Periodicities of T-systems and Y-systems

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joint with

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Quantum Integrable Discrete Systems

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Contents

What are T-systems and Y-systems?

Systems of difference equations among commuting variables

$$T_m^{(a)}(u) \quad \text{and} \quad Y_m^{(a)}(u)$$

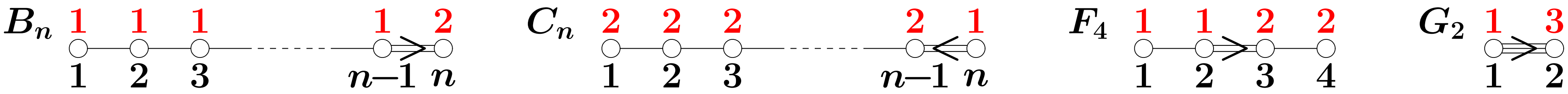
related to root system.

$$a \in \{\text{nodes of Dynkin diagram of } \mathfrak{g}\}$$
$$(\mathfrak{g} = A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2)$$

$$m \in \mathbb{Z}_{\geq 1}$$

$$u \in \mathbb{C} \quad \text{(spectral parameter)}$$

$$t_a := |\text{long root}|^2/|\alpha_a|^2 \quad (= 1 \ \text{ for ADE})$$



$$T^{(a)}_m(u - \frac{1}{t_a})T^{(a)}_m(u + \frac{1}{t_a}) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + \text{product of } T\text{'s},$$

$$Y^{(a)}_m(u - \frac{1}{t_a})Y^{(a)}_m(u + \frac{1}{t_a}) = \frac{\text{product of } (1+Y)\text{'s}}{(1+Y^{(a)}_{m-1}(u)^{-1})(1+Y^{(a)}_{m+1}(u)^{-1})}.$$

Structure of products in the RHS is dependent on $m \mod t_a\mathbb{Z}$.

$\mathfrak{g} = A_n, D_n, E_n$ case

$C = (C_{ab})_{1 \le a,b \le n}$: **Cartan matrix**

T-system

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{b:\, C_{ab}=-1} T_m^{(b)}(u)$$

Example A_n

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u)$$

$$(T_0^{(a)}(u) = T_m^{(0)}(u) = T_m^{(n+1)}(u) = 1.)$$

A version of Hirota-Miwa or Toda-field equation on discrete space-time.

B_n

$$\begin{aligned}
T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad (1 \le a \le n-2),\\
T^{(n-1)}_m(u-1)T^{(n-1)}_m(u+1) &= T^{(n-1)}_{m-1}(u)T^{(n-1)}_{m+1}(u) + T^{(n-2)}_m(u)T^{(n)}_{2m}(u),\\
T^{(n)}_{2m}(u-\frac{1}{2})T^{(n)}_{2m}(u+\frac{1}{2}) &= T^{(n)}_{2m-1}(u)T^{(n)}_{2m+1}(u) + T^{(n-1)}_m(u-\frac{1}{2})T^{(n-1)}_m(u+\frac{1}{2}),\\
T^{(n)}_{2m+1}(u-\frac{1}{2})T^{(n)}_{2m+1}(u+\frac{1}{2}) &= T^{(n)}_{2m}(u)T^{(n)}_{2m+2}(u) + T^{(n-1)}_m(u)T^{(n-1)}_{m+1}(u).
\end{aligned}$$

C_n

$$\begin{aligned}
T^{(a)}_m(u-\frac{1}{2})T^{(a)}_m(u+\frac{1}{2}) &= T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad (1 \le a \le n-2),\\
T^{(n-1)}_{2m}(u-\frac{1}{2})T^{(n-1)}_{2m}(u+\frac{1}{2}) &= T^{(n-1)}_{2m-1}(u)T^{(n-1)}_{2m+1}(u) + T^{(n-2)}_{2m}(u)T^{(n)}_m(u-\frac{1}{2})T^{(n)}_m(u+\frac{1}{2}),\\
T^{(n-1)}_{2m+1}(u-\frac{1}{2})T^{(n-1)}_{2m+1}(u+\frac{1}{2}) &= T^{(n-1)}_{2m}(u)T^{(n-1)}_{2m+2}(u) + T^{(n-2)}_{2m+1}(u)T^{(n)}_m(u)T^{(n)}_{m+1}(u),\\
T^{(n)}_m(u-1)T^{(n)}_m(u+1) &= T^{(n)}_{m-1}(u)T^{(n)}_{m+1}(u) + T^{(n-1)}_{2m}(u).
\end{aligned}$$

F_4

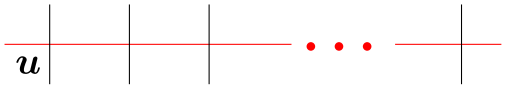
$$
\begin{aligned}
T^{(1)}_m(u-1)T^{(1)}_m(u+1) &= T^{(1)}_{m-1}(u)T^{(1)}_{m+1}(u) + T^{(2)}_m(u),\\
T^{(2)}_m(u-1)T^{(2)}_m(u+1) &= T^{(2)}_{m-1}(u)T^{(2)}_{m+1}(u) + T^{(1)}_m(u)T^{(3)}_{2m}(u),\\
T^{(3)}_{2m}(u-\frac{1}{2})T^{(3)}_{2m}(u+\frac{1}{2}) &= T^{(3)}_{2m-1}(u)T^{(3)}_{2m+1}(u) + T^{(2)}_m(u-\frac{1}{2})T^{(2)}_m(u+\frac{1}{2})T^{(4)}_{2m}(u),\\
T^{(3)}_{2m+1}(u-\frac{1}{2})T^{(3)}_{2m+1}(u+\frac{1}{2}) &= T^{(3)}_{2m}(u)T^{(3)}_{2m+2}(u) + T^{(2)}_m(u)T^{(2)}_{m+1}(u)T^{(4)}_{2m+1}(u),\\
T^{(4)}_m(u-\frac{1}{2})T^{(4)}_m(u+\frac{1}{2}) &= T^{(4)}_{m-1}(u)T^{(4)}_{m+1}(u) + T^{(3)}_m(u).
\end{aligned}
$$

