000000000000\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4}00000000000000000000000000 \cdots \\
\cdots 000000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4}00000000000000000000000 \cdots \\
\cdots 000000000000000000000003\bar{0}\bar{0}\bar{3}\bar{4}\bar{4}000000000000000000 \cdots \\
\cdots 00000000000000000000000000311\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}0000000000000 \cdots \\
\cdots 0000000000000000000000000000031100\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}00000000 \cdots \\
\cdots 000000000000000000000000000000003110000\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}000 \cdots
\end{array}
$$

- Particles and antiparticles undergo pair-creations/annihilations.

- Soliton & scattering most naturally captured in quantum group framework.


Classical
integrable system
Ultradiscrete
integrable system
Quantum
integrable system
Nonlinear waves
Soliton equations
UD
Cellular automata
Box-ball systems
0 ← q
Lattice statistical models
Solvable vertex models

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	$\xrightarrow{\text{UD}}$	Cellular automata Box-ball systems	$\xleftarrow{0 \leftarrow q}$	Lattice statistical models Solvable vertex models
Inverse scattering method		KKR bijection		Bethe ansatz

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- **Kerov-Kirillov-Reshetikhin (KKR) bijection** (1986) asserts "formal completeness" of the hypothetical string solutions to the Bethe equation at combinatorial level. Its remarkable connection to BBS was discovered in 2002.

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Example: XXX chain with $L = 6$ sites 3 down spins.

$$\left(\frac{u_1 + i}{u_1 - i}\right)^6 = \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)},$$
$$\left(\frac{u_2 + i}{u_2 - i}\right)^6 = \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)},$$
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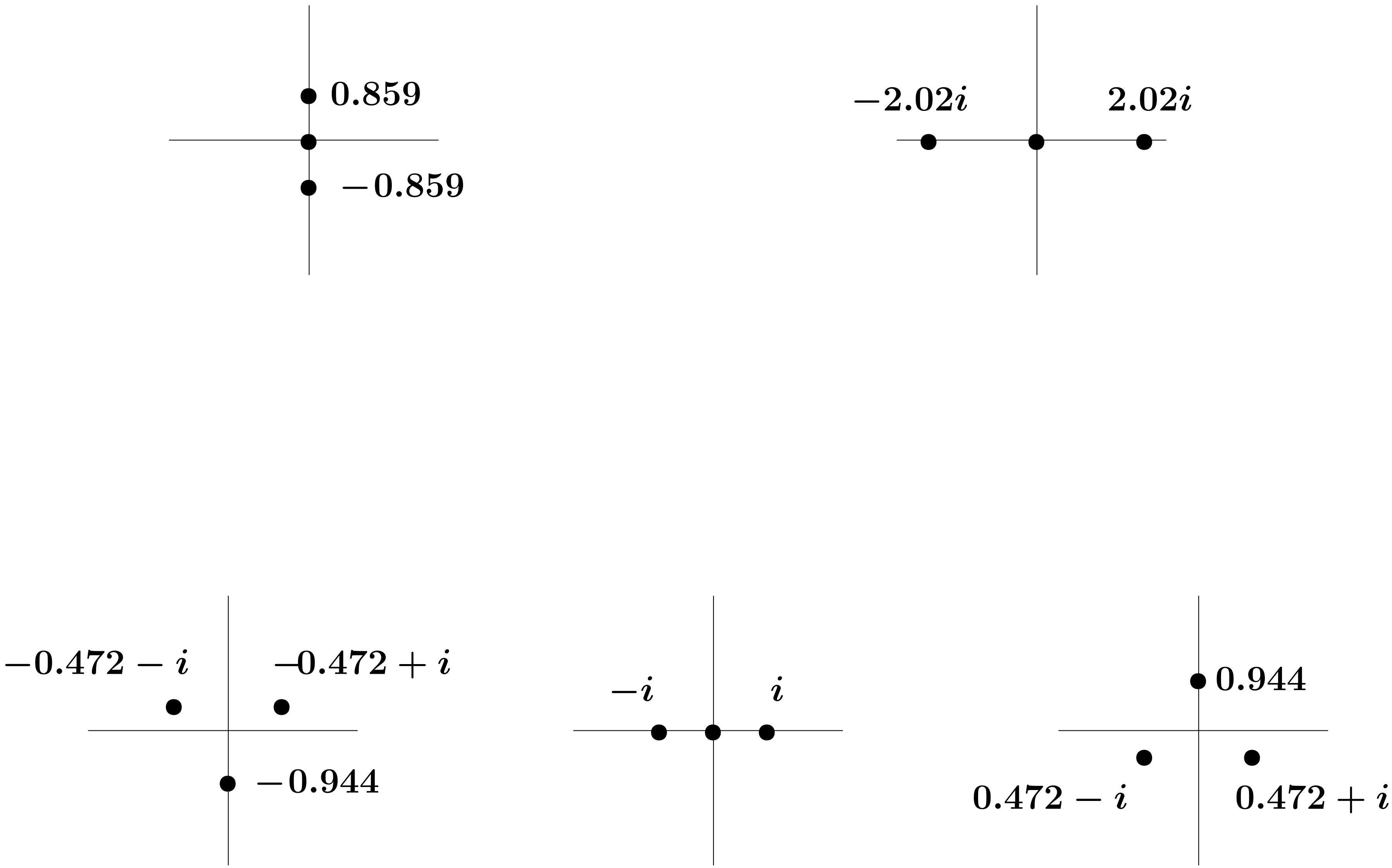
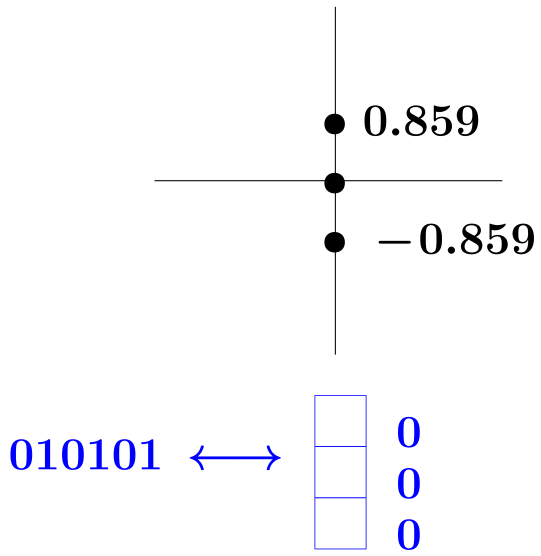
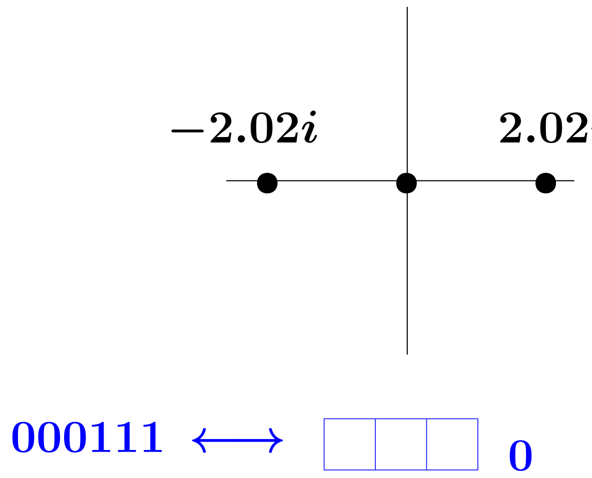
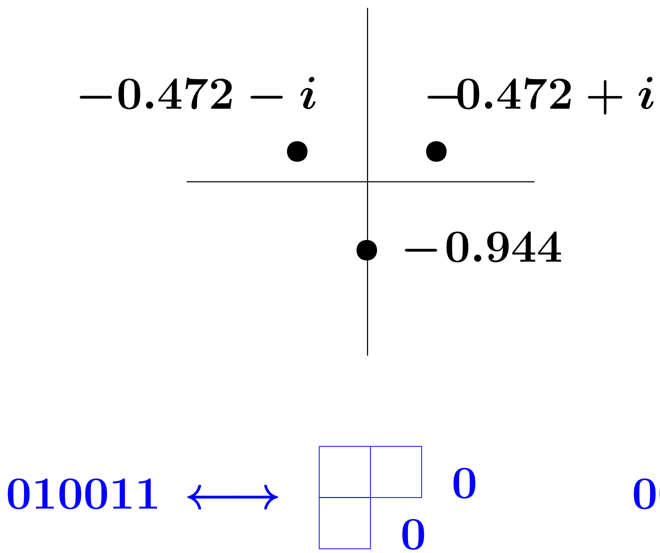
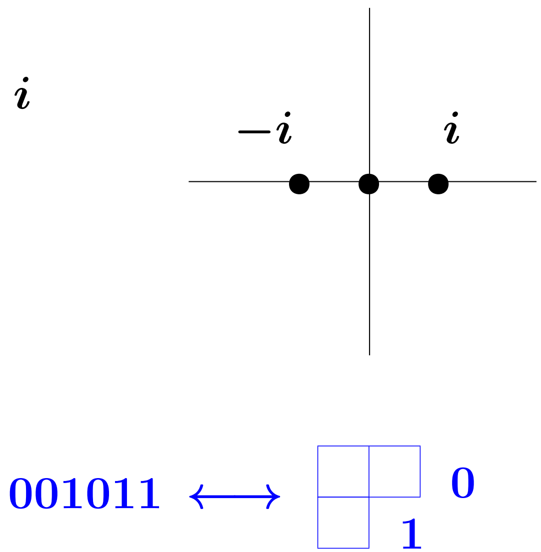

