Tetrahedron equation and quantum R matrices for q-oscillator representations

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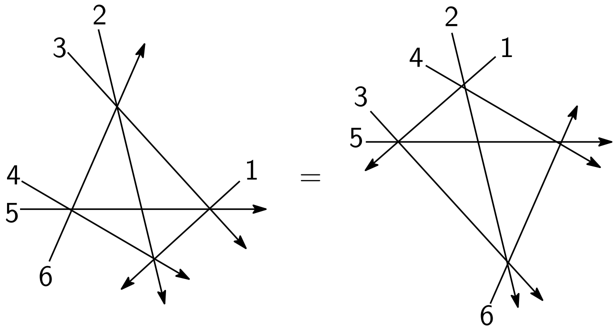
Joint work with Masato Okado

Tetrahedron equation

$$R : F \otimes F \otimes F \to F \otimes F \otimes F \qquad \text{(3D } R\text{)}$$

$$R_{1,2,4}R_{1,3,5}R_{2,3,6}R_{4,5,6} = R_{4,5,6}R_{2,3,6}R_{1,3,5}R_{1,2,4} \in \mathrm{End}(F^{\otimes 6})$$

(A.B. Zamolodchikov 1980)



3D R: q-oscillator solutions

$F = \oplus_{m \geq 0} \mathbb{C}|m\rangle$ Fock space   (q: generic).

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c \geq 0} R^{a,b,c}_{i,j,k} |a\rangle \otimes |b\rangle \otimes |c\rangle,$$

$$R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j} \delta^{b+c}_{j+k} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}.$$

$$\delta^a_j = \begin{cases} 1 & a = j \\ 0 & a \neq j, \end{cases} \qquad (q)_m = \prod_{j=1}^m (1 - q^j), \qquad \binom{m}{j}_q = \frac{(q)_m}{(q)_j (q)_{m-j}}.$$

Remark. $R^{a,b,c}_{i,j,k}$ is a polynomial in q. For example,

$$R^{041}_{314} = -q^2(1-q^4)(1-q^6)(1-q^8),$$
$$R^{132}_{314} = (1-q^6)(1-q^8)(1-q^4-q^6-q^8-q^{10}),$$
$$R^{223}_{314} = q^2(1+q^2)(1+q^4)(1-q^6)(1-q^6-q^{10}), \text{ etc.}$$

Origin of 3D R

(1) Kapranov-Voevodsky (1994):

Intertwiner of quantized coordinate ring $A_q(sl_3)$

(2) Bazhanov-Sergeev (2006): Quantization of Miquel's theorem (1838)

(3) Sergeev (2008): (2)=Transition coefficients of PBW bases of $U_q^+(sl_3)$

(4) K-Okado (2012): (1)=(2) $\subset$ solution of 3d reflection equation

Remark: 3D R has Combinatorial and Birational counterparts

$$\text{3D } R \ \overset{q\to 0}{\longrightarrow}\ \text{Combinatorial 3D } R \ \overset{\text{ultradiscretization}}{\longleftarrow}\ \text{Birational 3D } R$$

Reduction to 2D

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$

$\Downarrow$ 2d reduction (eliminate spaces 4,5,6)

$$S_{12}(x)S_{13}(xy)S_{23}(y) = S_{23}(y)S_{13}(xy)S_{12}(x) \quad \cdots \text{Yang-Baxter equation}$$

Prescription : $\langle\chi_s(x,y)|R_{124}R_{135}R_{236}R_{456}|\chi_t(1,1)\rangle$
$= \langle\chi_s(x,y)|R_{456}R_{236}R_{135}R_{124}|\chi_t(1,1)\rangle$

by the boundary vectors

$$\langle\chi_s(x,y)| = \langle\chi_s(x)| \otimes \langle\chi_s(xy)| \otimes \langle\chi_s(y)| \in \overset{4}{F^*} \otimes \overset{5}{F^*} \otimes \overset{6}{F^*},$$

$$|\chi_t(x,y)\rangle = |\chi_t(x)\rangle \otimes |\chi_t(xy)\rangle \otimes |\chi_t(y)\rangle \in \overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F}$$

satisfying $\langle\chi_s(x,y)|R_{456} = \langle\chi_s(x,y)|, \quad R_{456}|\chi_t(x,y)\rangle = |\chi_t(x,y)\rangle.$

Then $S_{12}(x) = \langle\chi_s(x)|R_{123}|\chi_t(1)\rangle \in \mathrm{End}(F \otimes F)$, etc.