G_2

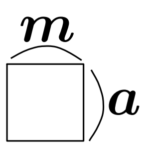
$$
\begin{aligned}
T^{(1)}_m(u-1)T^{(1)}_m(u+1) &= T^{(1)}_{m-1}(u)T^{(1)}_{m+1}(u) + T^{(2)}_{3m}(u),\\
T^{(2)}_{3m}(u-\frac{1}{3})T^{(2)}_{3m}(u+\frac{1}{3}) &= T^{(2)}_{3m-1}(u)T^{(2)}_{3m+1}(u) + T^{(1)}_m(u-\frac{2}{3})T^{(1)}_m(u)T^{(1)}_m(u+\frac{2}{3}),\\
T^{(2)}_{3m+1}(u-\frac{1}{3})T^{(2)}_{3m+1}(u+\frac{1}{3}) &= T^{(2)}_{3m}(u)T^{(2)}_{3m+2}(u) + T^{(1)}_m(u-\frac{1}{3})T^{(1)}_m(u+\frac{1}{3})T^{(1)}_{m+1}(u),\\
T^{(2)}_{3m+2}(u-\frac{1}{3})T^{(2)}_{3m+2}(u+\frac{1}{3}) &= T^{(2)}_{3m+1}(u)T^{(2)}_{3m+3}(u) + T^{(1)}_m(u)T^{(1)}_{m+1}(u-\frac{1}{3})T^{(1)}_{m+1}(u+\frac{1}{3}).
\end{aligned}
$$

Origin of T-system: $T_m^{(a)}(u)$ stands for

Phys: commuting transfer matrices in Yang-Baxter solvable lattice models.

$T_m^{(a)}(u) = \mathrm{Tr}_{W_m^{(a)}(u)} \Big($  $\Big), \qquad [T_m^{(a)}(u), T_{m'}^{(a')}(u')] = 0.$

Math: q-characters of Kirillov-Reshetikhin modules $W_m^{(a)}(u)$ of quantum affine algebra $U_q(\hat{\mathfrak{g}})$.

For A_n $\quad W_m^{(a)}(u) \simeq$ 

$$0 \to W_m^{(a-1)}(u) \otimes W_m^{(a+1)}(u) \to W_m^{(a)}(u-1) \otimes W_m^{(a)}(u+1) \to W_{m-1}^{(a)}(u) \otimes W_{m+1}^{(a)}(u) \to 0$$

Proposed in the former context by K-Nakanishi-Suzuki (1994).

Proved in the latter context by Nakajima for ADE (2003) and Hernandez for $\forall \mathfrak{g}$ (2006).

- **Solutions in A_n case** (Bazhanov-Reshetikhin 1991)

Jacobi-Trudi type formula

$$T^{(a)}_m(u) = \det_{1\le i,j\le m}(T^{(a-i+j)}_1(u+i+j-m-1)).$$

Tableau-sum type formula

$$A_3 \text{ example}: \quad T^{(2)}_3(u) = \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2&2\\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2&3\\ \hline\end{array} + \cdots + \begin{array}{|c|c|c|}\hline 3&3&3\\ \hline 4&4&4\\ \hline\end{array}$$

each tableaux = ratio of products of Baxter Q functions.

- **Generalizations to $\mathfrak{g}$**

BCD analogue of Jacobi-Trudi including Pfaffians: K-Nakamura-Hirota (1996).
Casorati like determinants (A type): Krichever-Lipan-Wiegmann-Zabrodin (1997).
(C type): K-Okado-Suzuki-Yamada (2002).

- **Tableau-sum formula for various $\mathfrak{g}$**

Chari-Pressley, K-Ohta-Suzuki, Kleber, Frenkel-Reshetikhin, Nakajima, Nakai-Nakanishi, Hernandez, etc.

Y-system

$\mathfrak{g} = A_n, D_n, E_n$ case $\quad (Y_0^{(a)}(u)^{-1} = 0)$

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{\prod_{b:\,C_{ab}=-1}(1+Y_m^{(b)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}$$

B_n

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})} \quad (1 \le a \le n-2),$$

$$Y_m^{(n-1)}(u-1)Y_m^{(n-1)}(u+1)$$
$$= \frac{(1+Y_m^{(n-2)}(u))(1+Y_{2m}^{(n)}(u+\frac{1}{2}))(1+Y_{2m}^{(n)}(u-\frac{1}{2}))(1+Y_{2m-1}^{(n)}(u))(1+Y_{2m+1}^{(n)}(u))}{(1+Y_{m-1}^{(n-1)}(u)^{-1})(1+Y_{m+1}^{(n-1)}(u)^{-1})},$$

$$Y_{2m}^{(n)}(u-\frac{1}{2})Y_{2m}^{(n)}(u+\frac{1}{2}) = \frac{1+Y_m^{(n-1)}(u)}{(1+Y_{2m-1}^{(n)}(u)^{-1})(1+Y_{2m+1}^{(n)}(u)^{-1})},$$

$$Y_{2m+1}^{(n)}(u-\frac{1}{2})Y_{2m+1}^{(n)}(u+\frac{1}{2}) = \frac{1}{(1+Y_{2m}^{(n)}(u)^{-1})(1+Y_{2m+2}^{(n)}(u)^{-1})}.$$

$\boldsymbol{C_n}$

$$Y_m^{(a)}(u-\frac{1}{2})Y_m^{(a)}(u+\frac{1}{2})=\frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}\quad(1\le a\le n-2),$$

$$Y_{2m}^{(n-1)}(u-\frac{1}{2})Y_{2m}^{(n-1)}(u+\frac{1}{2})=\frac{(1+Y_{2m}^{(n-2)}(u))(1+Y_m^{(n)}(u))}{(1+Y_{2m-1}^{(n-1)}(u)^{-1})(1+Y_{2m+1}^{(n-1)}(u)^{-1})},$$

$$Y_{2m+1}^{(n-1)}(u-\frac{1}{2})Y_{2m+1}^{(n-1)}(u+\frac{1}{2})=\frac{1+Y_{2m+1}^{(n-2)}(u)}{(1+Y_{2m}^{(n)}(u)^{-1})(1+Y_{2m+2}^{(n)}(u)^{-1})},$$

$$Y_m^{(n)}(u-1)Y_m^{(n)}(u+1)$$
$$=\frac{(1+Y_{2m}^{(n-1)}(u+\frac{1}{2}))(1+Y_{2m}^{(n-1)}(u-\frac{1}{2}))(1+Y_{2m-1}^{(n-1)}(u))(1+Y_{2m+1}^{(n-1)}(u))}{(1+Y_{m-1}^{(n)}(u)^{-1})(1+Y_{m+1}^{(n)}(u)^{-1})}.$$