0.859
−0.859
−2.02i
2.02i
−0.472 − i
−0.472 + i
−0.944
−i
i
0.944
0.472 − i
0.472 + i


0.859
−0.859
010101 ⟷
0
0
0


−2.02i
2.02i
000111 ⟷
0


−0.472 − i
−0.472 + i
−0.944
010011 ⟷
0
0


−i
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001011 ⟷
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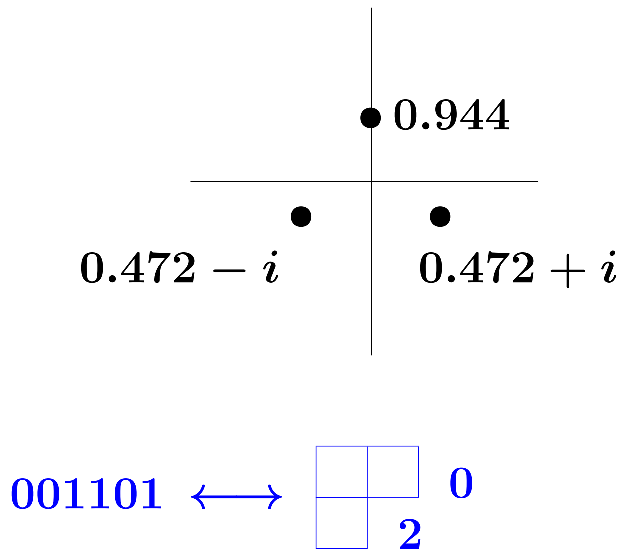
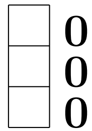
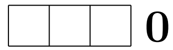
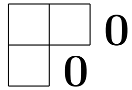
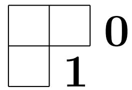
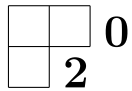
1


0.944
0.472 − i
0.472 + i
001101 ⟷
0
2

KKR bijection

{highest states} $\overset{1:1}{\longleftrightarrow}$ {rigged configurations}

010101	$\longleftrightarrow$	
000111	$\longleftrightarrow$	
010011	$\longleftrightarrow$	
001011	$\longleftrightarrow$	
001101	$\longleftrightarrow$	

“Bethe vectors” “Bethe roots”

Solitons Strings (bound states of magnons)

KKR bijection

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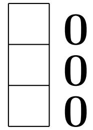
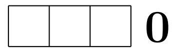
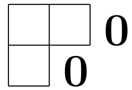
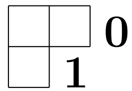
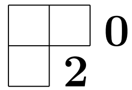
010101 $\longleftrightarrow$ 

