Quantized six-vertex model on a torus

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Dedicated to the memory of Professor Rodney Baxter

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Quantized six-vertex model is a 3D lattice model with the following features:

Defined on **admissible graphs** G on a torus

Layer transfer matrix $T_G(x,y)$ with two spectral parameters x, y forms a commuting family:

$$[T_G(x,y), T_G(u,w)] = 0$$

Commutativity assured by several kinds of **tetrahedron equations**

Formulated also as quantized dimer models, includes **Free Parafermions**, Relativistic Toda etc.

References

[IKTY25] Inoue, K, Terashima, Yagi,
Quantized six-vertex model on a torus, arXiv:2505.08924

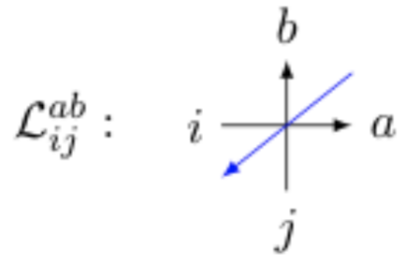
[IKSTY24] Inoue, K, Sun, Terashima, Yagi,
Solutions of tetrahedron equation from quantum cluster algebra associated with symmetric butterfly quiver, SIGMA (2024)

[KMY23] K, Matsuike, Yoneyama,
New solutions to the tetrahedron equation associated with quantized six-vertex models, CMP (2023)

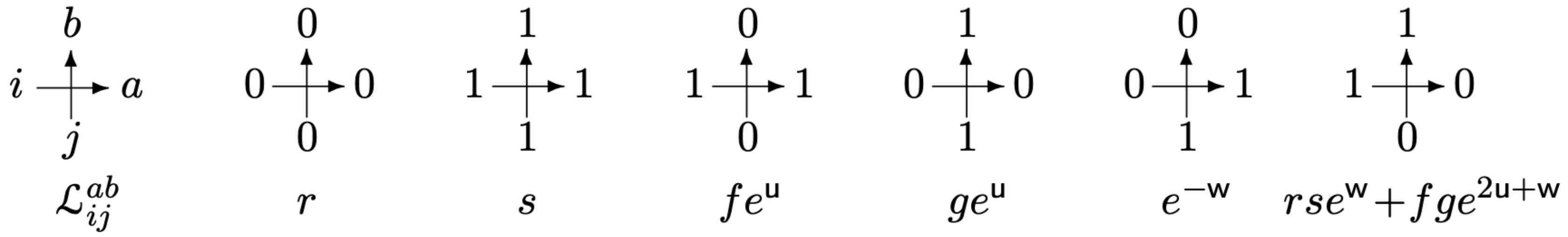
3D L-operator (building block of quantized 6V model)

Bazhanov-Sergeev06, KMY23

$$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1,\ \mathcal{W}(q) = \langle e^{\pm \mathsf{u}}, e^{\pm \mathsf{w}} \rangle :\ q\text{-Weyl algebra } e^{\mathsf{u}}e^{\mathsf{w}} = qe^{\mathsf{w}}e^{\mathsf{u}}$$



$$\mathcal{L} = \mathcal{L}(r,s,f,g;q) = \sum_{a,b,i,j=0,1} E_{ai} \otimes E_{bj} \otimes \mathcal{L}_{ij}^{ab} \in \mathrm{End}(V \otimes V) \otimes \mathcal{W}(q)$$



V: black arrow

$\mathcal{W}(q)$: blue arrow

r, s, f, g are parameters

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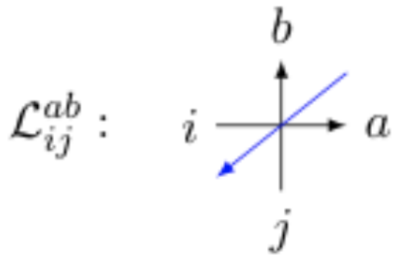

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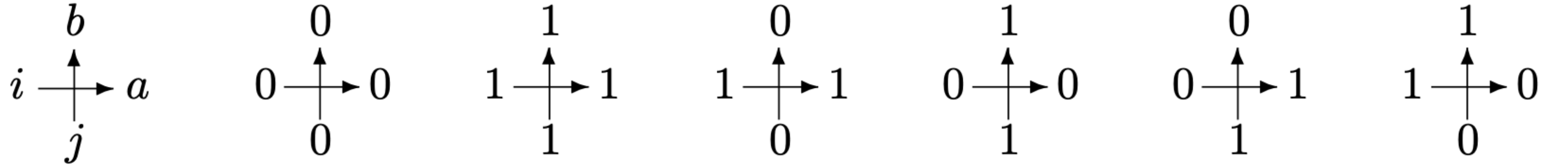


$\mathcal{L}_{ij}^{ab}$ | r | s | fe^{u} | ge^{u} | $e^{-\mathsf{w}}$ | $rse^{\mathsf{w}} + fge^{2\mathsf{u}+\mathsf{w}}$

V: black arrow

$\mathcal{W}(q)$: blue arrow

r, s, f, g are parameters

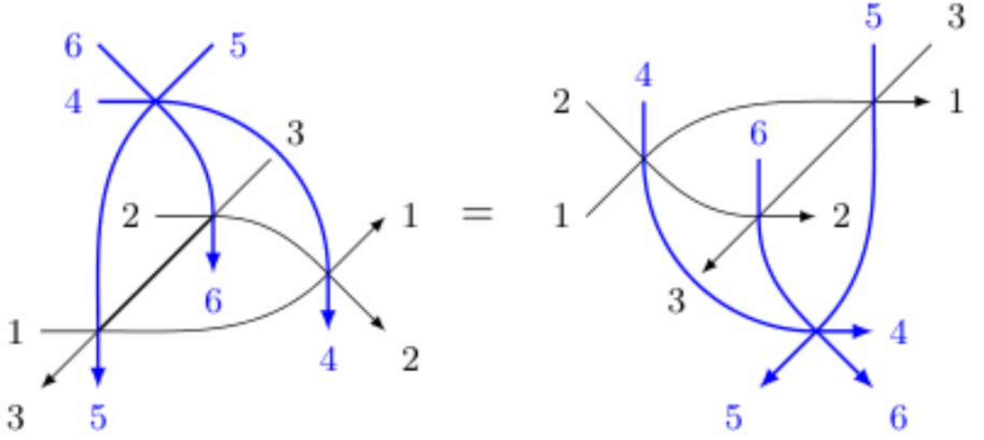
$\mathcal{L}$ acting on $\overset{i}{V} \otimes \overset{j}{V} \otimes \overset{k}{\mathcal{W}}(q)$ will be denoted by $\mathcal{L}_{ijk}$.

It satisfies the tetrahedron equation of $RLLL$ type:

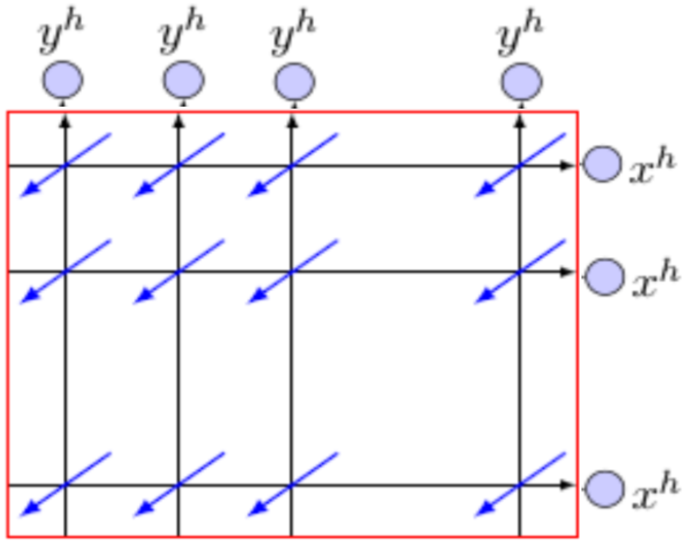
$$R_{456}\mathcal{L}_{236}\mathcal{L}_{135}\mathcal{L}_{124} = \mathcal{L}_{124}\mathcal{L}_{135}\mathcal{L}_{236}R_{456}$$

for some $\mathrm{Ad}(R_{456}) \in \mathrm{End}(\mathcal{W}(q)^{\otimes 3})$

… Yang-Baxter equation *up to conjugation*
(Quantized Yang-Baxter equation)