$\boldsymbol{F_4}$

$$\boldsymbol{Y}^{(1)}_m(u-1)\boldsymbol{Y}^{(1)}_m(u+1)=\frac{1+\boldsymbol{Y}^{(2)}_m(u)}{(1+\boldsymbol{Y}^{(1)}_{m-1}(u)^{-1})(1+\boldsymbol{Y}^{(1)}_{m+1}(u)^{-1})},$$

$$\boldsymbol{Y}^{(2)}_m(u-1)\boldsymbol{Y}^{(2)}_m(u+1)$$
$$=\frac{(1+\boldsymbol{Y}^{(1)}_m(u))(1+\boldsymbol{Y}^{(3)}_{2m}(u-\frac{1}{2}))(1+\boldsymbol{Y}^{(3)}_{2m}(u+\frac{1}{2}))(1+\boldsymbol{Y}^{(3)}_{2m-1}(u))(1+\boldsymbol{Y}^{(3)}_{2m+1}(u))}{(1+\boldsymbol{Y}^{(2)}_{m-1}(u)^{-1})(1+\boldsymbol{Y}^{(2)}_{m+1}(u)^{-1})},$$

$$\boldsymbol{Y}^{(3)}_{2m}(u-\frac{1}{2})\boldsymbol{Y}^{(3)}_{2m}(u+\frac{1}{2})=\frac{(1+\boldsymbol{Y}^{(2)}_m(u))(1+\boldsymbol{Y}^{(4)}_{2m}(u))}{(1+\boldsymbol{Y}^{(3)}_{2m-1}(u)^{-1})(1+\boldsymbol{Y}^{(3)}_{2m+1}(u)^{-1})},$$

$$\boldsymbol{Y}^{(3)}_{2m+1}(u-\frac{1}{2})\boldsymbol{Y}^{(3)}_{2m+1}(u+\frac{1}{2})=\frac{1+\boldsymbol{Y}^{(4)}_{2m+1}(u)}{(1+\boldsymbol{Y}^{(3)}_{2m}(u)^{-1})(1+\boldsymbol{Y}^{(3)}_{2m+2}(u)^{-1})},$$

$$\boldsymbol{Y}^{(4)}_m(u-\frac{1}{2})\boldsymbol{Y}^{(4)}_m(u+\frac{1}{2})=\frac{1+\boldsymbol{Y}^{(3)}_m(u)}{(1+\boldsymbol{Y}^{(4)}_{m-1}(u)^{-1})(1+\boldsymbol{Y}^{(4)}_{m+1}(u)^{-1})}.$$

G_2

$$
\begin{aligned}
Y^{(1)}_m(u-1)Y^{(1)}_m(u+1) = {} & (1+Y^{(2)}_{3m}(u-\tfrac{2}{3}))(1+Y^{(2)}_{3m}(u))(1+Y^{(2)}_{3m}(u+\tfrac{2}{3})) \\
& \times(1+Y^{(2)}_{3m-1}(u-\tfrac{1}{3}))(1+Y^{(2)}_{3m-1}(u+\tfrac{1}{3})) \\
& \times(1+Y^{(2)}_{3m+1}(u-\tfrac{1}{3}))(1+Y^{(2)}_{3m+1}(u+\tfrac{1}{3})) \\
& \times(1+Y^{(2)}_{3m-2}(u))(1+Y^{(2)}_{3m+2}(u)) \\
& \times\left((1+Y^{(1)}_{m-1}(u)^{-1})(1+Y^{(1)}_{m+1}(u)^{-1})\right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
Y^{(2)}_{3m}(u-\tfrac{1}{3})Y^{(2)}_{3m}(u+\tfrac{1}{3}) &= \frac{1+Y^{(1)}_m(u)}{(1+Y^{(2)}_{3m-1}(u)^{-1})(1+Y^{(2)}_{3m+1}(u)^{-1})}, \\
Y^{(2)}_{3m+1}(u-\tfrac{1}{3})Y^{(2)}_{3m+1}(u+\tfrac{1}{3}) &= \frac{1}{(1+Y^{(2)}_{3m}(u)^{-1})(1+Y^{(2)}_{3m+2}(u)^{-1})}, \\
Y^{(2)}_{3m+2}(u-\tfrac{1}{3})Y^{(2)}_{3m+2}(u+\tfrac{1}{3}) &= \frac{1}{(1+Y^{(2)}_{3m+1}(u)^{-1})(1+Y^{(2)}_{3m+3}(u)^{-1})}.
\end{aligned}
$$

Y-system is an algebraic form of thermodynamic Bethe ansatz equation of type $\mathfrak{g}$ under string hypothesis.

$Y_m^{(a)}(u)$ ∼**Boltzmann factor of string/hole excitation with color a, length m, rapidity u.**

A_1 **example:** $(Y_m(u) = Y_m^{(1)}(u)^{-1})$

$$\log Y_m(u) = \text{known fcn.} + \int_{-\infty}^{\infty} \frac{\log(1+Y_{m-1}(v))(1+Y_{m+1}(v))}{4\cosh\frac{\pi(u-v)}{2}} dv$$

$$\rightsquigarrow Y_m(u-i)Y_m(u+i) = (1+Y_{m-1}(u))(1+Y_{m+1}(u)).$$

Y-system was proposed by

ADE: Al. Zamolodchikov (1991), Ravanini-Tateo-Valleriani (1993).

$\forall\mathfrak{g}$: K-Nakanishi (1992).

Relation of T and Y-systems

$\boldsymbol{A_1}$ **example**

$$\begin{aligned} Y_m(u-1)Y_m(u+1) &= (1+Y_{m-1}(u))(1+Y_{m+1}(u)), \\ T_m(u-1)T_m(u+1) &= T_{m-1}(u)T_{m+1}(u)+1. \end{aligned}$$

Formally setting $Y_m(u) = T_{m-1}(u)T_{m+1}(u)$,

$$\begin{aligned} Y_m(u-1)Y_m(u+1) &= T_{m-1}(u-1)T_{m+1}(u-1)T_{m-1}(u+1)T_{m+1}(u+1) \\ &= T_{m+1}(u-1)T_{m+1}(u+1)T_{m-1}(u-1)T_{m-1}(u+1) \\ &= (T_{m+2}(u)T_m(u)+1)(T_{m-2}(u)T_m(u)+1) \\ &= (Y_{m+1}(u)+1)(Y_{m-1}(u)+1). \end{aligned}$$

Similarly for $\forall \mathfrak{g}$,

T-system solves Y-system.

(Yet to be understood why.)

Restriction and Periodicity conjecture

Introduce $\ell \in \mathbb{Z}_{\geq 2}$ called level.