000111 $\longleftrightarrow$ 

010011 $\longleftrightarrow$ 

001011 $\longleftrightarrow$ 

001101 $\longleftrightarrow$ 

"Bethe vectors"	"Bethe roots"
Solitons	Strings (bound states of magnons)

- highest states $= i_1 i_2 \ldots i_L$ $(i_k = 0, 1)$ satisfying the highest condition:

$$\#_0\{i_1, \ldots, i_k\} \geq \#_1\{i_1, \ldots, i_k\} \quad (\forall k)$$

- rigged configuration $= (\mu, r)$

$$\left.\begin{array}{l} \mu : \text{configuration} = \text{Young diagram} \\ r : \text{rigging} = \text{integers assigned to each row} \end{array}\right\} + \text{selection rule}$$

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0:$ 000011110000111000011000000000000000000000000

$t = 1:$ 000000001111000111000110000000000000000000000

$t = 2:$ 000000000000111000111001110000000000000000000

$t = 3:$ 000000000000000111000110001111000000000000000

$t = 4:$ 000000000000000000111001100000111100000000000

$t = 5:$ 000000000000000000000110011100000011110000000

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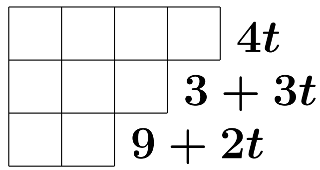
$\xrightarrow{\text{KKR}}$



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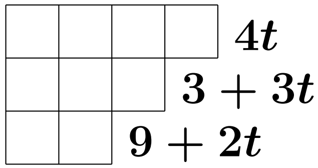

- **Configuration is conserved (action variable)**

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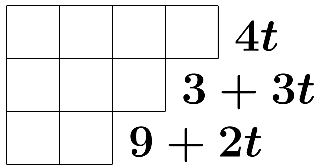

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- Rigging flows linearly (angle variable)

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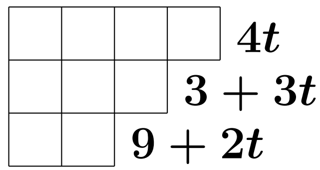

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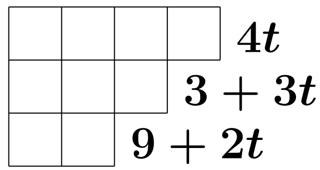

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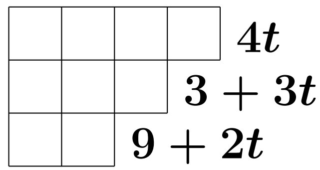

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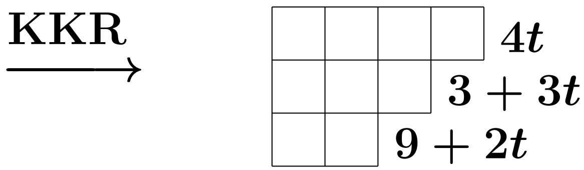

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Configuration (= list of amplitude of solitons) will be called a soliton content.

Randomized box-ball system

BBS state $\qquad\qquad$ **Soliton content**

$$\alpha_1\alpha_2\ldots\alpha_L 00000\ldots \quad \overset{\text{KKR}}{\longmapsto} \quad \text{Young diagram } \mu$$

Randomize $\alpha_1\alpha_2\ldots\alpha_L$ by introducing the i.i.d. measure on local states:

$$\mathbb{P}(\alpha_j = 0) = \frac{1}{1+q}, \qquad \mathbb{P}(\alpha_j = 1) = \frac{q}{1+q} \qquad (0 < q < 1).$$

Randomized box-ball system

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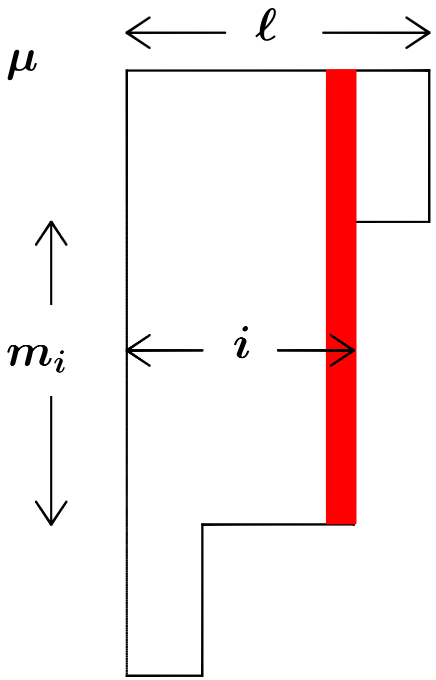
This can be solved by Thermodynamic Bethe ansatz based on Bethe's formula counting the "number of string solutions" (so-called "Fermionic formula").

$m_i :=$ number of i-solitons, $\rho_i := \frac{1}{L} m_i =$ density of i-solitons $(L \to \infty)$.

