Classical and quantum aspects of ultradiscrete solitons

Atsuo Kuniba (Univ. Tokyo)

2 April 2009, Glasgow

Tau function of KP hierarchy

$$\tau_i(x) = \langle i| e^{H(x)} \exp\Big(\sum_{j=1}^{N} c_j \psi(p_j)\psi^*(q_j)\Big)|i\rangle$$

$(e^{H(x)}$ = time evolution op. involving $\beta_1, \beta_2, \ldots)$

$$\tau_i(x) = \det(1+F)$$

$$= 1 + \sum_{1\le j\le N} F_{jj} + \sum_{1\le j_1<j_2\le N} \begin{vmatrix} F_{j_1 j_1} & F_{j_1 j_2} \\ F_{j_2 j_1} & F_{j_2 j_2} \end{vmatrix} + \cdots ,$$

$$F_{jl} = \frac{c_j q_j}{p_j - q_l} \left(\frac{p_j}{q_j}\right)^{i-1} \prod_m \frac{\beta_m - q_j}{\beta_m - p_j}$$

Ultradiscrete (tropical) limit

$$\lim_{\epsilon \to +0} \epsilon \log \tau_i(x)$$

with an elaborate ϵ-tuning of the parameters

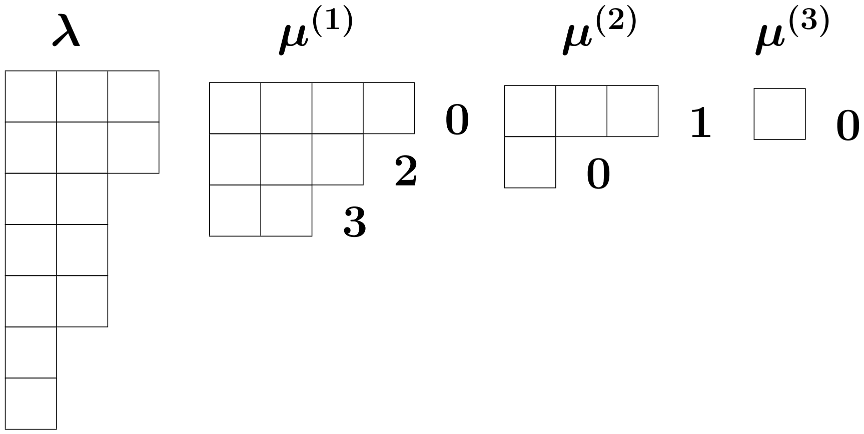
$$c_j, p_j, q_j, \beta_m$$

leads to a tropical tau function associated with combinatorial data called Rigged Configuration in Bethe ansatz.

Solitons in tau function $\leftrightarrow$ Strings in Bethe ansatz

$$c_j \psi(p_j) \psi^*(q_j)$$

Example from $sl_{n=4}$



Rigged configuration

$$(\mu, r) = (\lambda, (\mu^{(1)}, r^{(1)}), \ldots, (\mu^{(n-1)}, r^{(n-1)}))$$

$\mu^{(a)}$: configuration (Young diagram)

$r^{(a)}$: rigging (integers attached to $\mu^{(a)}$)

(+ selection rule)

Charge of rigged configuration

$$c(\mu, r) := \frac{1}{2} \sum_{a,b=1}^{n-1} C_{ab} \min(\mu^{(a)}, \mu^{(b)}) - \min(\lambda, \mu^{(1)}) + \sum_{a=1}^{n-1} |r^{(a)}|$$

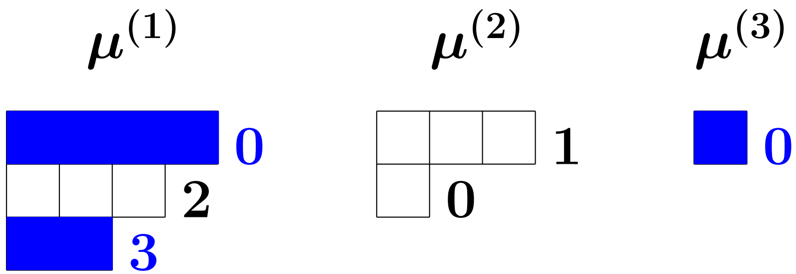
$$\left(\begin{array}{c} \min(\lambda, \mu) = \sum_{ij} \min(\lambda_i, \mu_j), \quad |r| = \sum_i r_i \\ (C_{ab}) = \text{Cartan matrix of } sl_n \end{array} \right)$$

Regard rigged conf. = {strings}, string = row attached with rigging.