Quantized 6V model on a torus（G = *m* by *n* square lattice case）



Each vertex is a 3D L-operator

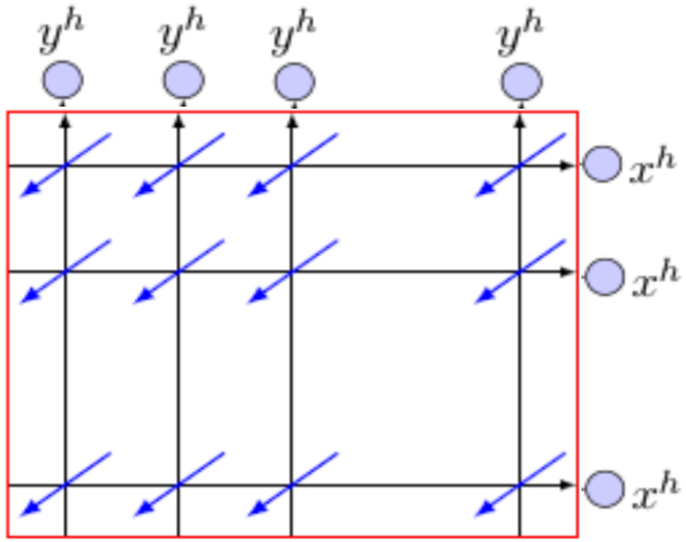
"Boltzmann weights" are **q-Weyl algebra valued**

x, y are boundary "magnetic fields" (serve as **spectral parameters**)

$h \curvearrowright V, \mathcal{W}(q); \;\; h \cdot v_k = k v_k, \;\; [h, e^{\mathsf{u}}] = 0, \;\; [h, e^{-\mathsf{w}}] = e^{-\mathsf{w}}$

: a number op. (counts the number of v_1 at the boundary)

Quantized 6V model on a torus (G = *m* by *n* square lattice case)



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2D Partition function with fixed boundaries yields

Monodromy matrix

$$\mathcal{T}(x, y) \in \mathrm{End}(V^{\otimes m} \otimes V^{\otimes n}) \otimes \mathcal{W}(q)^{\otimes mn}$$

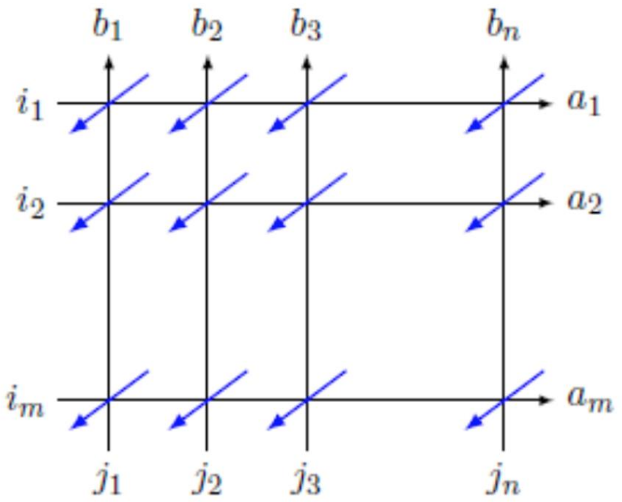
Summing over 2D boundaries under PBC yields

Layer transfer matrix

$$T_G(x, y) = \mathrm{Tr}_{V^{\otimes m} \otimes V^{\otimes n}}\big(\mathcal{T}(x, y)\big) \in \mathcal{W}(q)^{\otimes mn}$$

$$\mathcal{T}(x,y)(v_{\mathbf{i}} \otimes v_{\mathbf{j}}) = \sum_{\mathbf{a}\in\{0,1\}^m, \mathbf{b}\in\{0,1\}^n} x^{|\mathbf{a}|} y^{|\mathbf{b}|} v_{\mathbf{a}} \otimes v_{\mathbf{b}} \otimes T_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}}$$

$$T_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}} = \sum_{\{0,1\}^{\text{inner edges}}}$$



$\mathbf{a} = (a_1, \ldots, a_m)$

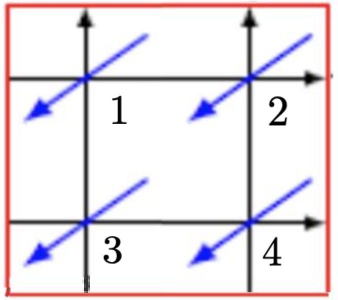
$v_{\mathbf{a}} = v_{a_1} \otimes \cdots \otimes v_{a_m} \in V^{\otimes m}$

$|\mathbf{a}| = a_1 + \cdots + a_m$ etc

$$T_G(x,y) = \mathrm{Tr}_{V^{\otimes m} \otimes V^{\otimes n}} \left(\mathcal{T}(x,y)\right) = \sum_{\mathbf{i}\in\{0,1\}^m, \mathbf{j}\in\{0,1\}^n} T_{\mathbf{i},\mathbf{j}}^{\mathbf{i},\mathbf{j}} x^{|\mathbf{i}|} y^{|\mathbf{j}|} \in \mathcal{W}(q)^{\otimes mn}$$

m=n=2 example

G=



$$\begin{aligned} T_G(x,y) =& f^4 x^2 e^{\mathrm{u}_1+\mathrm{u}_2+\mathrm{u}_3+\mathrm{u}_4} + f^2 s^2 x^2 y (e^{\mathrm{u}_1+\mathrm{u}_3} + e^{\mathrm{u}_2+\mathrm{u}_4}) + s^4 x^2 y^2 + f^2 r^2 x (e^{\mathrm{u}_1+\mathrm{u}_2} + e^{\mathrm{u}_3+\mathrm{u}_4}) \\ &+ f^2 g^2 xy \left(e^{2\mathrm{u}_2+2\mathrm{u}_3-\mathrm{w}_1+\mathrm{w}_2+\mathrm{w}_3-\mathrm{w}_4} + e^{2\mathrm{u}_1+2\mathrm{u}_4+\mathrm{w}_1-\mathrm{w}_2-\mathrm{w}_3+\mathrm{w}_4}\right) \\ &+ fgrsxy \left(e^{2\mathrm{u}_2-\mathrm{w}_1+\mathrm{w}_2+\mathrm{w}_3-\mathrm{w}_4} + e^{2\mathrm{u}_3-\mathrm{w}_1+\mathrm{w}_2+\mathrm{w}_3-\mathrm{w}_4} + e^{2\mathrm{u}_1+\mathrm{w}_1-\mathrm{w}_2-\mathrm{w}_3+\mathrm{w}_4} + e^{2\mathrm{u}_4+\mathrm{w}_1-\mathrm{w}_2-\mathrm{w}_3+\mathrm{w}_4}\right) \\ &+ 2fgrsxy (e^{\mathrm{u}_2+\mathrm{u}_3} + e^{\mathrm{u}_1+\mathrm{u}_4}) + r^2 s^2 xy (e^{-\mathrm{w}_1+\mathrm{w}_2+\mathrm{w}_3-\mathrm{w}_4} + e^{\mathrm{w}_1-\mathrm{w}_2-\mathrm{w}_3+\mathrm{w}_4}) \\ &+ g^2 s^2 x y^2 (e^{\mathrm{u}_1+\mathrm{u}_2} + e^{\mathrm{u}_3+\mathrm{u}_4}) + g^2 r^2 y (e^{\mathrm{u}_1+\mathrm{u}_3} + e^{\mathrm{u}_2+\mathrm{u}_4}) + g^4 y^2 e^{\mathrm{u}_1+\mathrm{u}_2+\mathrm{u}_3+\mathrm{u}_4} + r^4 \end{aligned}$$

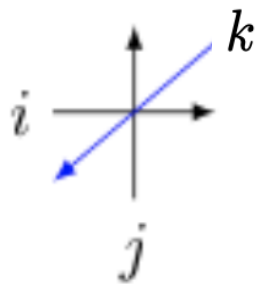
Preparation for showing the commutativity

$V = \mathbb{C}v_0 \otimes \mathbb{C}v_1$, $\mathcal{W}(q) = \langle e^{\pm \mathsf{u}}, e^{\pm \mathsf{w}} \rangle$: q-Weyl alg. $e^{\mathsf{u}} e^{\mathsf{w}} = q e^{\mathsf{w}} e^{\mathsf{u}}$

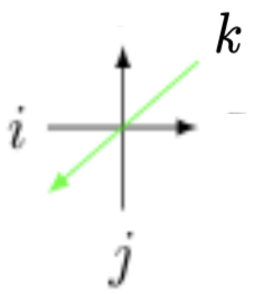
$\mathcal{L} = \mathcal{L}(r, s, f, g; q) \in \mathrm{End}(V \otimes V) \otimes \mathcal{W}(q)$; **3D L-operator**

$\mathcal{M} = \mathcal{L}(r', s', f', g'; -\mathbf{q}) \in \mathrm{End}(V \otimes V) \otimes \mathcal{W}(-\mathbf{q})$; **Another companion 3D L-operator**