Result. The limit shape of the soliton content μ is given by



Density of solitons

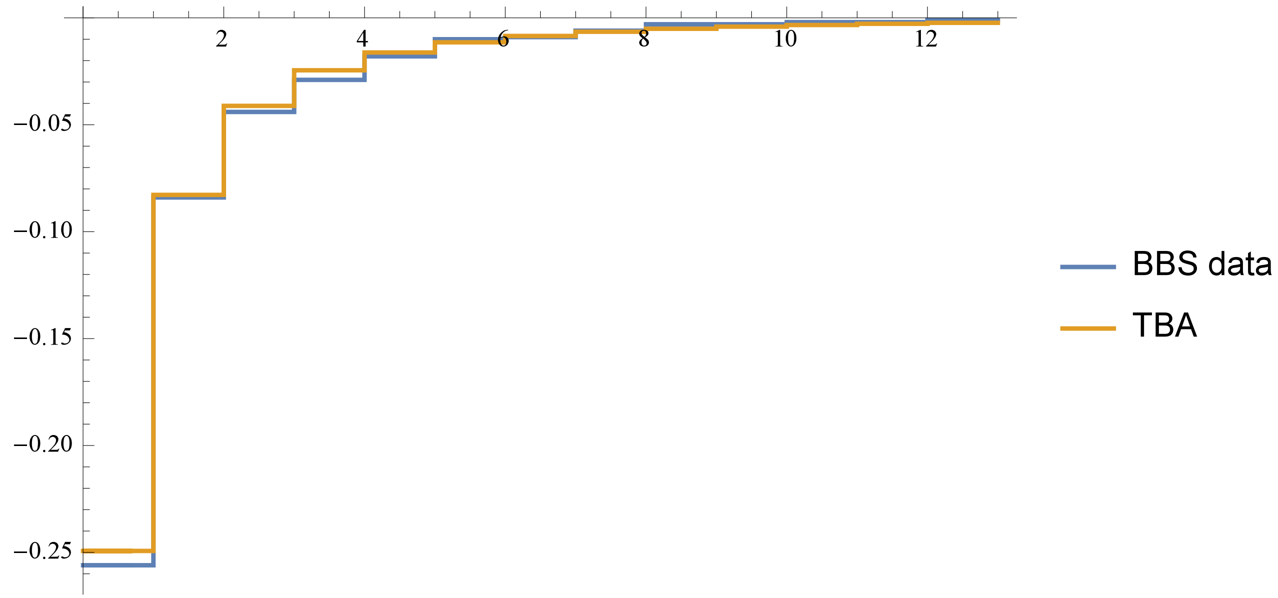
$$\rho_i = \lim_{L \to \infty} \frac{1}{L} m_i = \frac{q^i (1-q)^3 (1+q^{i+1})}{(1+q)(1-q^i)(1-q^{i+1})(1-q^{i+2})}$$

$$\lim_{L \to \infty} \frac{1}{L} (\text{Length of the } i \text{ th column}) = \frac{q^i (1-q)^2}{(1+q)(1-q^i)(1-q^{i+1})}$$

$$\text{Maximal amplitude } \ell \simeq -\frac{\log L}{\log q} \qquad (L \to \infty)$$

More general results are available for $\widehat{sl}_n$ [K-Lyu-Okado, 2018], which clarifies an intrinsic connection to logarithmic derivative of a deformed character of KR-modules.

Limit shape of vertically $1/L$ scaled soliton content for $L = 1000$ and $q = 0.9$



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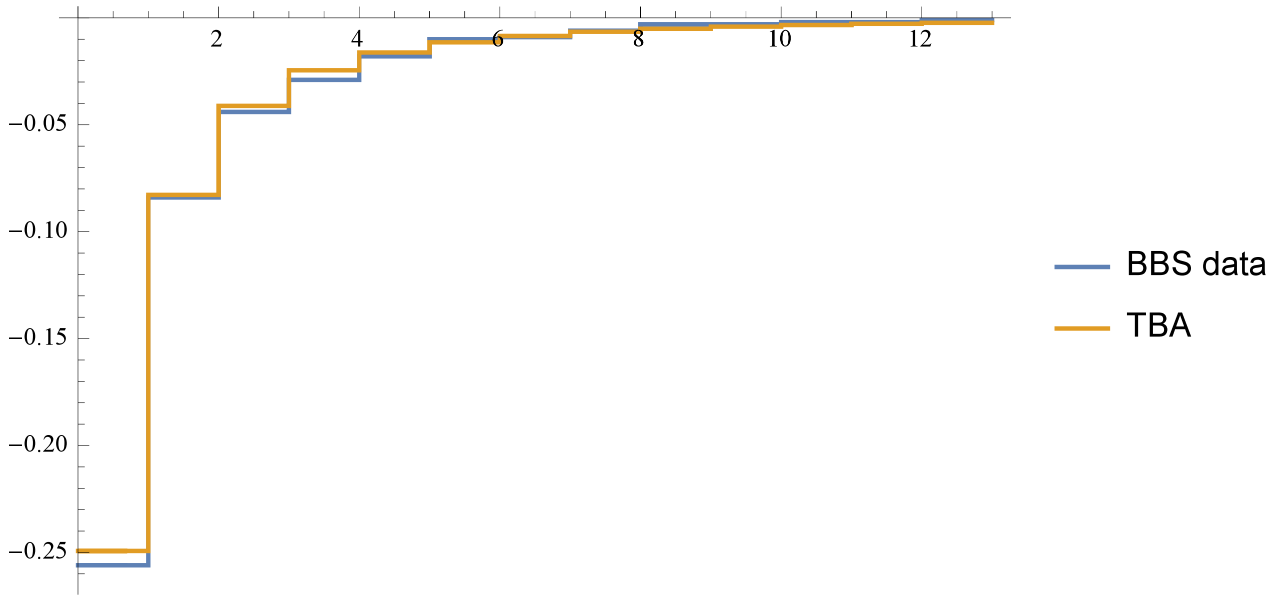

Effective speed $v_j^{(l)}$ of amplitude j-solitons under T_l is governed by

$$v_j^{(l)} = \min(j, l) + 2\sum_{k=1}^{\infty} \min(j, k)(v_j^{(l)} - v_k^{(l)})\rho_k$$

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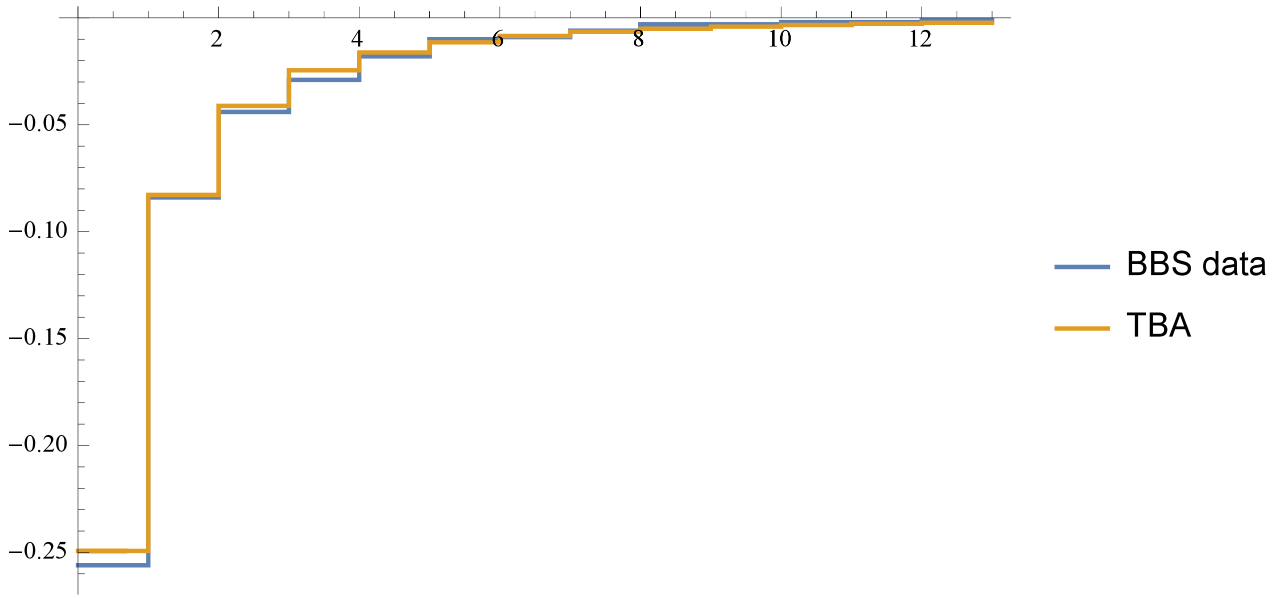