Tropical tau function

$$\tau_i(\lambda) = -\min_{(\nu,s)}\{c(\nu,s) + |\nu^{(i)}|\} \quad (1 \le i \le n)$$

$\min_{(\nu,s)}$ extends over the power set of (μ, r). e.g.,



<u>Proposition</u> ([K-Sakamoto-Yamada 2007] "Tropical Hirota eq".)

$$\bar{\tau}_{k,i-1} + \tau_{k-1,i} = \max(\bar{\tau}_{k,i} + \tau_{k-1,i-1}, \bar{\tau}_{k-1,i-1} + \tau_{k,i} - \lambda_k),$$

where

$$\tau_{k,i} = \tau_i(\lambda_1, \ldots, \lambda_k), \quad \bar{\tau}_{k,i} = \tau_{k,i}|_{r^{(a)} \to r^{(a)} + \delta_{a1}\mu^{(1)}}$$

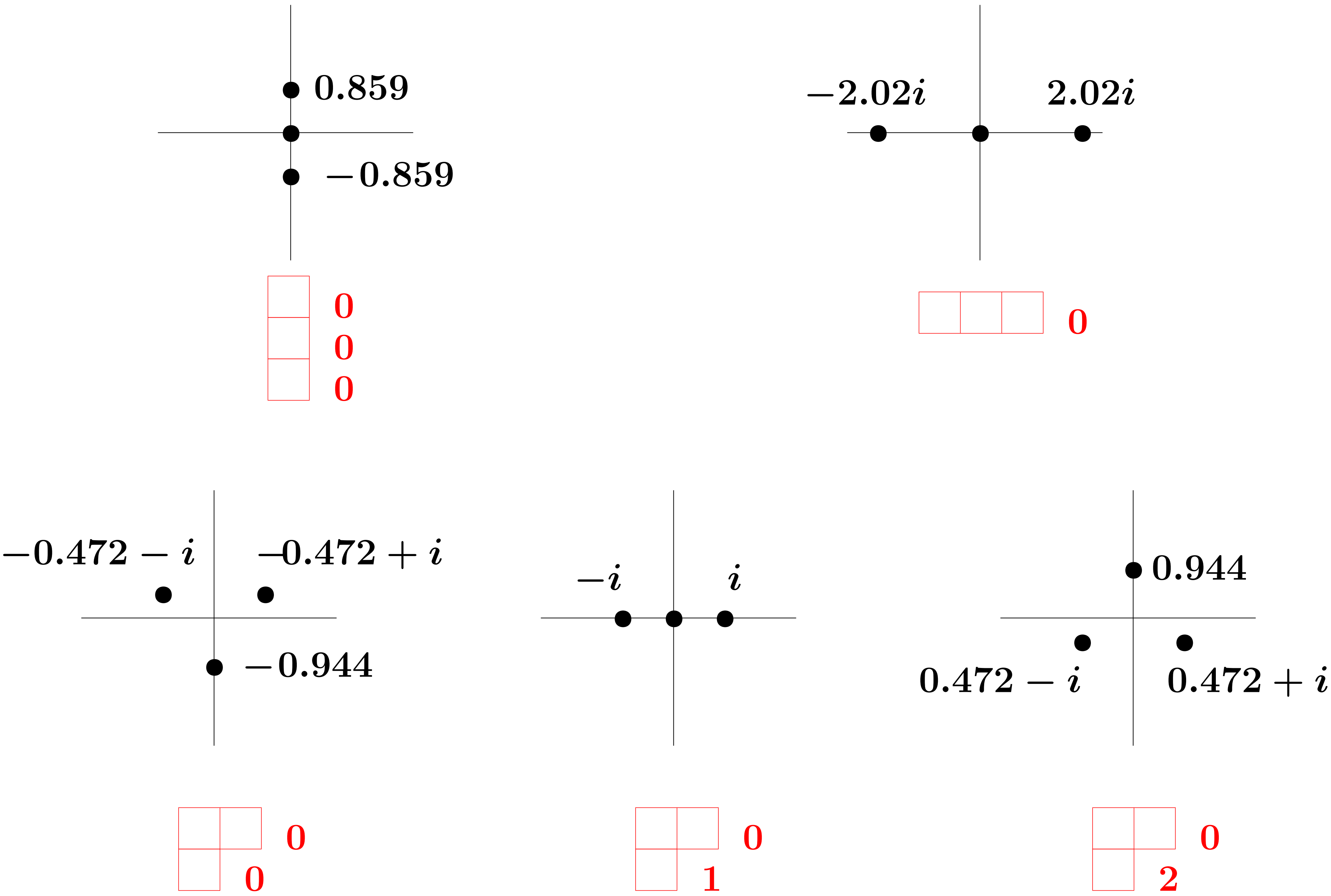
Rigged configuration originates in string hypothesis in Bethe ansatz