$\mathcal{L}_{ijk}$



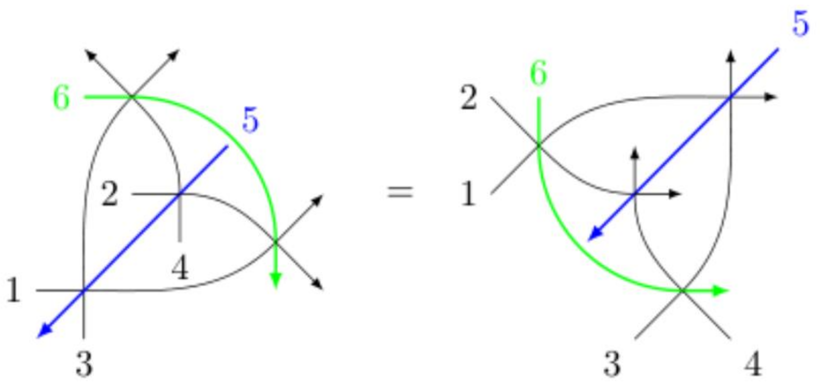
$\mathcal{M}_{ijk}$



They satisfy a tetrahedron equation of MMLL type:

Bazhanov-Sergeev06

$$\mathcal{M}_{126}\mathcal{M}_{346}\mathcal{L}_{135}\mathcal{L}_{245} = \mathcal{L}_{245}\mathcal{L}_{135}\mathcal{M}_{346}\mathcal{M}_{126}$$



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$\mathcal{L}=\mathcal{L}(r,s,f,g;q)\in \mathrm{End}(V\otimes V)\otimes\mathcal{W}(q)$; 3D L-operator

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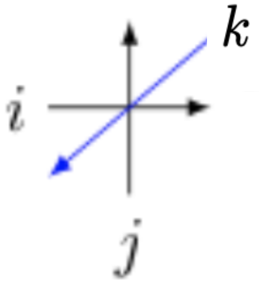

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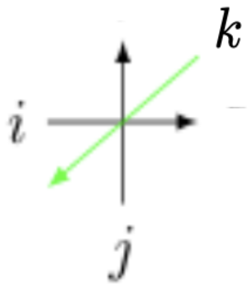
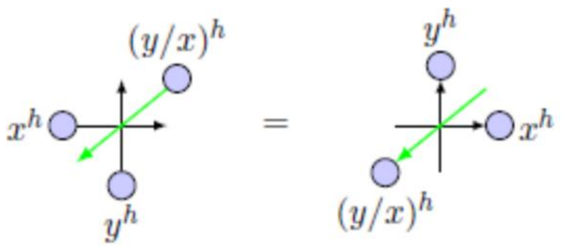


$\mathcal{L}$ and $\mathcal{M}$ are weight preserving:

$$[x^{h_1}y^{h_2}(\tfrac{y}{x})^{h_3},\mathcal{L}_{123}]=0,\ [x^{h_1}y^{h_2}(\tfrac{y}{x})^{h_3},\mathcal{M}_{123}]=0.$$



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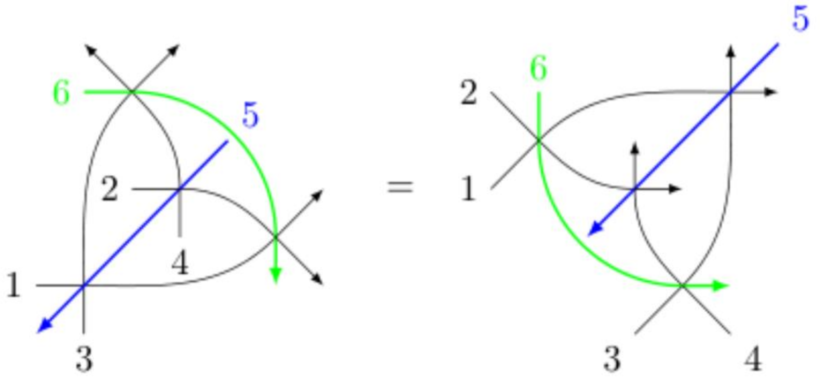
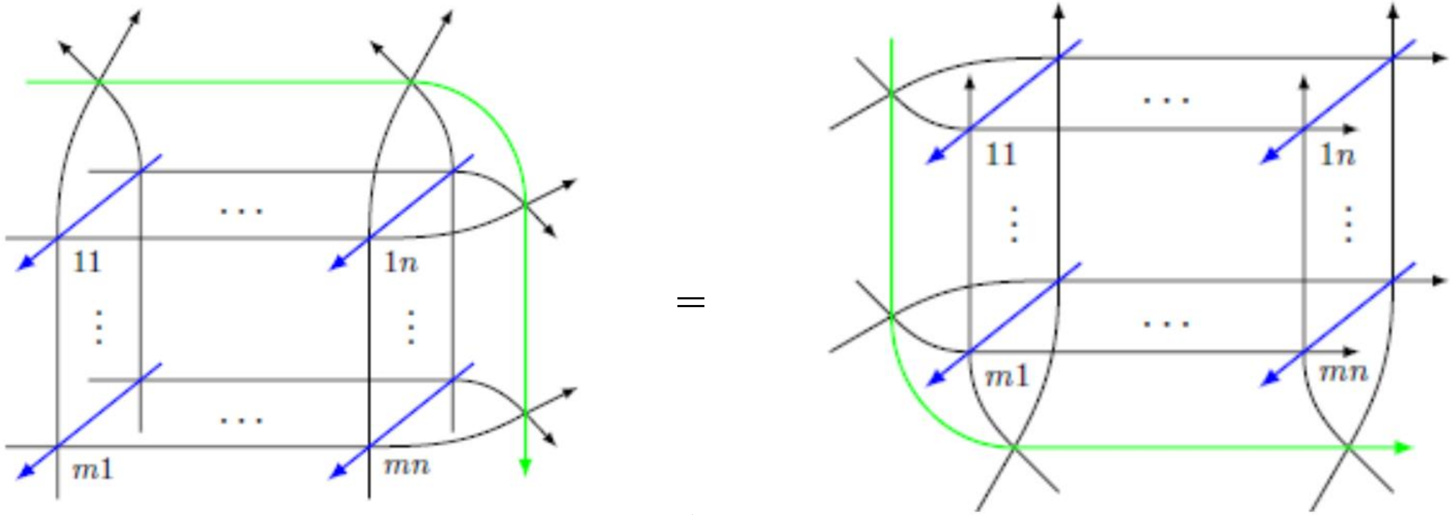
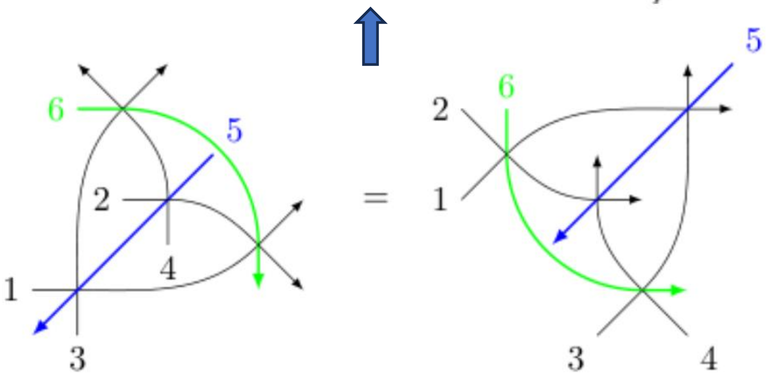


Thm. $[T_G(x,y), T_G(u,w)] = 0$

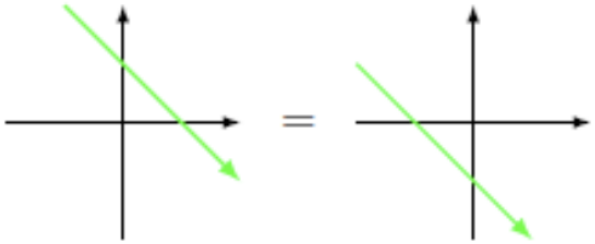
Proof.

(i) Combine $2mn$ copies of $\mathcal{L}$ (blue arrow) and $m+n$ copies of $\mathcal{M}$ (green arrow), and apply the tetrahedron equation to move the green arrow from NE to SW.