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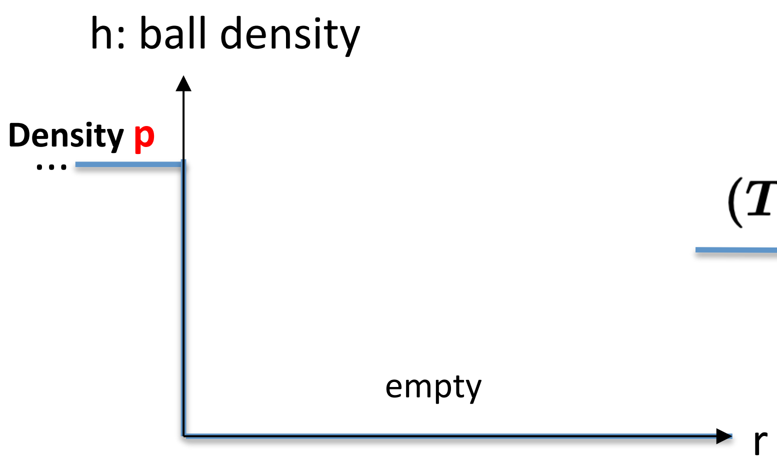
Solution:

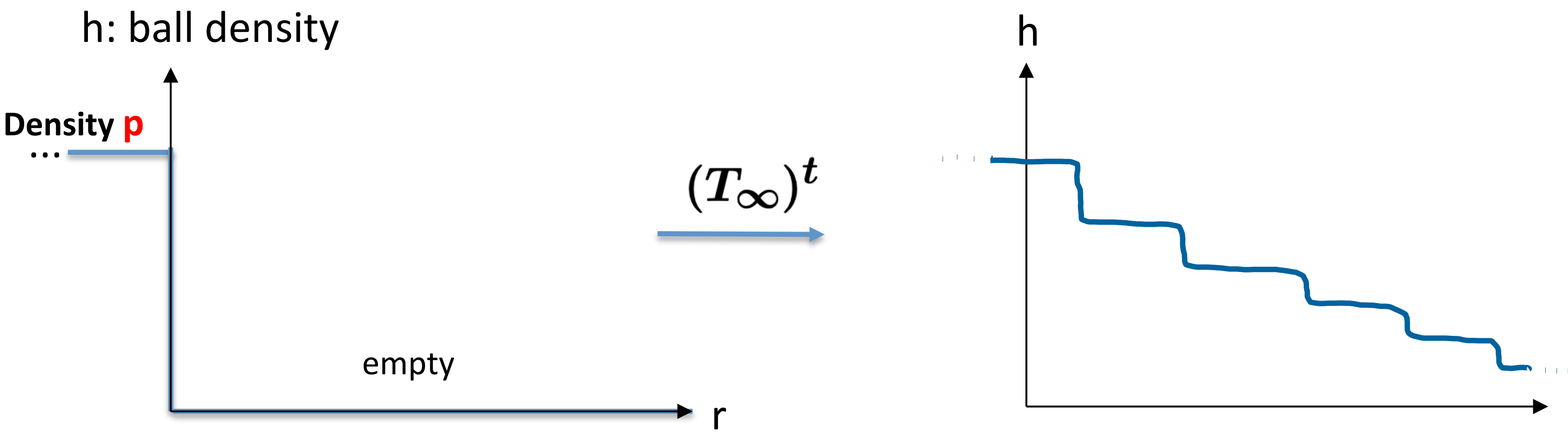
$$v_j^{(l)} = \frac{1 + q^{l+1}}{1 - q^{l+1}} v_{\min(j,l)}, \qquad v_j = j\frac{1 + q}{1 - q} - \frac{2q(1 + q)(1 - q^j)}{(1 - q)^2(1 + q^{j+1})}$$

Density Plateaux emerging from domain wall initial condition



$$(T_\infty)^t$$

Density Plateaux emerging from domain wall initial condition



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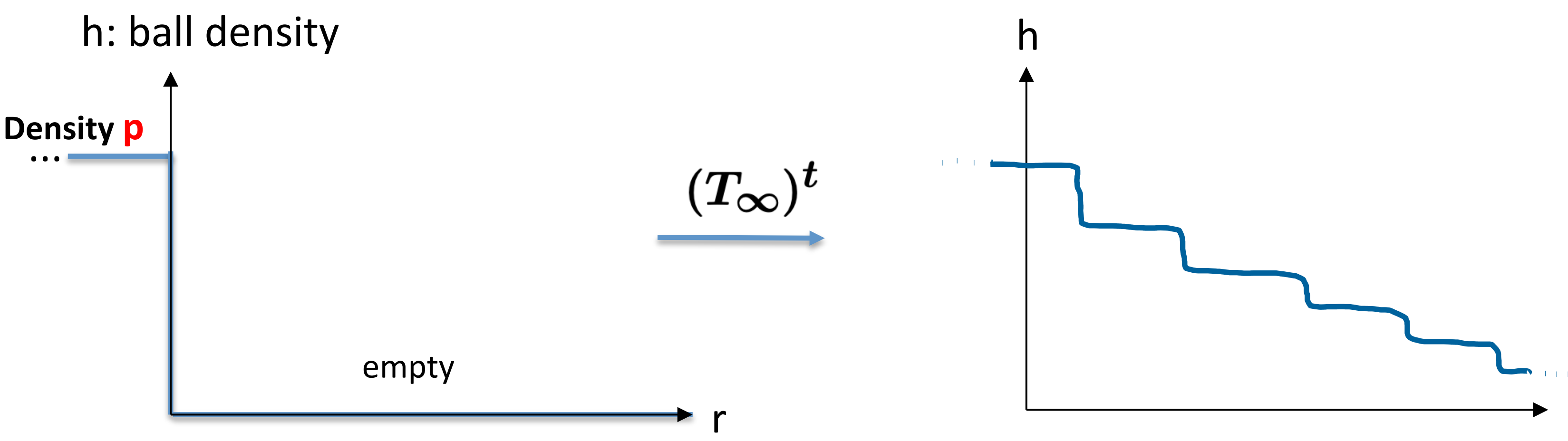