Level ℓ restricted T and Y-system are those closing among

$$T_m^{(a)}(u) \text{ and } Y_m^{(a)}(u) \text{ with } 1 \leq m \leq t_a\ell - 1,$$

obtained respectively by imposing

$$T_{t_a\ell}^{(a)}(u) = 1 \text{ and } Y_{t_a\ell}^{(a)}(u)^{-1} = 0.$$

C_2 **example.** $(t_1, t_2) = (2, 1)$

$$\begin{aligned} T^{(1)}_{2m+1}(u-\tfrac{1}{2})T^{(1)}_{2m+1}(u+\tfrac{1}{2}) &= T^{(1)}_{2m}(u)T^{(1)}_{2m+2}(u) + T^{(2)}_{m}(u)T^{(2)}_{m+1}(u), \\ T^{(1)}_{2m}(u-\tfrac{1}{2})T^{(1)}_{2m}(u+\tfrac{1}{2}) &= T^{(1)}_{2m-1}(u)T^{(1)}_{2m+1}(u) + T^{(2)}_{m}(u-\tfrac{1}{2})T^{(2)}_{m}(u+\tfrac{1}{2}), \\ T^{(2)}_{m}(u-1)T^{(2)}_{m}(u+1) &= T^{(2)}_{m-1}(u)T^{(2)}_{m+1}(u) + T^{(1)}_{2m}(u). \end{aligned}$$

Level 2 restriction: $T^{(1)}_{4}(u) = T^{(2)}_{2}(u) = 1.$ (b.c. $T^{(a)}_{0}(u) = 1$)

$$\begin{aligned} T^{(1)}_{1}(u-\tfrac{1}{2})T^{(1)}_{1}(u+\tfrac{1}{2}) &= T^{(1)}_{2}(u) + T^{(2)}_{1}(u), \\ T^{(1)}_{2}(u-\tfrac{1}{2})T^{(1)}_{2}(u+\tfrac{1}{2}) &= T^{(1)}_{1}(u)T^{(1)}_{3}(u) + T^{(1)}_{1}(u-\tfrac{1}{2})T^{(1)}_{1}(u+\tfrac{1}{2}), \\ T^{(1)}_{3}(u-\tfrac{1}{2})T^{(1)}_{3}(u+\tfrac{1}{2}) &= T^{(1)}_{2}(u) + T^{(2)}_{1}(u), \\ T^{(2)}_{1}(u-1)T^{(2)}_{1}(u+1) &= 1 + T^{(1)}_{2}(u), \end{aligned}$$

which closes among $T^{(1)}_{1}(u), T^{(1)}_{2}(u), T^{(1)}_{3}(u), T^{(2)}_{1}(u)$.

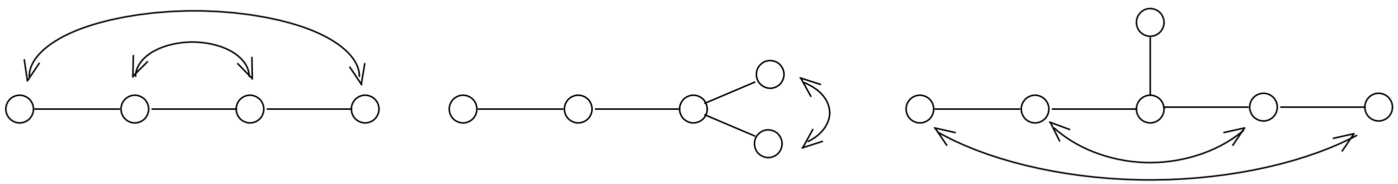
Restricted T and Y-systems $\cdots$ evolution eqs. in the u direction.

Periodicity conjecture

Level ℓ restricted T-system and Y-system obey

$$T_m^{(a)}(u+h^\vee+\ell) = T_{t_a\ell-m}^{(\omega(a))}(u) \;\text{ and }\; Y_m^{(a)}(u+h^\vee+\ell) = Y_{t_a\ell-m}^{(\omega(a))}(u).$$

ω is an involution whose only non-trivial cases are



A_n $\quad$ D_n (n:odd) $\quad$ E_6

$h^\vee$ = dual Coxeter number

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$h^\vee$	$n+1$	$2n-1$	$n+1$	$2n-2$	12	18	30	9	4

(Full periodicity: $T_m^{(a)}(u+2(h^\vee+\ell)) = T_m^{(a)}(u)$ and same for $Y_m^{(a)}(u)$.)

Example: $(A_2, 2)$

Write $T^{(a)}(u) = T_1^{(a)}(u)$.

$$T^{(1)}(u-1)T^{(1)}(u+1) = 1 + T^{(2)}(u),$$
$$T^{(2)}(u-1)T^{(2)}(u+1) = 1 + T^{(1)}(u).$$

Periodicity reads

$$T^{(1)}(u+5) = T^{(2)}(u), \quad T^{(2)}(u+5) = T^{(1)}(u).$$

$T^{(1)}(0) = a$

$T^{(2)}(1) = b$

$$T^{(1)}(2) = \frac{1+T^{(2)}(1)}{T^{(1)}(0)} = \frac{1+b}{a}$$

$$T^{(2)}(3) = \frac{1+T^{(1)}(2)}{T^{(2)}(1)} = \frac{1+\frac{1+b}{a}}{b} = \frac{1+a+b}{ab}$$

$$T^{(1)}(4) = \frac{1+T^{(2)}(3)}{T^{(1)}(2)} = \frac{1+\frac{1+a+b}{ab}}{\frac{1+b}{a}} = \frac{1+a}{b}$$

$$T^{(2)}(5) = \frac{1+T^{(1)}(4)}{T^{(2)}(3)} = \frac{1+\frac{1+a}{b}}{\frac{1+a+b}{ab}} = a = T^{(1)}(0)$$

$$T^{(1)}(6) = \frac{1+T^{(2)}(5)}{T^{(1)}(4)} = \frac{1+a}{\frac{1+a}{b}} = b = T^{(2)}(1)$$

$0\{10, 30, 50, 70\}$ $\qquad (E_8, 2): \quad \{T_1^{(1)}(u), T_1^{(3)}(u), T_1^{(5)}(u), T_1^{(7)}(u)\}_{u=0}^{32}$

$$2\left\{\frac{11}{5}, \frac{431}{15}, \frac{101291}{25}, \frac{31}{35}\right\}$$

$$4\left\{\frac{83}{45}, \frac{69696833}{230625}, \frac{45718438593497}{22157296875}, \frac{103041}{1525}\right\}$$

$$6\left\{\frac{102041}{1025}, \frac{8821291833971}{66471890625}, \frac{36034246310779729463 9}{3463462467773437 5}, \frac{14562107}{415125}\right\}$$

$$8\left\{\frac{1061807}{1441125}, \frac{527621002287915653}{153931665234375}, \frac{144652414821069001465529527}{6161870815433349609375}, \frac{2176297573}{492384375}\right\}$$

$$10\left\{\frac{15241182}{312625}, \frac{1741858802351675418 4}{1335906951855468 75}, \frac{65852952390687824418240896525206}{1926354863674850921630859375}, \frac{32206227374}{211021875}\right\}$$