Boundary vectors

There are 2 such boundary vectors [K-Sergeev (2013)]:

$$\langle\chi_1(z)| = \sum_{m\geq 0} \frac{z^m}{(q)_m}\langle m| \quad \langle\chi_2(z)| = \sum_{m\geq 0} \frac{z^m}{(q^4)_m}\langle 2m|,$$

$$|\chi_1(z)\rangle = \sum_{m\geq 0} \frac{z^m}{(q)_m}|m\rangle, \quad |\chi_2(z)\rangle = \sum_{m\geq 0} \frac{z^m}{(q^4)_m}|2m\rangle.$$

So far: 1-layer version of reduction

Possible to extend it to n-layer version

n-layer version of the tetrahedron equation

$$\prod^{\rightarrow}_{1\le i\le n} (R_{1_i 2_i 4} R_{1_i 3_i 5} R_{2_i 3_i 6})\, R_{456} = R_{456} \prod^{\rightarrow}_{1\le i\le n} (R_{2_i 3_i 6} R_{1_i 3_i 5} R_{1_i 2_i 4})$$

$1_1, \ldots, 1_n, 2_1, \ldots, 2_n, 3_1, \ldots, 3_n, 4, 5, 6$: copies of the Fock space F

The same reduction $\langle \chi_s(x,y)|(\cdots)|\chi_t(1,1)\rangle$ works.

$\Longrightarrow$ Solution of the Yang-Baxter equation constructed as

$$S^{s,t}(z) = \langle \chi_s(z)| R_{1_1 2_1 3} R_{1_2 2_2 3} \cdots R_{1_n 2_n 3} |\chi_t(1)\rangle \in \mathrm{End}(F^{\otimes n} \otimes F^{\otimes n}).$$

(The evaluation is done in the space 3.)

Matrix elements of $S^{s,t}(z)$ $(s, t = 1, 2)$

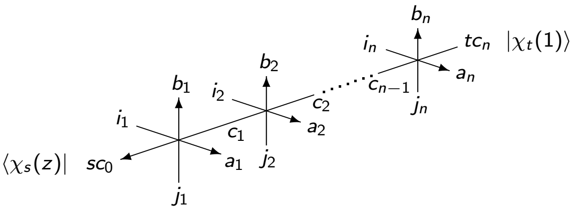
Notations:

$$|\mathbf{a}\rangle = |a_1\rangle \otimes \cdots \otimes |a_n\rangle \in F^{\otimes n} \quad \text{for} \quad \mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$$

$$\mathbf{e}_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0), \quad \mathbf{0} = (0, 0, \ldots, 0)$$

$$S^{s,t}(z)\big(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle\big) = \sum_{\mathbf{a},\mathbf{b}} S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle,$$

$$S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \sum_{c_0,\ldots,c_n \geq 0} \frac{z^{c_0}(q^2)_{sc_0}}{(q^{s^2})_{c_0}(q^{t^2})_{c_n}} R^{a_1,b_1,sc_0}_{i_1,j_1,c_1} R^{a_2,b_2,c_1}_{i_2,j_2,c_2} \cdots R^{a_n,b_n,c_{n-1}}_{i_n,j_n,tc_n}$$



Examples

Substitute the matrix elements of 3D R

$$R^{a,b,c}_{i,0,k} = q^{ac}\frac{(q^2)_i(q^2)_k}{(q^2)_a(q^2)_b(q^2)_c}\delta^{a+b}_i\delta^{b+c}_k,\ R^{0,b,c}_{i,j,k} = (-1)^j q^{j(c+1)}\frac{(q^2)_k}{(q^2)_c}\delta^b_{i+j}\delta^{b+c}_{j+k}.$$