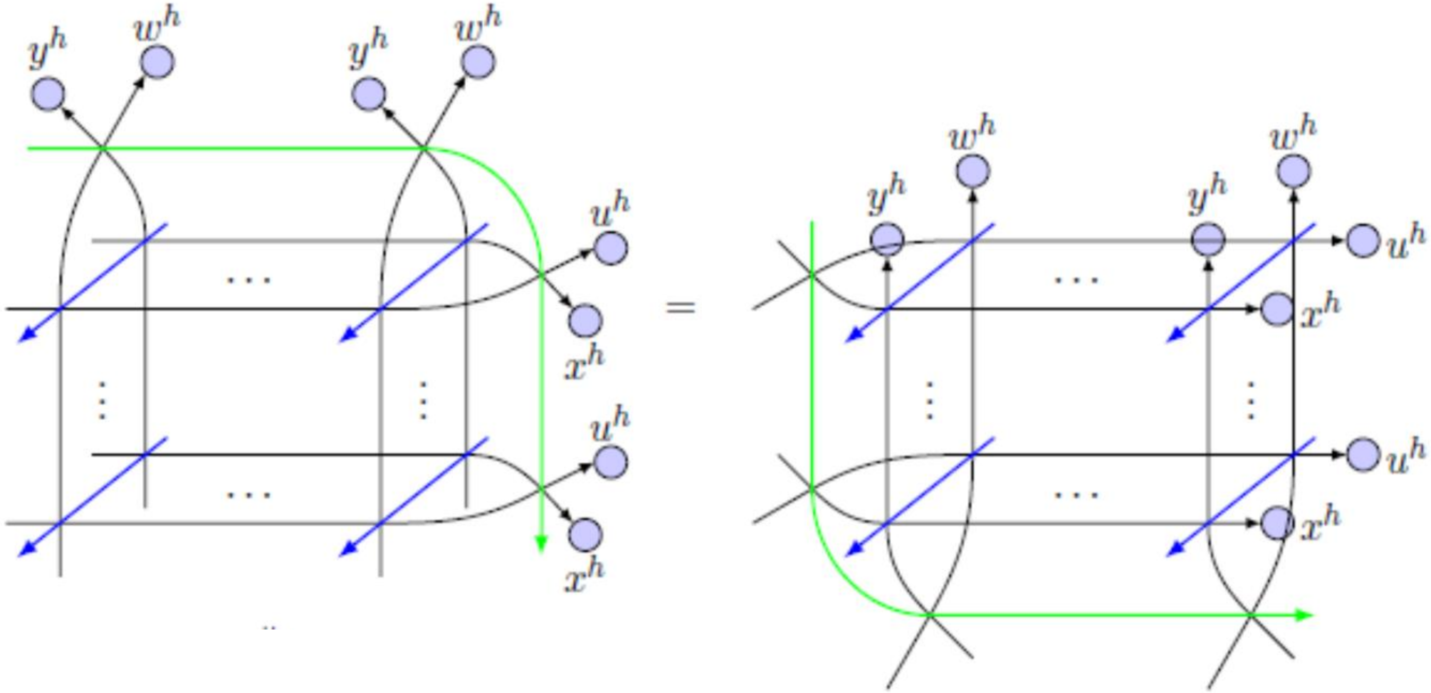


Tetrahedron eq.



2D projection of the tetrahedron eq.

(ii) Multiply x^h, y^h, u^h, v^h to the NE boundaries, and apply weight conservation on the LHS.



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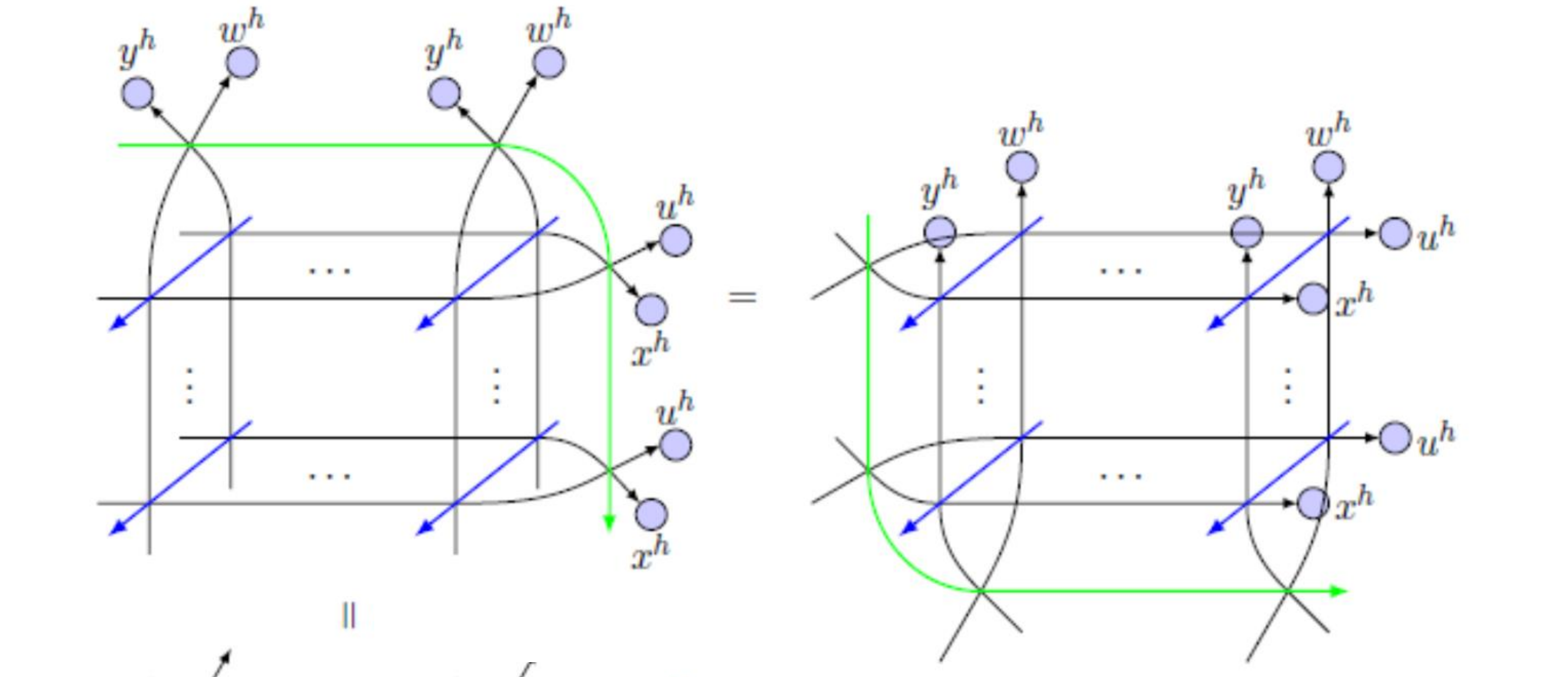
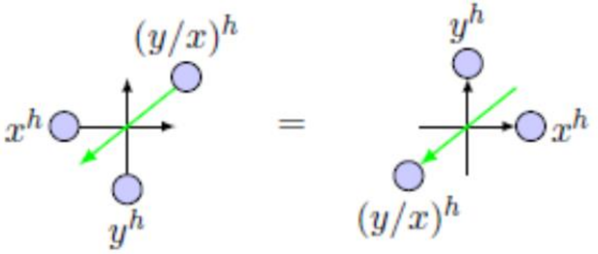
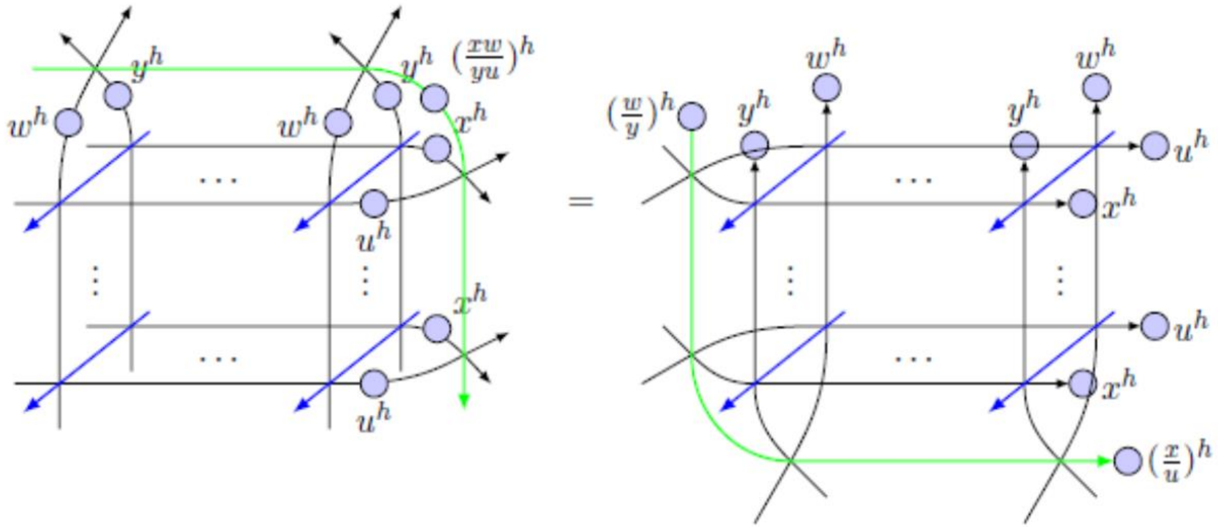




weight conservation

(iii) Multiply $(x/u)^h$ from left $(w/y)^h$ from right on the green arrow.