$$12\left\{\frac{23381761}{6226875}, \frac{443940578926110770904 1}{9128697504345703125}, \frac{255396681651083275452908699280166448}{8813073501312442966461181640625}, \frac{6587423634821}{129778453125}\right\}$$

$$14\left\{\frac{289412993}{98476875}, \frac{240117200327845738829501 9}{2875539713868896484375}, \frac{113421595121251725116844505024021577713}{542004020330715242437362670898437 5}, \frac{8472179120234}{2252658515625}\right\}$$

$$16\left\{\frac{391949128}{4689375}, \frac{739726316179777413222704 9}{5846930751533422851562 5}, \frac{1290705517162033306270461619591091257193}{569104221347251004559230804443359375}, \frac{14335608965944}{129778453125}\right\}$$

$$18\left\{\frac{66998956}{126613125}, \frac{210714979567782348600241}{92808424627514648437 5}, \frac{1724707384403205750584318844948339136 63}{113820844269450200911846160888671875}, \frac{74693044181731}{1362673757812 5}\right\}$$

$$20\left\{\frac{11232037}{1563125}, \frac{157625994240195774364 7}{2464748326173339843 75}, \frac{2104768617341673326572332456823959011}{25293520948766711313743591308593 75}, \frac{1211696207719}{450531703125}\right\}$$

$$22\left\{\frac{3077201}{1245375}, \frac{1154015828669881829 27}{426005883536132812 5}, \frac{23354104411061828987973549647671}{2467660580367484030609130859375}, \frac{175786811543}{2883965625}\right\}$$

$$24\left\{\frac{4476646}{2188375}, \frac{47183886310350193}{12468464883984375}, \frac{35939455246726991953433003}{1712315434377645263671875}, \frac{991662341}{13294378125}\right\}$$

$$26\left\{\frac{7058}{4575}, \frac{1842216632119}{879609515625}, \frac{1312893318319321061 59}{115021588554755859375}, \frac{7222892}{312625}\right\}$$

$$28\left\{\frac{1181}{615}, \frac{61893029}{42204375}, \frac{156275914764469}{18471799828125}, \frac{46522}{27675}\right\}$$

$$30\left\{\frac{23}{15}, \frac{32333}{1845}, \frac{4966808}{187575}, \frac{1781}{2135}\right\}$$

$32\{10, 30, 50, 70\}$

Periodicity of Y-system for $(\mathfrak{g}, \ell)$ was proposed:

Al. Zamolodchikov (1991) (ADE, 2),

Ravanini-Tateo-Valleriani (1993) (ADE, ℓ),

K-Nakanishi-Suzuki (1994) $(\mathfrak{g}, \ell)$.

It has been proved:

$(A_n, 2)$: Frenkel-Szenes (1995), Gliozzi-Tateo (1996),

(A_n, ℓ): Volkov, Henriques (2007),

$(\mathrm{ADE}, 2)$: Fomin-Zelevinsky (2003) Cluster algebra,

(ADE, ℓ): Keller (arXiv:0807.1960) Cluster algebra/category.

⋆ Periodicity of T-system:

Proposed in Inoue-Iyama-K-Nakanishi-Suzuki (arXiv:0812.0667), where (ACDE, ℓ) case was proved.

(A_n, ℓ) case: proof also contained in Henriques (2007).

∘ Periodicities of T and Y-systems do not follow from each other in general.

Origin of the periodicity conjecture

Phys: Level ℓ restricted solid-on-solid model

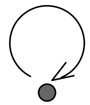
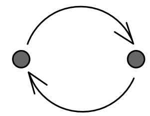
[T-sys] Transfer mat. $T^{(a)}_m(u)$ is $2(h^\vee + \ell)$-periodic by construction.

[Y-sys] String hypothesis works by assuming $m \le t_a \ell$.

Math: $2(h^\vee + \ell)$-periodicity of q-characters holds in the quotient

$$\frac{\text{Ring of } q\text{-characters}}{\text{Ideal including } T^{(1)}_\ell(u) - 1} \quad \text{for type A.}$$

Quivers and Cluster algebra formulation

Q: quiver (finite oriented graph without loop  and 2-cycle )

$I = \{1, \ldots, N\}$: vertex set, $x = (x_1, \ldots, x_N)$: I-tuple of variables

x_i: cluster variable, (Q, x): seed.

Cluster algebra $\mathcal{A}_Q$ is defined by (i)–(iv). Fomin-Zelevinsky (2002)

(i) Start from the (initial) seed (Q, x) as above.

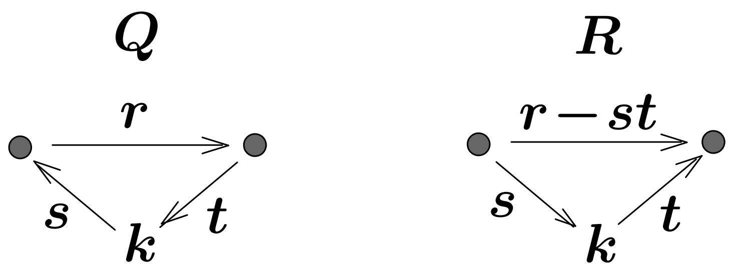
(ii) For each $k \in I$, define another seed (R, y)
by $(R, y) = \mu_k(Q, x)$ ('mutation' at k, def. next page).

(iii) Iterate mutations for every new seed at every k,
and collect all (possibly infinite) seeds.

(iv) $\mathcal{A}_Q = \mathbb{Z}$-subalgebra of $\mathbb{Q}(x_1, \ldots, x_N)$ gen. by $\forall$cluster variables.