Example from sl_2 (Heisenberg chain)

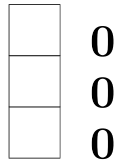
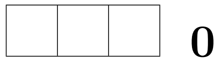
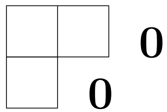
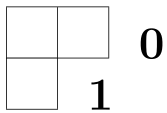
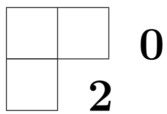
$$H = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z)$$

Bethe equation for length $L = 6$ chain with 3 down spins.

$$\left(\frac{u_1 + i}{u_1 - i}\right)^6 = \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)},$$
$$\left(\frac{u_2 + i}{u_2 - i}\right)^6 = \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)},$$
$$\left(\frac{u_3 + i}{u_3 - i}\right)^6 = \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.$$


0.859
−0.859
0
0
0
−2.02i
2.02i
0
−0.472 − i
−0.472 + i
−0.944
0
0
−i
i
0
1
0.944
0.472 − i
0.472 + i
0
2

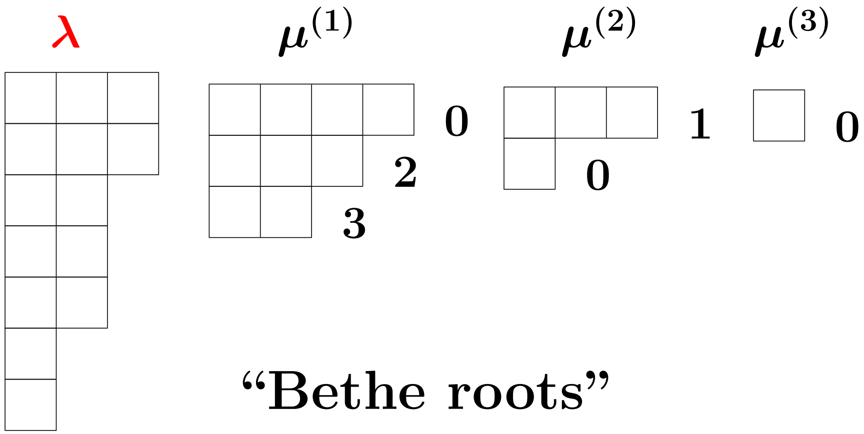
Kerov-Kirillov-Reshetikhin (1986) gave a canonical bijection

"Bethe root"		"Bethe vector"
{rigged configurations}	$\overset{\text{KKR}}{\longleftrightarrow}$	{highest paths}
		121212 $(1 = \uparrow,\ 2 = \downarrow)$
		111222
		121122
		112212
		112122

which may viewed as a combinatorial analogue of Bethe ansatz.