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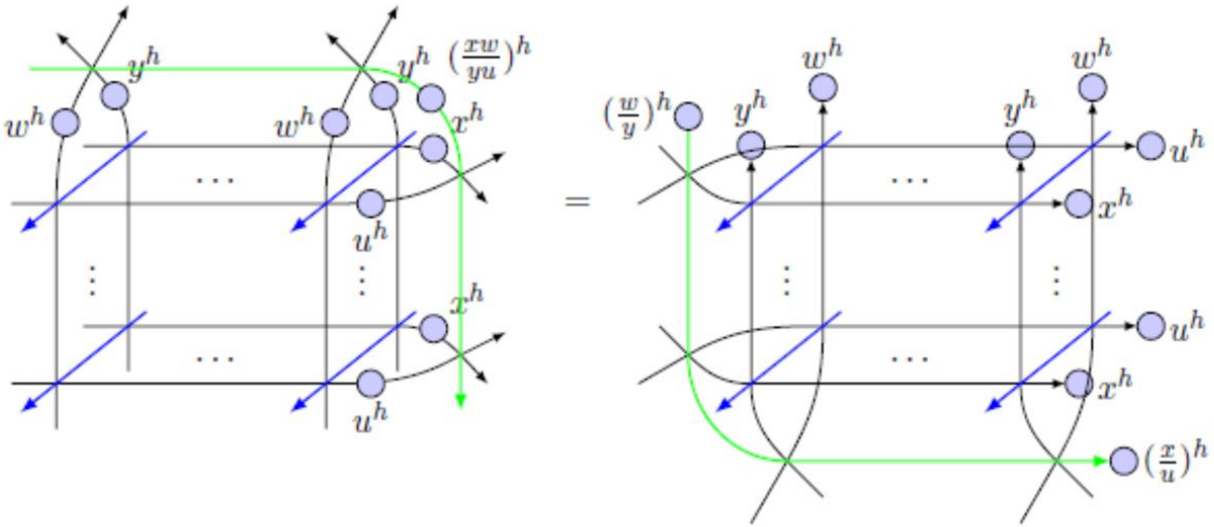
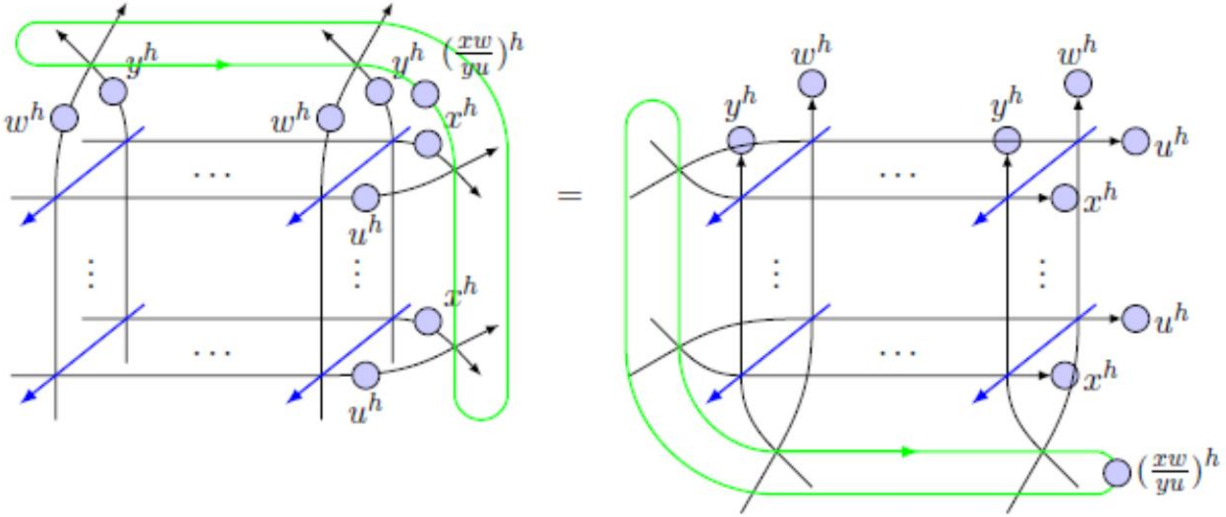
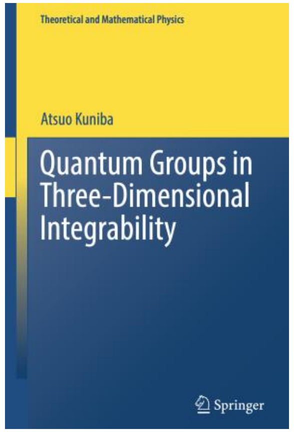


(iv) Take the trace over a 'non-negative mode rep'. of $\mathcal{W}(-q)$ living on the green arrow.



Remark
This R coincides with quantum R of $U_{-1/q}(\widehat{sl}_{n+m})$ on V^{n+m}.
Invertible!

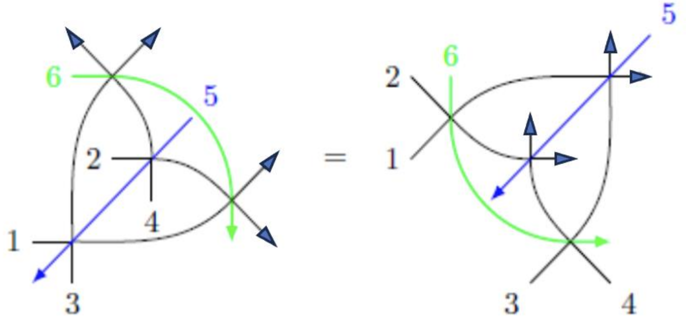


$$\rightsquigarrow \; R\left(\frac{xw}{yu}\right)\mathcal{T}(u,w)\mathcal{T}(x,y) = \mathcal{T}(x,y)\mathcal{T}(u,w)R\left(\frac{xw}{yu}\right) \qquad \blacksquare$$

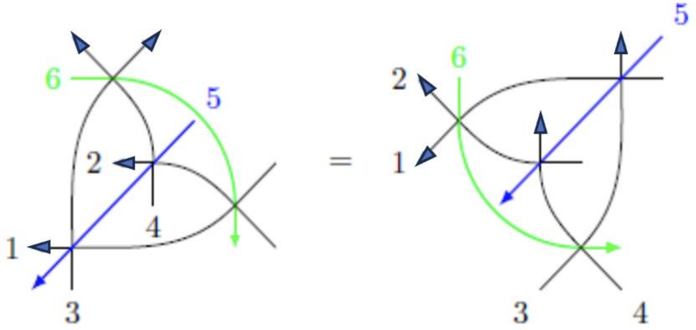
Generalization to admissible graphs

$\mathcal{L}$ and $\mathcal{M}$ satisfy 4-types of tetrahedron eq.

(o) Ordinary type:

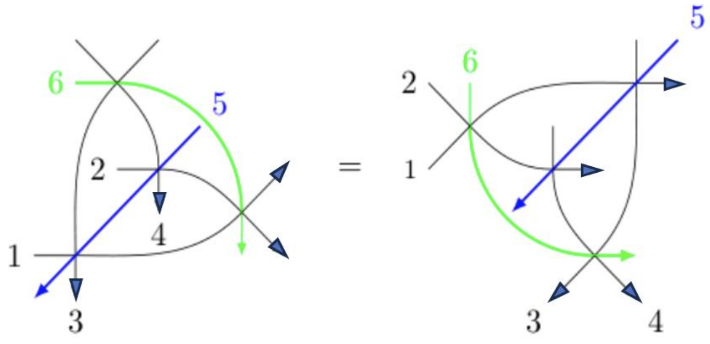


(h) Horizontally reversed type:



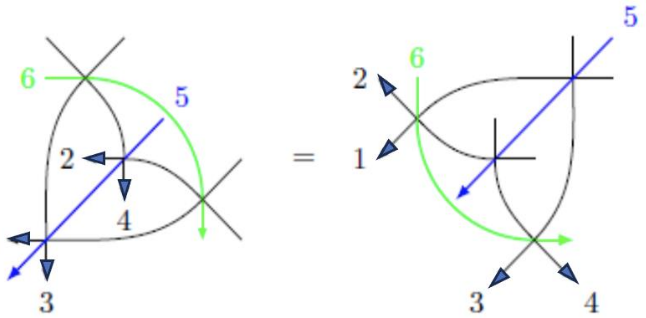
when $r' = s', g' = qf'$

(v) Vertically reversed type:



when $r' = s', g' = q^{-1}f'$

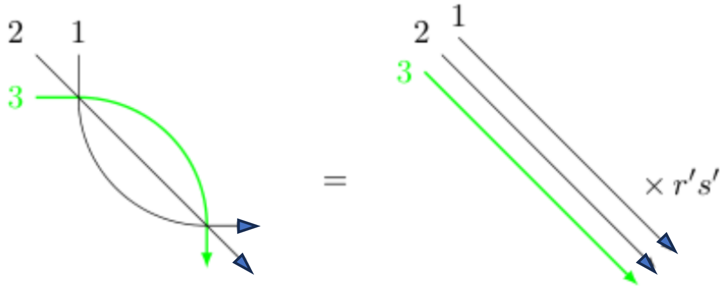
(t) Totally reversed type:



Recall $\begin{cases} \mathcal{L} \text{ (vertex including blue arrow) depends on } r, s, f, g \\ \mathcal{M} \text{ (vertex including green arrow) depends on } r', s', f', g' \end{cases}$

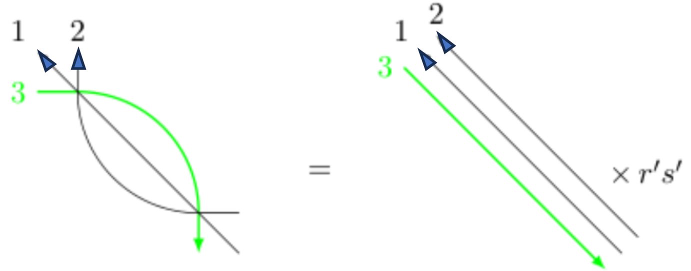
$\mathcal{M} = \mathcal{M}(r', s', f', g'; q)$ satisfies 2-types of inversion relations.