Mutation at k: $\quad \mu_k(Q, x) = (R, y)$

A new quiver R is obtained from Q by reversing $\forall$arrows incident with k and



$(s, t \geq 0, \quad r \gtreqless 0)$

$\xrightarrow{r}$ means an $|r|$-fold arrows $\begin{cases} \longrightarrow & (r \geq 0) \\ \longleftarrow & (r < 0) \end{cases}$

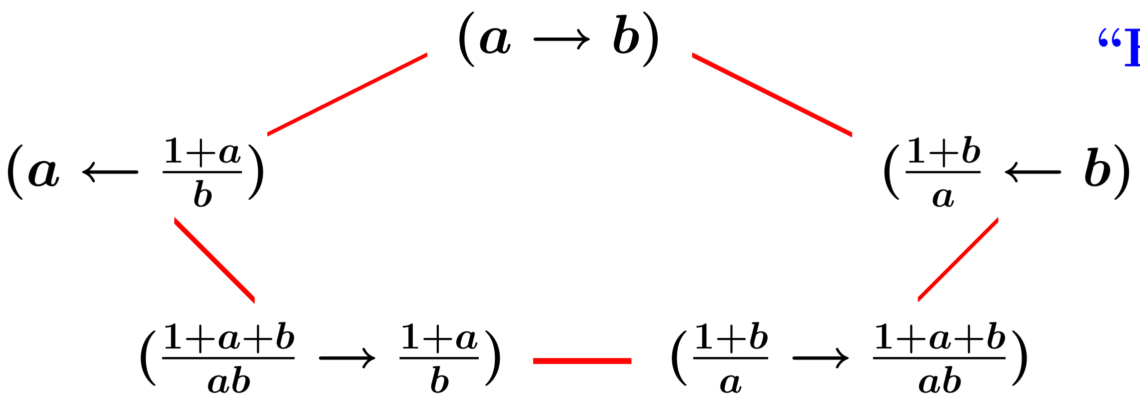
New cluster variables $y = (y_1, \ldots, y_n)$ are given by

$$y_i = \begin{cases} x_i & i \neq k, \\ \dfrac{1}{x_k}\left(\displaystyle\prod_{\text{arrows } j \to k \text{ of } Q} x_j + \prod_{\text{arrows } k \to j \text{ of } Q} x_j \right) & i = k, \end{cases}$$

$\mu_k^2 = \mathrm{id}, \quad \mu_j \mu_k = \mu_k \mu_j$ for j, k not connected by an arrow.

Example. $I = \{1, 2\}$. Initial seed $(Q, x) = (1 \to 2, \{a, b\})$.

Seeds denoted by $(a \to b)$, and mutation μ_1, μ_2 by ——.



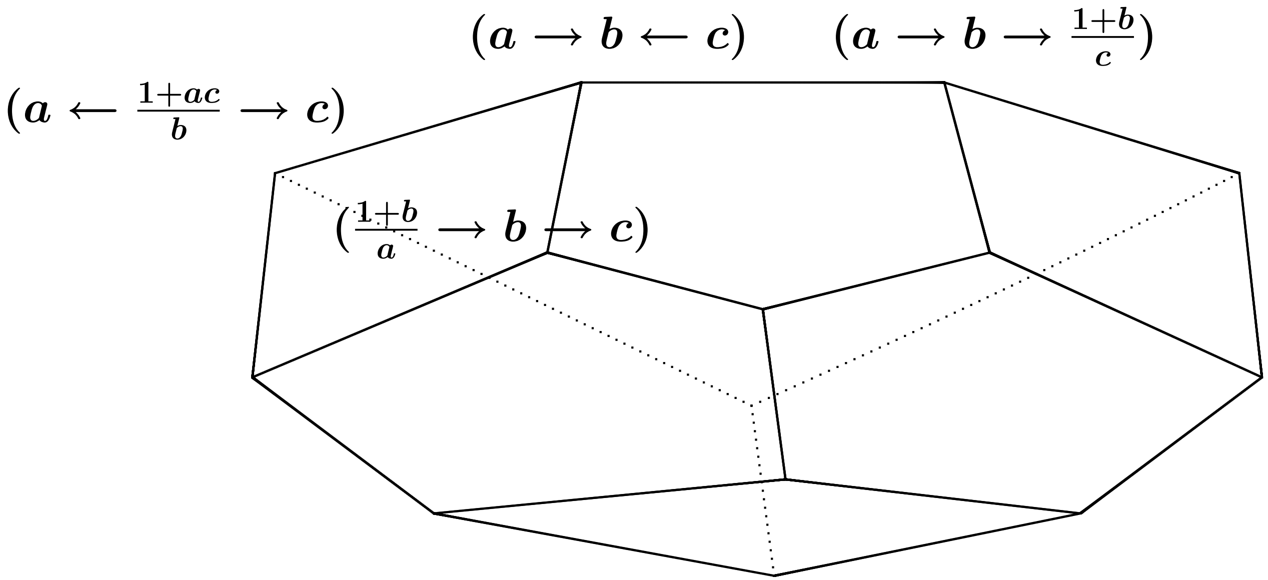
"Exchange graph"

Fomin-Zelevinsky theorem (2003) $\begin{cases} \text{(1) Laurent phenomenon,} \\ \text{(2) Finite type classification.} \end{cases}$

(1) $\forall$cluster variables are Laurent polynomials.

(2) $\sharp\{\text{cluster var.}\} < \infty \Leftrightarrow Q \overset{\text{mutations}}{\sim}$ orientation on ADE Dynkin diag.

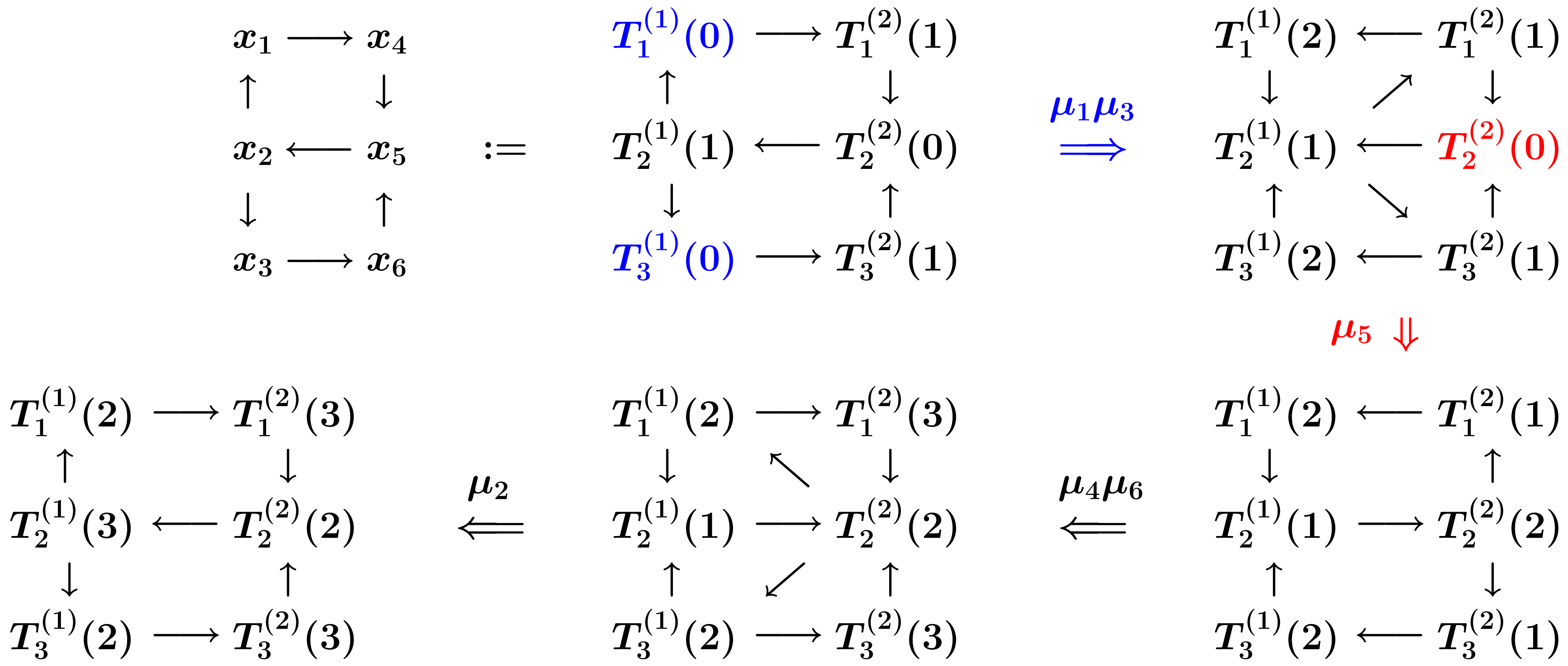
Example with initial seed $(a \to b \leftarrow c)$ closes among 14 seeds.