Up to an overall factor, the following formulas are valid ($t = 1, 2$):

$$S^{1,t}(z)^{\mathbf{a},\mathbf{0}}_{\mathbf{a},\mathbf{0}} = (-q)^{-|\mathbf{a}|}S^{1,t}(z)^{\mathbf{0},\mathbf{a}}_{\mathbf{0},\mathbf{a}} = \frac{(z^t;q^t)_{|\mathbf{a}|}}{(-z^t q;q^t)_{|\mathbf{a}|}} \quad (|\mathbf{a}| = a_1 + \cdots + a_n),$$

$$S^{1,1}(z)^{2\mathbf{e}_1,\mathbf{0}}_{\mathbf{e}_1,\mathbf{e}_1} = (-q)^{-1}S^{1,1}(z)^{\mathbf{0},2\mathbf{e}_1}_{\mathbf{e}_1,\mathbf{e}_1} = \frac{(1+q)(1-z)}{(1+zq)(1+zq^2)},$$

where $(z;q)_m = \prod_{j=1}^m(1 - zq^{j-1})$.

Proposition (summary so far)

$S(z) = S^{s,t}(z) \in \mathrm{End}(F^{\otimes n} \otimes F^{\otimes n})$ satisfies the Yang-Baxter equation

$$S_{12}(x)S_{13}(xy)S_{23}(y) = S_{23}(y)S_{13}(xy)S_{12}(x)$$

Problem:

Find a characterization of $S^{1,1}(z), S^{1,2}(z), S^{2,2}(z)$ in the framework of the quantum group theory. ($S^{2,1}(z)$ is simply related to $S^{1,2}(z)$.)

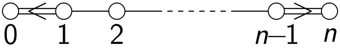
Result

They are quantum R-matrices intertwining the q-oscillator representations of $U_q(D^{(2)}_{n+1}), U_q(A^{(2)}_{2n}), U_q(C^{(1)}_n)$.

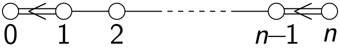
Affine Lie algebras $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, $C_n^{(1)}$

Dynkin diagrams

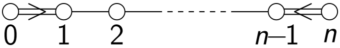
$D_{n+1}^{(2)}$



$A_{2n}^{(2)}$



$C_n^{(1)}$



q-oscillator representations

$V_x := F^{\otimes n}[x, x^{-1}]$ (x : spectral parameter).

Let $\langle e_j, f_j, k_j^{\pm 1} \rangle_{0 \le j \le n}$ act on V_x by ($[m] = (q^m - q^{-m})/(q - q^{-1})$)

$$
\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m} + \mathbf{e}_1\rangle \\
f_0|\mathbf{m}\rangle &= \sqrt{-1}\kappa[m_1]x^{-1}|\mathbf{m} - \mathbf{e}_1\rangle \qquad \kappa = (q+1)/(q-1) \\
k_0|\mathbf{m}\rangle &= -\sqrt{-1}q^{m_1+\frac{1}{2}}|\mathbf{m}\rangle \\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m} - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \qquad (0 < j < n) \\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m} + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle \qquad (0 < j < n) \\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle \qquad (0 < j < n) \\
e_n|\mathbf{m}\rangle &= \sqrt{-1}\kappa[m_n]|\mathbf{m} - \mathbf{e}_n\rangle \\
f_n|\mathbf{m}\rangle &= |\mathbf{m} + \mathbf{e}_n\rangle \\
k_n|\mathbf{m}\rangle &= \sqrt{-1}q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle.
\end{aligned}
$$

$\mathbf{e}_j = (0, \ldots, \overset{j}{1}, \ldots, 0)$, $\mathbf{m} = \sum_{j=1}^n m_j \mathbf{e}_j \in \mathbb{Z}^n$, $|\mathbf{m}\rangle = |m_1\rangle \otimes \cdots \otimes |m_n\rangle \in F^{\otimes n}$

Proposition

V_x is an irreducible representation (q-oscillator representation) of the Drinfeld-Jimbo quantum affine algebra $U_q(D^{(2)}_{n+1}) = \langle e_j, f_j, k_j^{\pm 1}\rangle_{0\le j\le n}$.