(I) Ordinary type:



when $r' = s', g' = q^{-1}f'$

(I') Reversed type:

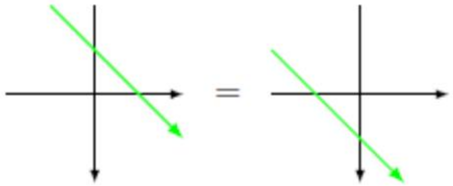


when $r' = s', g' = qf'$

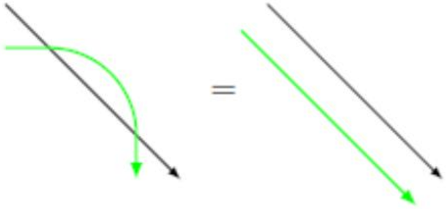
Tetrahedron equations and inversion relations are grouped into 3 types (depicted in 2D projection)

Type A: (v) (I) (iI); $r' = s', g' = q^{-1} f'$

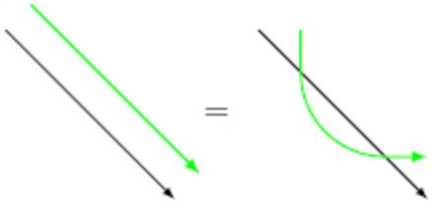
(v)



(I)

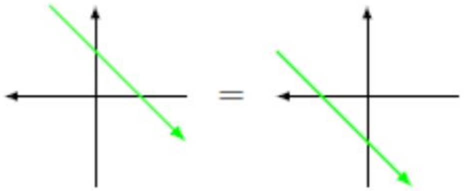


(iI)

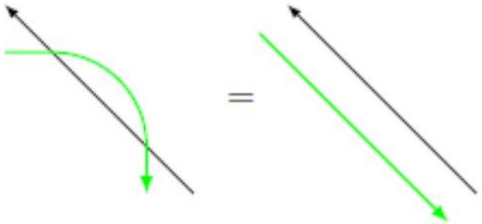


Type B: (h) (I') (iI'); $r' = s', g' = q f'$

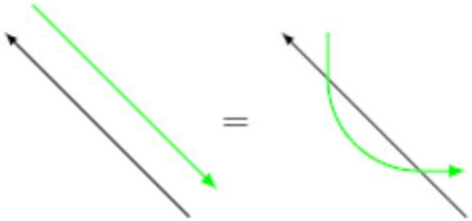
(h)



(I')

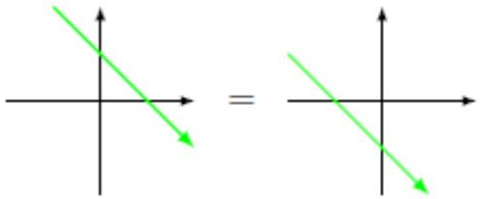


(iI')

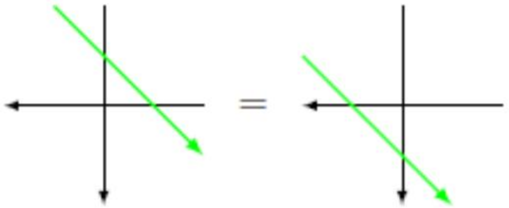


Type C: (o) (t); no condition

(o)



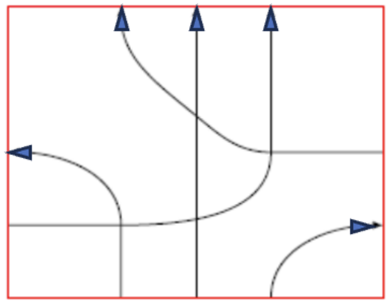
(t)



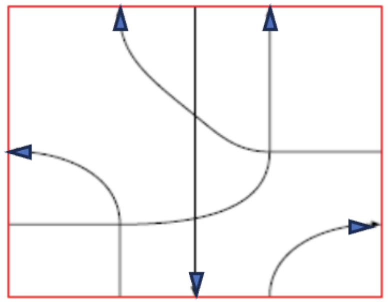
The fact that the conditions on the parameters in Type A and B are different motivates our definition of *admissible graph* on the next page.

Def **Admissible graphs on a torus**

G: a directed graph on a torus; exactly two wires meet at each crossing
G is *admissible* if the initial NE arrow can be moved to the final SW arrow by applying 8 moves, where type A and B do not coexist.



admissible



non-admissible

Thm

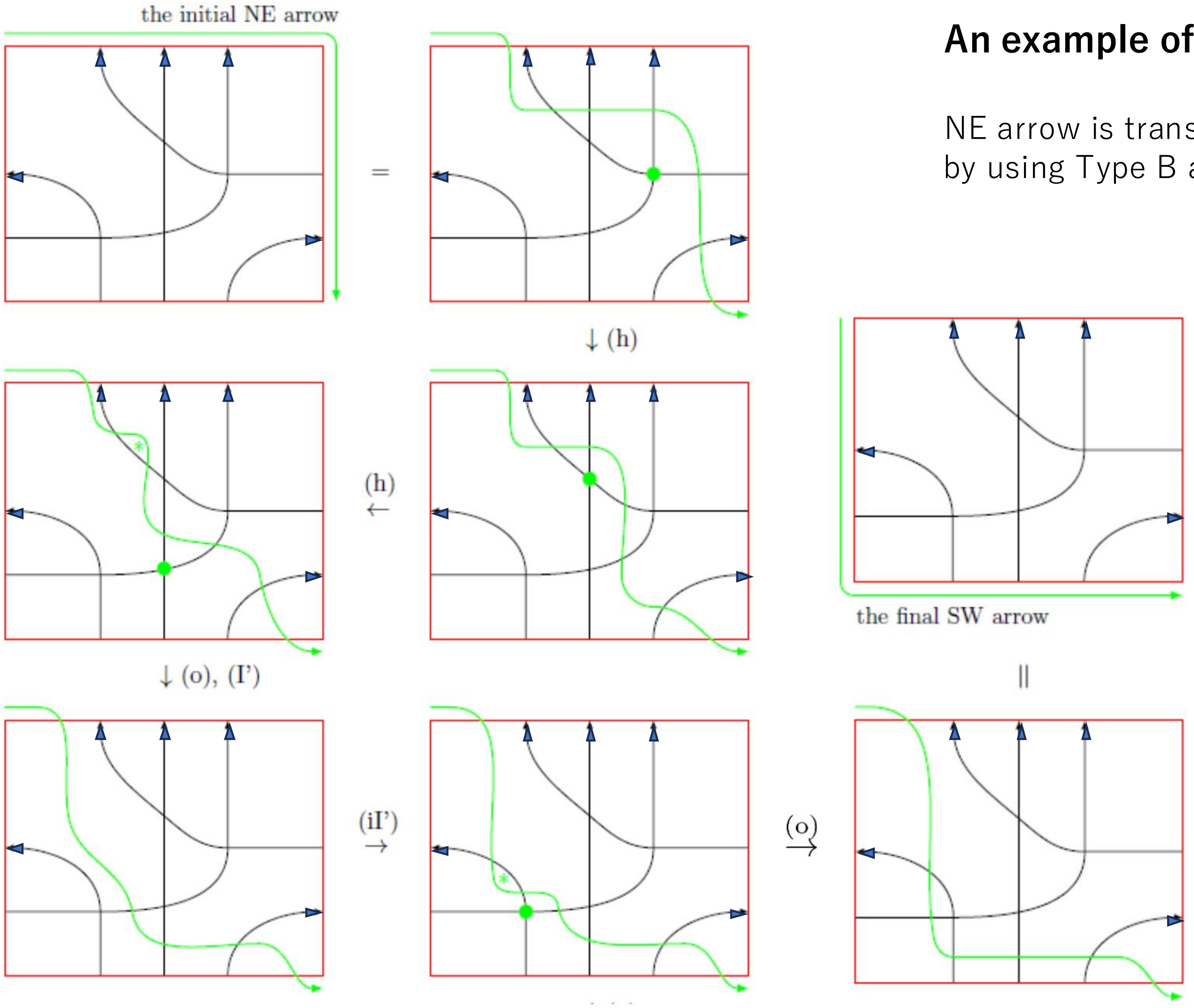
If G is admissible, then $[T_G(x,y),\ T_G(u,w)] = 0$.