Plateaux broaden linearly in time t. The plot against $\zeta = r/t$ collapses into a single curve.

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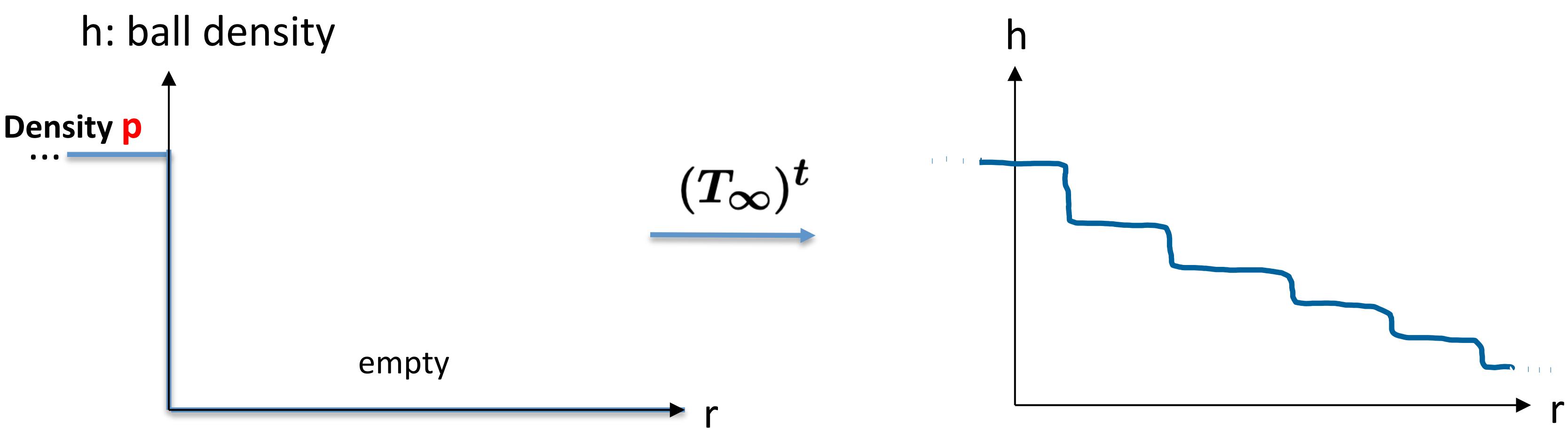

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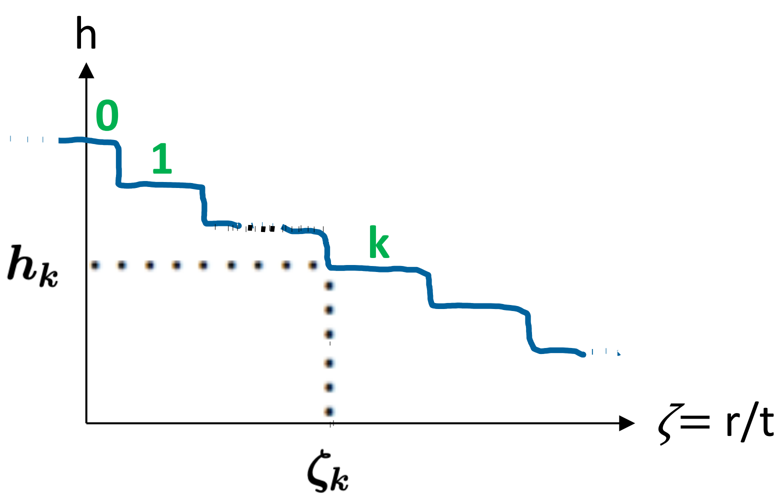

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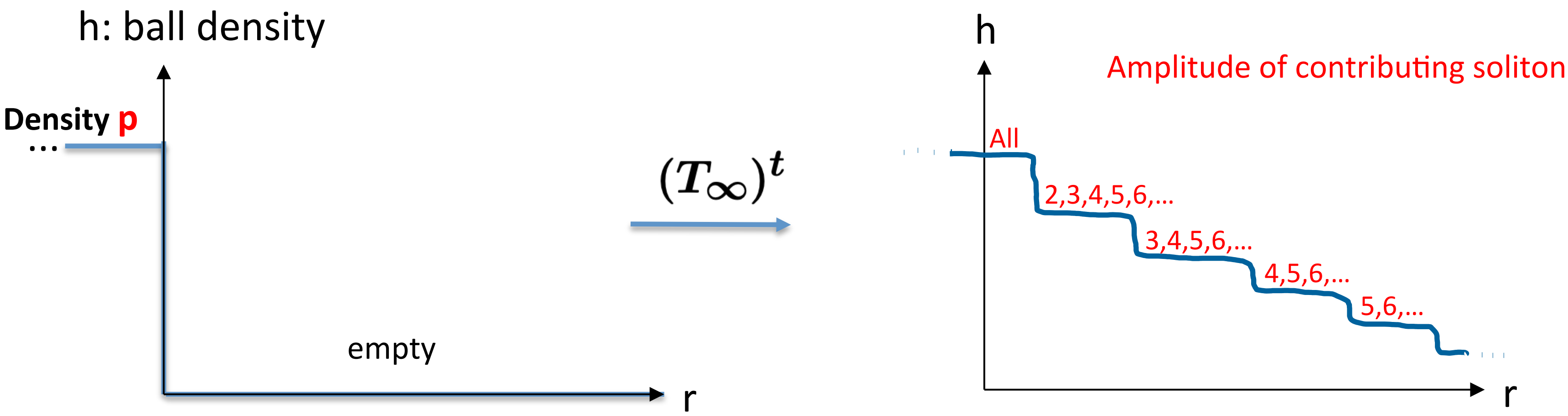