- $U_q(A^{(2)}_{2n})$ and $U_q(C^{(1)}_n)$ also have similar q-oscillator representations. ($U_q(D^{(2)}_2)$ and $U_q(C^{(1)}_1)$ are regarded as $U_q(A^{(1)}_1)$.)
- For $U_q(D^{(2)}_{n+1})$ and $U_q(A^{(2)}_{2n})$, the q-oscillator representation is singular at $q = 1$ due to $\kappa = (q+1)/(q-1)$.
- The q-oscillator representations for $U_q(A^{(1)}_n)$, $U_q(C_n)$ were known by Hayashi (1990).

Quantum R matrix for q-oscillator representation

For simplicity, consider $U_q = U_q(D^{(2)}_{n+1})$ for the time being.

$R(z) \in \mathrm{End}(V_x \otimes V_y) \quad (z = x/y)$ is characterized by

(i) Commutativity: $[PR(z), \Delta(g)] = 0 \quad \forall g \in U_q$

(Δ: coproduct of U_q, $P(u \otimes v) = v \otimes u$)

(ii) Normalization: $R(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = \frac{(-zq;q)_\infty}{(z;q)_\infty} |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$

Introduce a gauge transformed $R(z)$:

$$\tilde{R}(z) := (K^{-1} \otimes 1) R(z) (1 \otimes K)$$

$$K|\mathbf{m}\rangle = (-\sqrt{-1}q^{\frac{1}{2}})^{m_1 + \cdots + m_n} |\mathbf{m}\rangle$$

Properties of the R-matrix

- The both $R(z)$ and $\tilde{R}(z)$ satisfy the Yang-Baxter equation.

- The following spectral decomposition holds:

$$\frac{(z;q)_\infty}{(-zq;q)_\infty} PR(z) = \bigoplus_{k=0}^{\infty} \left(\prod_{j=1}^{k} \frac{z+q^j}{1+zq^j} \right) P_k$$

$P_k =$ projector onto the irreducible component with respect to the classical subalgebra $U_q(B_n) \subset U_q(D^{(2)}_{n+1})$ labeled by k.

- For $U_q(C^{(1)}_n)$, $V_x \otimes V_y$ consists of 4 irreducible components and a slightly finer characterization is necessary.

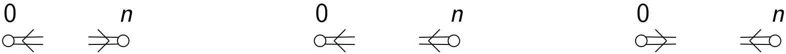
Main result

$\tilde{R}_{\mathfrak{g}}(z) := \tilde{R}(z)$ of the q-oscillator representation of $U_q(\mathfrak{g})$

Theorem (K-Okado, arXiv:1311.4258)

$$S^{1,1}(z) = \tilde{R}_{D^{(2)}_{n+1}}(z), \quad S^{1,2}(z) = \tilde{R}_{A^{(2)}_{2n}}(z), \quad S^{2,2}(z) = \tilde{R}_{C^{(1)}_{n}}(z).$$

Remark: Boundary vector $\Longleftrightarrow$ End shape of the Dynkin diagram of $\mathfrak{g}$



$\langle \chi_1(z)|$ $\quad |\chi_1(1)\rangle$ $\quad \langle \chi_1(z)|$ $\quad |\chi_2(1)\rangle$ $\quad \langle \chi_2(z)|$ $\quad |\chi_2(1)\rangle$

Proof: Can check the commutativity of $S^{s,t}(z)$ with U_q. $\square$

Related results and outlook

- Bazhanov-Sergeev (JPA 2006) (L = 3D L-operator satisfying $RLLL = LLLR$)

  $\mathrm{Tr}(R \cdots R)$, $\mathrm{Tr}(L \cdots L) = \oplus$ (R for type A sym or anti-sym tensor rep.)

- K-Sergeev (CMP 2013)

  $\langle \chi_s(z) | L \cdots L | \chi_t(1) \rangle$ = R-matrix for spin representation

- K-Okado (to appear in CMP, this talk)

  $\langle \chi_s(z) | R \cdots R | \chi_t(1) \rangle$ = R-matrix for q-oscillator representation

- K-Okado (in preparation)

  Mixture of 3D R and L like $\mathrm{Tr}(RLLRL)$, $\langle \chi_s(z) | LRLLR | \chi_t(1) \rangle$, etc.

  Commutant ⇒ examples of generalized quantum group

  (Lusztig, Heckenberger, Batra-Yamane, etc.)