Conjecture

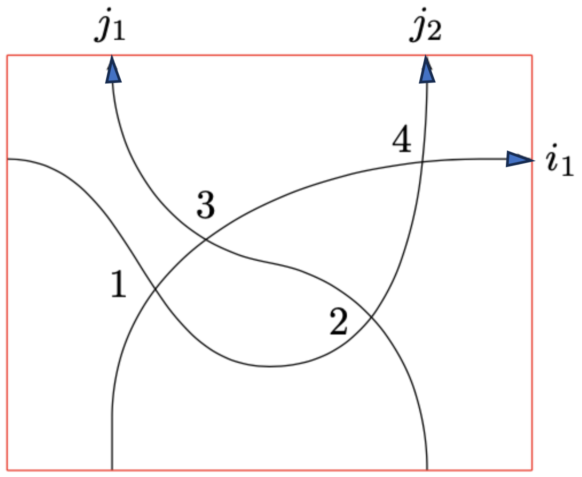
If G includes no oriented face, then it is admissible.

An example of admissible G

NE arrow is transformed to SW
by using Type B and C moves only.



Another example of admissible G and the associated commuting $T_G(x,y)$



G

$$T_G(x,y) = \sum_{i_1,j_1,j_2} T^{\mathrm{i,j}}_{\mathrm{i,j}}\, x^{i_1} y^{j_1+j_2} = r^4 + s^4 xy^2 + B_{0,1}\, y + B_{0,2}\, y^2 + B_{1,0}\, x + B_{1,1}\, xy,$$

$$B_{0,1} = e^{2\mathrm{u}_2+\mathrm{u}_3-\mathrm{w}_1+\mathrm{w}_2} fg^2r + e^{\mathrm{u}_2+2\mathrm{u}_4-\mathrm{w}_3+\mathrm{w}_4} fg^2r + e^{\mathrm{u}_4-\mathrm{w}_2} gr^2 + e^{\mathrm{u}_1+2\mathrm{u}_3+\mathrm{w}_3} fg^2r^2 + e^{\mathrm{u}_3-\mathrm{w}_1+\mathrm{w}_2} gr^2s + e^{\mathrm{u}_2-\mathrm{w}_3+\mathrm{w}_4} gr^2s + e^{\mathrm{u}_1+\mathrm{w}_3} gr^3s,$$

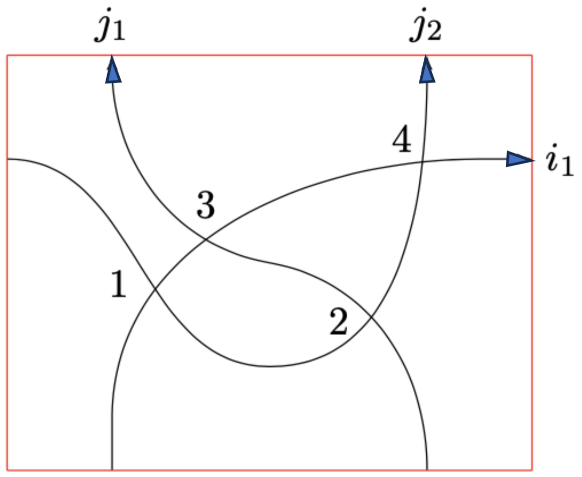
$$B_{0,2} = e^{\mathrm{u}_1+2\mathrm{u}_3+\mathrm{u}_4-\mathrm{w}_2+\mathrm{w}_3} fg^3 + e^{\mathrm{u}_3+\mathrm{u}_4-\mathrm{w}_1} g^2s + e^{\mathrm{u}_1+\mathrm{u}_2+2\mathrm{u}_4+\mathrm{w}_4} fg^3s + e^{\mathrm{u}_1+\mathrm{u}_4-\mathrm{w}_2+\mathrm{w}_3} g^2rs + e^{\mathrm{u}_1+\mathrm{u}_2+\mathrm{w}_4} g^2rs^2.$$

$$B_{1,0} = e^{\mathrm{u}_1+2\mathrm{u}_2+\mathrm{u}_4+\mathrm{w}_2-\mathrm{w}_3} f^3g + e^{\mathrm{u}_1+\mathrm{u}_2-\mathrm{w}_4} f^2r + e^{2\mathrm{u}_1+\mathrm{u}_3+\mathrm{u}_4+\mathrm{w}_1} f^3gr + e^{\mathrm{u}_1+\mathrm{u}_4+\mathrm{w}_2-\mathrm{w}_3} f^2rs + e^{\mathrm{u}_3+\mathrm{u}_4+\mathrm{w}_1} f^2r^2s.$$

$$B_{1,1} = e^{2\mathrm{u}_1+\mathrm{u}_3+\mathrm{w}_1-\mathrm{w}_2} f^2gs + e^{\mathrm{u}_2+2\mathrm{u}_3+\mathrm{w}_3-\mathrm{w}_4} f^2gs + e^{\mathrm{u}_1-\mathrm{w}_3} fs^2 + e^{2\mathrm{u}_2+\mathrm{u}_4+\mathrm{w}_2} f^2gs^2 + e^{\mathrm{u}_3+\mathrm{w}_1-\mathrm{w}_2} frs^2 + e^{\mathrm{u}_2+\mathrm{w}_3-\mathrm{w}_4} frs^2 + e^{\mathrm{u}_4+\mathrm{w}_2} frs^3.$$

$B_{i,j}$ are commuting "Hamiltonians" living in $\mathcal{W}(q)^{\otimes 4}$.

Another example of admissible G and the associated commuting $T_G(x,y)$



G

$$T_G(x,y) = \sum_{i_1,j_1,j_2} T^{\mathrm{i,j}}_{\mathrm{i,j}}\, x^{i_1} y^{j_1+j_2} = r^4 + s^4 x y^2 + B_{0,1}\, y + B_{0,2}\, y^2 + B_{1,0}\, x + B_{1,1}\, xy,$$

$$B_{0,1} = e^{2u_2+u_3-w_1+w_2} f g^2 r + e^{u_2+2u_4-w_3+w_4} f g^2 r + e^{u_4-w_2} g r^2 + e^{u_1+2u_3+w_3} f g^2 r^2 + e^{u_3-w_1+w_2} g r^2 s + e^{u_2-w_3+w_4} g r^2 s + e^{u_1+w_3} g r^3 s,$$

$$B_{0,2} = e^{u_1+2u_3+u_4-w_2+w_3} f g^3 + e^{u_3+u_4-w_1} g^2 s + e^{u_1+u_2+2u_4+w_4} f g^3 s + e^{u_1+u_4-w_2+w_3} g^2 r s + e^{u_1+u_2+w_4} g^2 r s^2.$$

$$B_{1,0} = e^{u_1+2u_2+u_4+w_2-w_3} f^3 g + e^{u_1+u_2-w_4} f^2 r + e^{2u_1+u_3+u_4+w_1} f^3 g r + e^{u_1+u_4+w_2-w_3} f^2 r s + e^{u_3+u_4+w_1} f^2 r^2 s.$$

$$B_{1,1} = e^{2u_1+u_3+w_1-w_2} f^2 g s + e^{u_2+2u_3+w_3-w_4} f^2 g s + e^{u_1-w_3} f s^2 + e^{2u_2+u_4+w_2} f^2 g s^2 + e^{u_3+w_1-w_2} f r s^2 + e^{u_2+w_3-w_4} f r s^2 + e^{u_4+w_2} f r s^3.$$

$B_{i,j}$ are commuting "Hamiltonians" living in $\mathcal{W}(q)^{\otimes 4}$.