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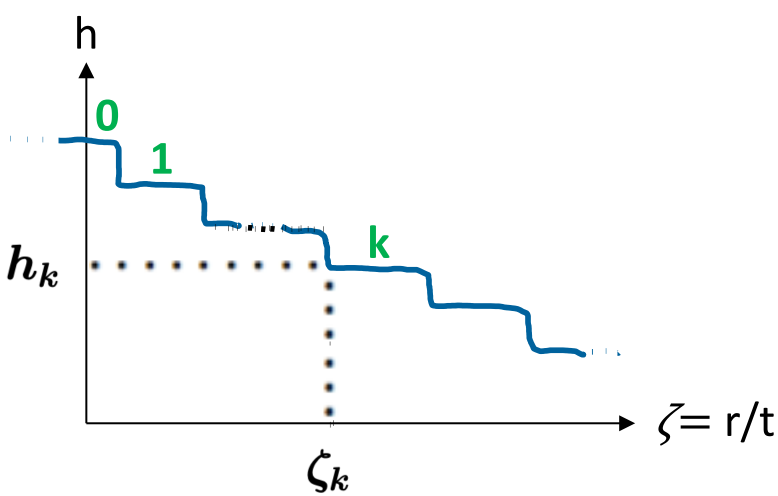

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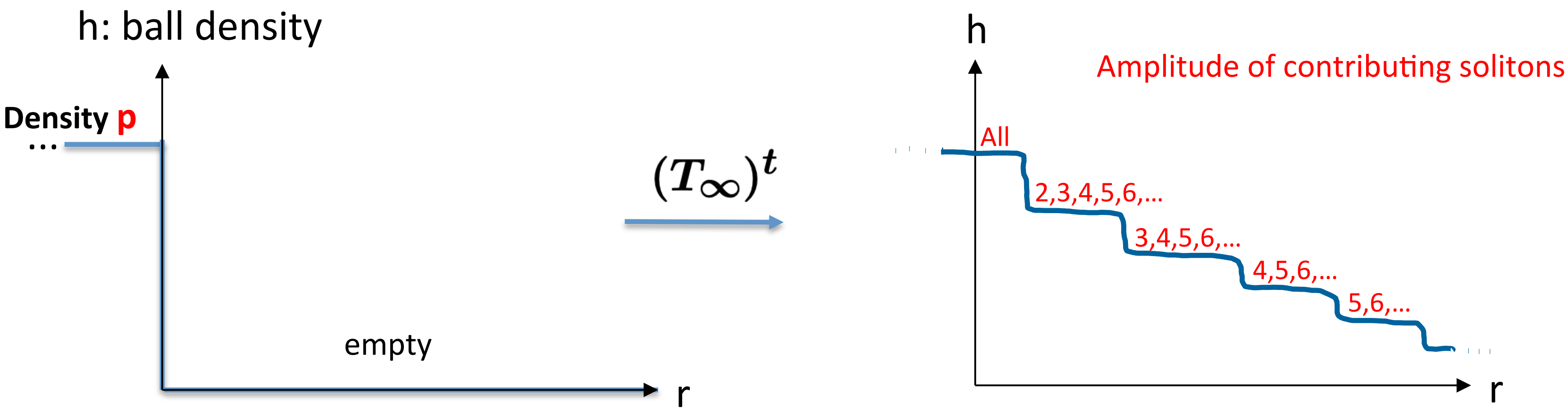

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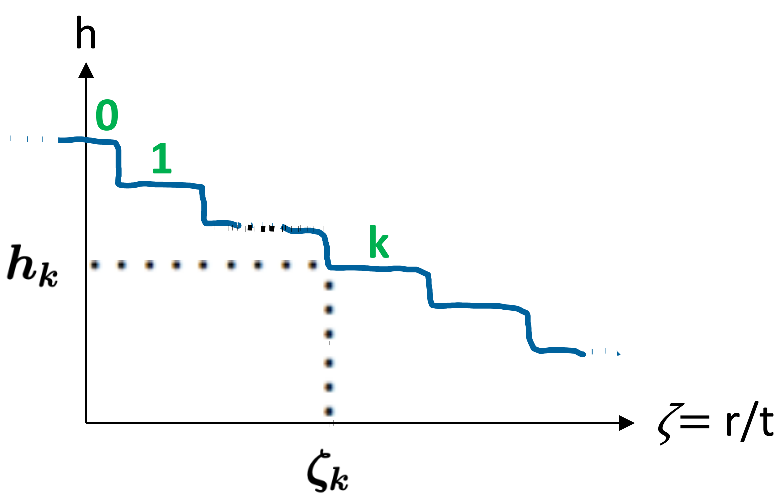


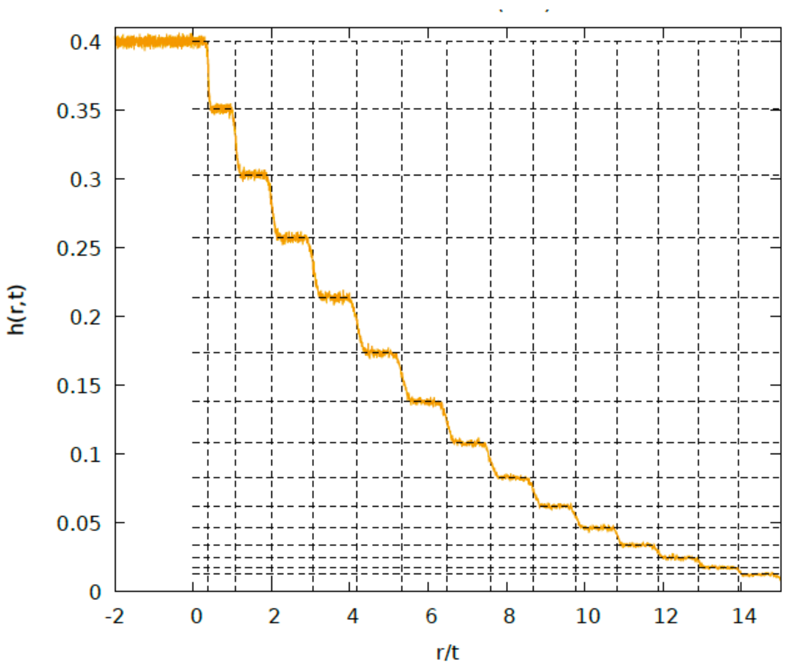
Generalized hydrodynamics (GHD) predicts