Exchange graph is Stasheff associahedron.

Cluster algebra formulation of T-system. $(A_2, 4)$ example

$$T_1^{(1)}(u-1)T_1^{(1)}(u+1) = T_2^{(1)}(u) + T_1^{(2)}(u),$$
$$T_2^{(2)}(u-1)T_2^{(2)}(u+1) = T_1^{(2)}(u)T_3^{(2)}(u) + T_2^{(1)}(u),$$
$$T_3^{(1)}(u-1)T_3^{(1)}(u+1) = T_2^{(1)}(u) + T_3^{(2)}(u).$$



$$(Q, x(u+2)) = \mu_2\mu_4\mu_6\mu_5\mu_3\mu_1(Q, x(u)) \quad \text{for} \quad x = \{T_m^{(a)}\}.$$

Similarly, T-system for any $(\mathfrak{g}, \ell)$ is formulated as

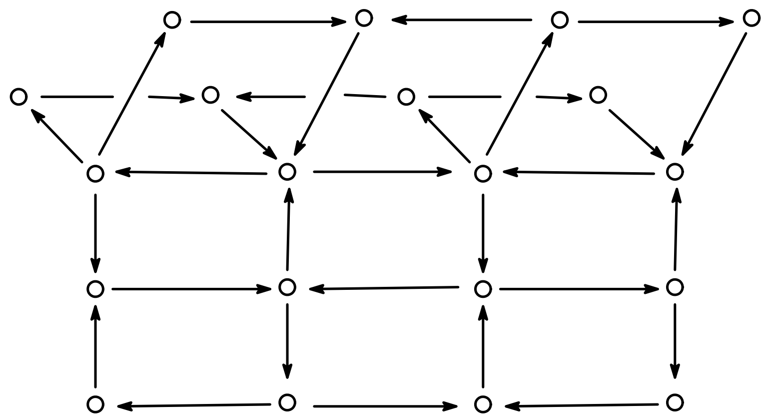
$(Q, x(u+2)) = \mu(Q, x(u))$ by an appropriate choice of

$$\begin{cases} Q & : \text{quiver}, \\ x(u) = \{x_i(u)\} & : \text{cluster variables suitably identified with } T_m^{(a)}(u)\text{'s}, \\ \mu = \mu_{i_1} \cdots \mu_{i_s} & : \text{composite mutation.} \end{cases}$$

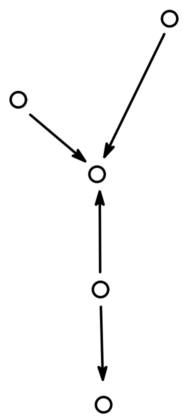
For $\mathfrak{g} = \text{ADE}$, $Q = Q_{\mathfrak{g}} \square Q_{A_{\ell-1}}$: (Keller's square product),
$Q_{\mathfrak{g}}, Q_{A_{\ell-1}}$ = Dynkin quivers with alternating source/sink.

Example $(\mathfrak{g}, \ell) = (D_5, 5)$

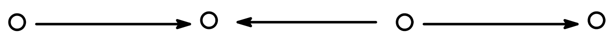
$Q_{D_5} \square Q_{A_4}$



Q_{D_5}



Q_{A_4}



(Full) periodicity for any $(\mathfrak{g}, \ell)$ is formulated as

$$(Q, x(u)) = \mu^{h^\vee + \ell}(Q, x(u)).$$

Theorem (Inoue-Iyama-K-Nakanishi-Suzuki 2008)
Periodicity conjecture of T-system is true for $\mathfrak{g}$ = ACDE, $\forall \ell$.

(Full periodicity for ADE is also proved by Keller.)

Our proof is by

C : direct method using determinant expressions,
ADE : Cluster category.

Sketch of the proof for $(\mathfrak{g}, \ell)$ = (ADE, 2).

$$Q = Q_{\mathfrak{g}} \square Q_{A_1} = Q_{\mathfrak{g}} = \text{alternating Dynkin quiver},$$

$$\mu = \prod_{i:\text{sink}} \mu_i \prod_{j:\text{source}} \mu_j, \quad h^\vee = h \text{ (Coxeter number)}.$$

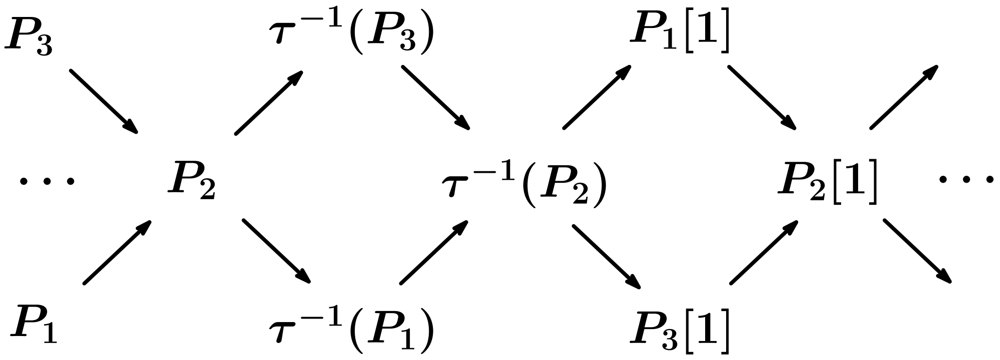
To show $\mu^{h+2}(Q, x) = (Q, x)$.