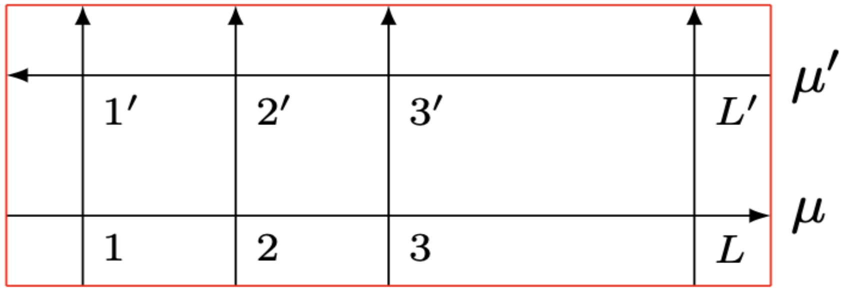
Inhomogeneous generalizations keeping the commutativity $[T_G(x,y), T_G(u,w)] = 0$

- Boundary magnetic field: $x \to (x\mu_1, \ldots, x\mu_m), y \to (y\nu_1, \ldots, y\nu_n)$
- r, s, f, g parameters: $(r,s,f,g) \to (r_i, s_i, f_i, g_i)$ for each vertex i of G

Free parafermions

Baxter1989, Au-Yang, Perk, Fendley, Alcaraz, Batchelor, Liu, Zhou, Pimenta, Henry, Lu,⋯

G = (admissible)



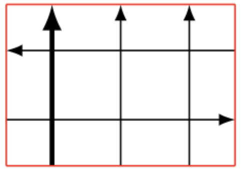
Quantized **5V** model corresponding to

$$\mathcal{L}(r_i = 1, s_i = 1, f_i = 0, g_{i'} = g; q)$$

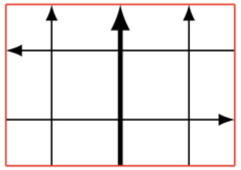
$$\mathcal{L}(r_{i'} = 1, s_{i'} = 1, f_{i'} = f, g_{i'} = 0; q)$$

Inhomogeneity $\mu = \mu' = 0$ case. $\quad T_G(x, y) \to T_G(y) = \sum_{m=0}^{L} y^m J^{(m)}$

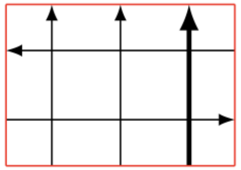
For $L = 3$, the first Hamiltonian is $J^{(1)} = h_1 + h_2 + h_3 + h_4$



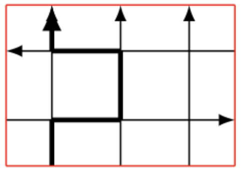
$h_1 = fgX_1$



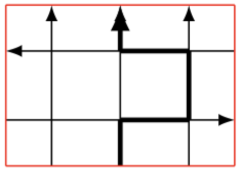
$h_3 = fgX_2$



$h_5 = fgX_3$



$h_2 = Z_1^{-1} Z_2$



$h_4 = Z_2^{-1} Z_3$

$$X_i = e^{\mathsf{u}_i + \mathsf{u}_{i'}}, \quad Z_i = e^{\mathsf{w}_i + \mathsf{w}_{i'}}$$

$$X_i Z_j = q^{2\delta_{i,j}} Z_j X_i.$$

For general L, set $h_{2i-1} = fgX_i$, $h_{2i} = Z_i^{-1} Z_{i+1}$ similarly, which satisfy

$$h_a h_{a+1} = q^{-2} h_{a+1} h_a, \quad h_a h_b = h_b h_a \ \text{ for } |a - b| > 1.$$

$$J^{(m)} = \sum_{1 \le b_i \prec \cdots \prec b_m \le 2L-1} h_{b_1} \cdots h_{b_m} \qquad (b \prec b' \overset{\text{def}}{\Longleftrightarrow} b + 2 \le b')$$

$$J^{(1)} = \sum_{a=1}^{2L-1} h_a = \sum_{i=1}^{L-1} Z_i^{-1} Z_{i+1} + fg \sum_{i=1}^{L} X_i \qquad [T_G(x), T_G(y)] = 0 \ \Rightarrow [J^{(m)}, J^{(n)}] = 0$$

$J^{(m)}$ reproduces the **Free Parafermion Hamiltonian** ($m = 1$) and its higher order conserved quantities (Fendley 2014), when q^2 is specialized to a primitive Nth root of unity and the following representation is taken:

$$X_i|\mathbf{m}\rangle = |\mathbf{m} - \mathbf{e}_i\rangle, \qquad Z_i|\mathbf{m}\rangle = q^{2m_i}|\mathbf{m}\rangle,$$

$$\mathbf{m} = (m_1, \ldots, m_L) \in (\mathbb{Z}_N)^L, \quad \mathbf{e}_i = (0, \ldots, \overset{i}{1}, \ldots, 0).$$

Summary +

Quantized 6V model on admissible graphs G on a torus

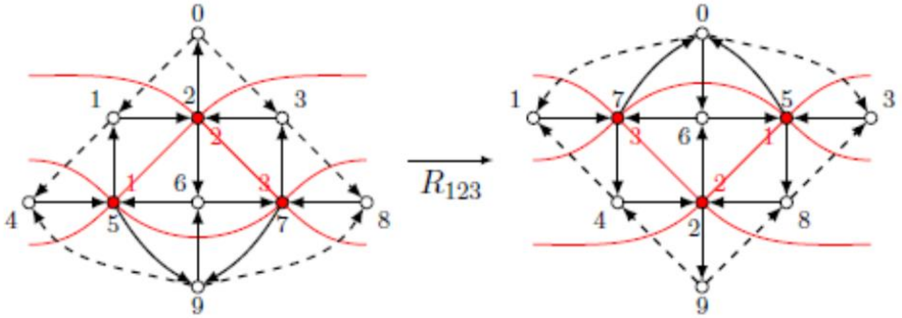
Commuting layer transfer matrix $T_G(x,y)$ with two spectral parameters

$T_G(x,y)$ belongs to tensor prod. of q-Weyl algebra $e^{\mathsf{u}_i}e^{\mathsf{w}_i} = qe^{\mathsf{w}_i}e^{\mathsf{u}_i}$ attached to each vertex i of G

Commutativity ensured by the tetrahedron equations for the 3D L-operators

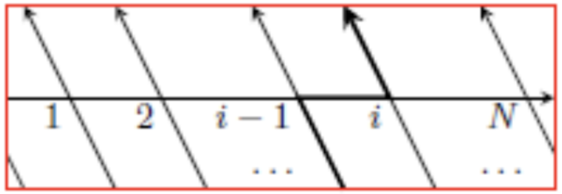
Relation to quantum cluster algebra [IKSTY 24]

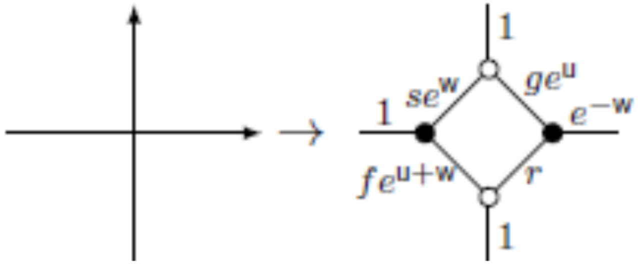
The R-operator in $RLLL = LLLR$ is obtained from the quantum cluster algebra for Symmetric Butterfly quiver



Relation to other models

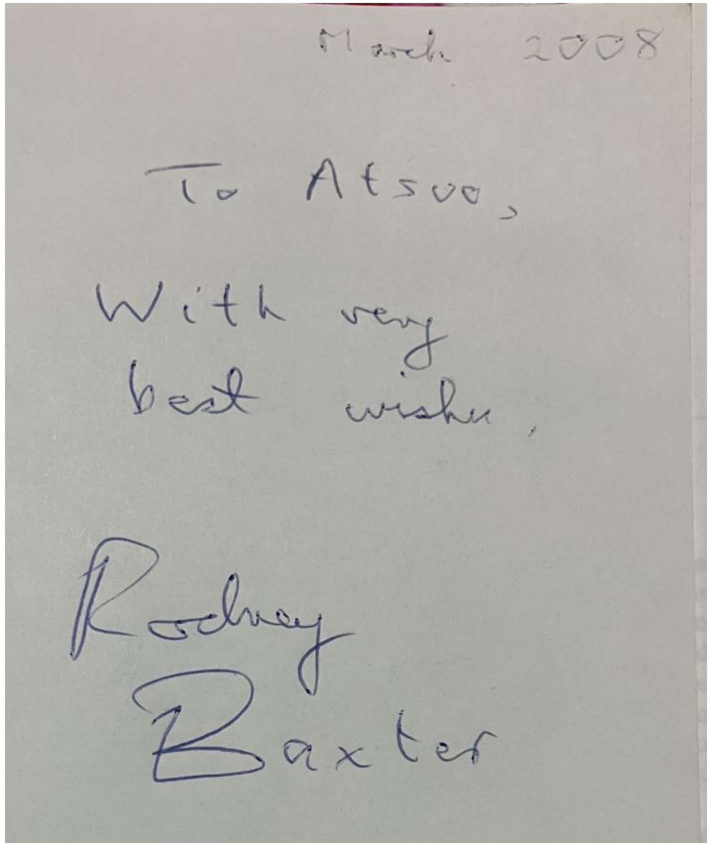
- relativistic Toda chain
- dimer model





Yang-Baxter photo

February 14, 1992



Rodney's signature in the 2007 edition of his book – his familiar hand, fondly remembered

March 2008

Thank you