$$h_k = \frac{q^{k+1}(1-q^{k+2}+k(1-q))}{1-q^{2k+3}+(2k+1)(1-q)q^{k+1}}$$

$$\zeta_k = \frac{k(1-q^{k+1})}{1+q^{k+1}} \qquad \left(p = \frac{q}{1+q}\right)$$

Simulation with $N_{samples}$ = 50000

(Plots of ball density vs ζ= r/t. Dotted lines are GHD predictions)



p=0.4, q=0.666…, t=500.

Width of plateau edge $\propto \sqrt{t}$ for finite t.

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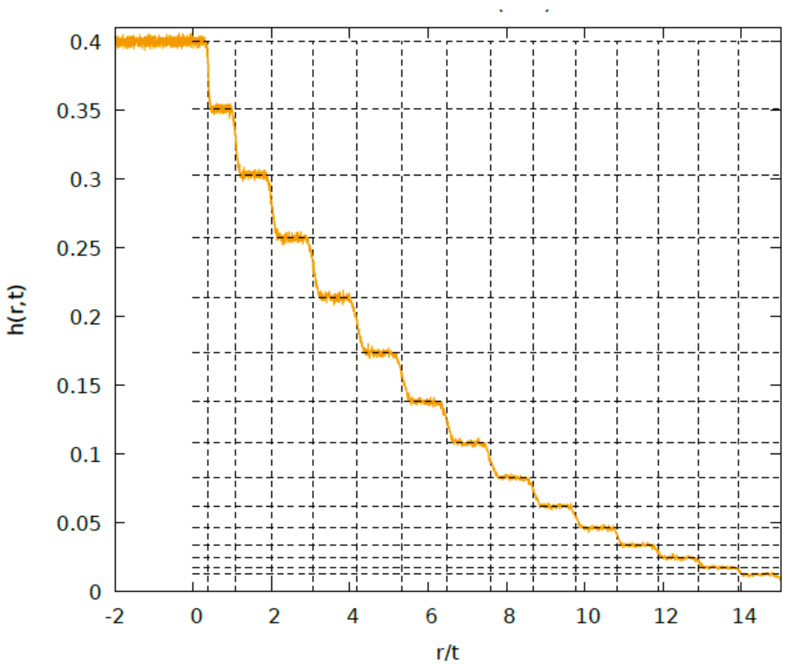

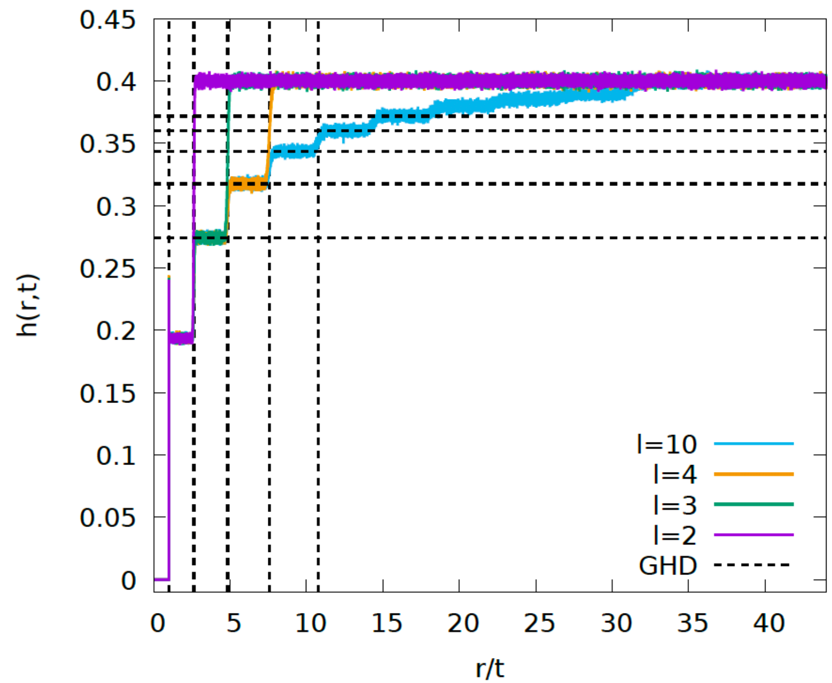

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Plateaux emerging from the *opposite domain wall* with empty left region by T_l (l=2,3,4,10)

Width of plateau edge $\propto \sqrt{t}$ for finite t.

Left edge of k th plateau $\zeta = \mathbf{effective\ speed}\ v_k^{(k)}$

Height of k th plateau $(1 \leq k < l)$

$$= \frac{q(1-q^{2k+2}-(k+1)(1-q^2)q^k)}{(1+q)\left(1-(2k+3)(1-q)q^{k+1}-q^{2k+3}\right)}$$

Summary

1. **BBS is a Yang-Baxter integrable cellular automaton with explicit action-angle variables originating in Bethe strings.**

2. **Limit shape of soliton content in randomized BBS is determined by TBA.**

3. **Density plateaux emerging from domain wall initial condition is analytically described by GHD.**

Reference

Review part:

R.Inoue, AK and T.Takagi
"Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry", JPA Topical Review (2012), arXiv:1109.5349.

Limit shape problem:

AK, H.Lyu and M.Okado
"Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz", NPB(2018), arXiv:1808.02626.

Generalized hydrodynamics of BBS:

AK, G.Misguich and V.Pasquier in preparation.