KQ: Path algebra gen. by ∀paths on Q. Product = composition.

$\mathcal{D}_Q$ = (bounded) derived category of finite dim. KQ-modules.

A_3 example: $Q = 1 \longleftarrow 2 \longrightarrow 3$

Auslander-Reiten (AR) quiver of $\mathcal{D}_Q$



P_i: projective KQ-module

τ^{-1}: AR translation (right shift)

[1]: suspention functor
(upside down· rightshift^2)

Cluster category (Buan-Marsh-Reineke-Reiten-Todorov 2006)

$$\mathcal{C}_Q := \mathcal{D}_Q/(\tau^{-1} \circ [1]).$$

In $\mathcal{C}_Q$, $\tau^{-h-2} \simeq \mathrm{id}$. $\cdots (\star)$ "categorical periodicity"

Cluster tilting object

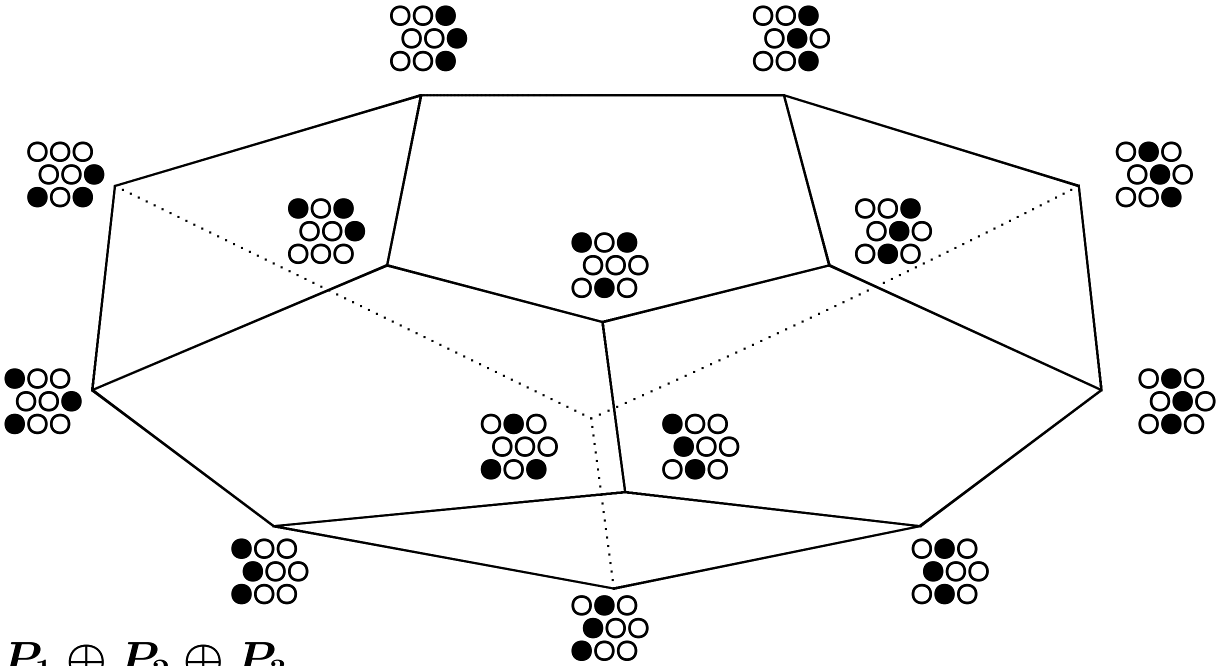
$\bigoplus_{a=1}^{\mathrm{rank}\,\mathfrak{g}}$ (indecomposables) satisfying certain conditions.

In the present example, rank $\mathfrak{g} = 3$, and there are 14 cluster tilting objects like

$$P_1 \oplus P_2 \oplus P_3 \quad \begin{matrix} \xrightarrow{\tilde{\mu}_1} & \tau^{-1}(P_1) \oplus P_2 \oplus P_3 \\ \xrightarrow{\tilde{\mu}_2} & P_1 \oplus P_2[1] \oplus P_3 \\ \xrightarrow{\tilde{\mu}_3} & P_1 \oplus P_2 \oplus \tau^{-1}(P_3) \end{matrix}$$

$\tilde{\mu}_a$: cluster tilting mutation

Cluster tilting objects are connected by cluster tilting mutations.



$P_1 \oplus P_2 \oplus P_3$

$P_1 \oplus P_2 \oplus \tau^{-1}(P_3)$

Theorem (Buan-Marsh-Reineke-Reiten-Todorov 2006)

$\exists$ bijection $\tilde{X}$ s.t. the following diagram is commutative:

$$\begin{array}{ccc} \text{Cluster algebra} & & \text{Cluster category} \\ \{\text{seeds in } \mathcal{A}_Q\} & \overset{\tilde{X}}{\longleftarrow} & \{\text{cluster tilting objects in } \mathcal{C}_Q\} \\ \text{mutation } \mu_a \big\downarrow & & \big\downarrow \tilde{\mu}_a \text{ cluster tilting mutation} \\ \{\text{seeds in } \mathcal{A}_Q\} & \overset{\tilde{X}}{\longleftarrow} & \{\text{cluster tilting objects in } \mathcal{C}_Q\} \end{array}$$

This is an example of "Categorification of cluster algebra".

Proof of full periodicity.

Take a cluster tilting object $P = \oplus_{a=1}^{\mathrm{rank}\,\mathfrak{g}} P_a$ and set $(Q, x) = \tilde{X}_P$.

$$\mu^{h+2}(Q,x) = \mu^{h+2}(\tilde{X}_P) \overset{\text{Th}}{=} \tilde{X}_{\tilde{\mu}^{h+2}(P)} \overset{\text{easy}}{=} \tilde{X}_{\tau^{-h-2}(P)} \overset{(\star)}{=} \tilde{X}_P = (Q,x). \quad \square$$

Higher level case: Idea is parallel although technically more involved.

Outlook

- **Y-system can be incorporated in Cluster algebra with coefficients introduced by Fomin-Zelevinsky (2007).**

- **$\exists$ T,Y-systems, periodicity conjectures for $\forall$ twisted affine Lie alg.**

- **Q-system (a degeneration of T-system) and its application are also discussed by Kedem-Di Francesco (2008).**

- **Besides periodicity, T,Y-systems have various aspects related to;**

  Fermionic character formula and Dilogarithm identities in CFT,

  Crystal base of quantum affine algebras,

  Integrable cellular automata and Ultradiscrete τ-functions, etc. (talk at Glasgow)