Higher rank example (sl_4)

$$\{\text{rigged configurations}\} \overset{1:1}{\longleftrightarrow} \{\text{highest paths}\}$$



$\longleftrightarrow$ $\boxed{111}$ $\boxed{122}$ $\boxed{22}$ $\boxed{13}$ $\boxed{23}$ $\boxed{4}$ $\boxed{3}$

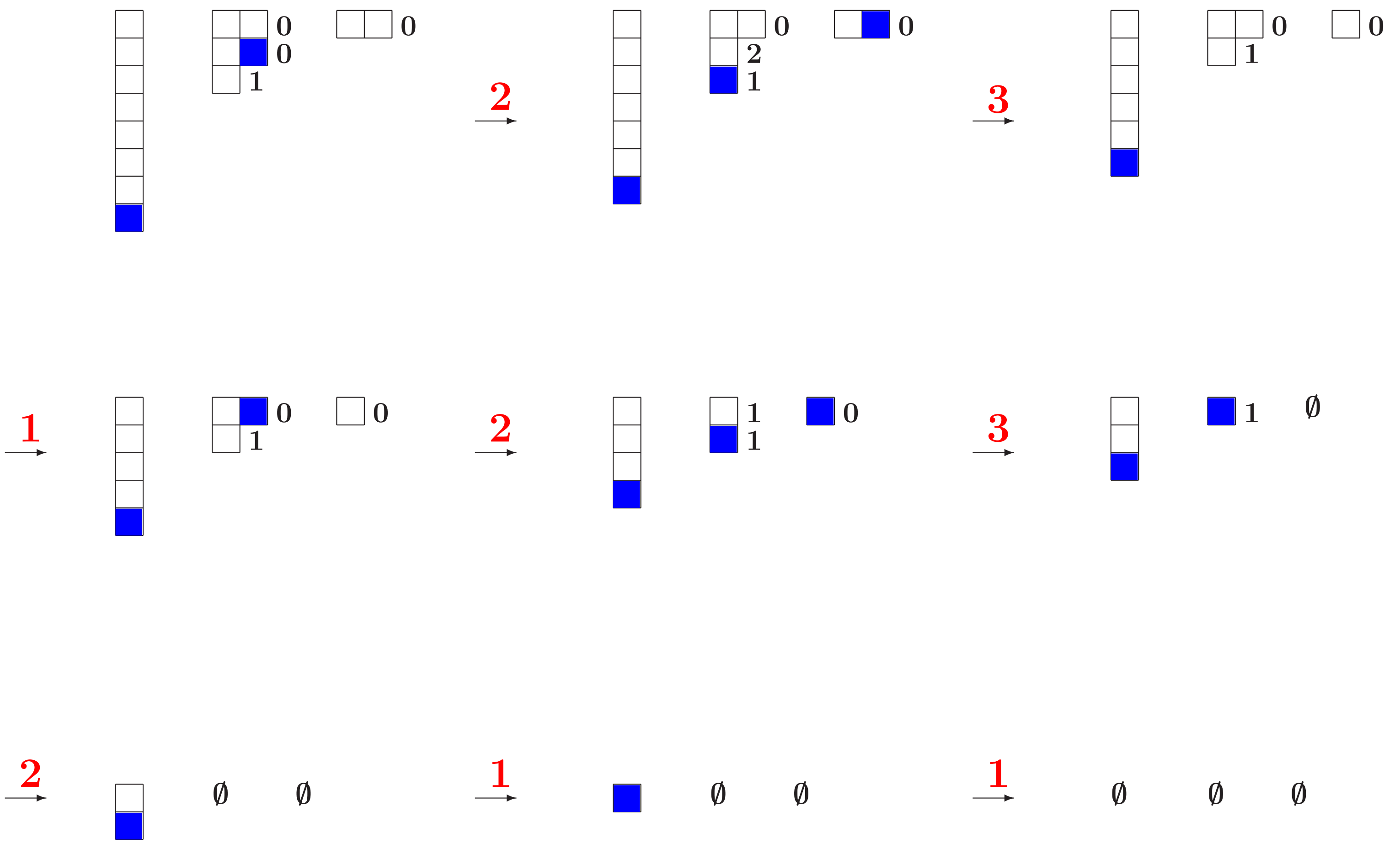
"Bethe roots" "Bethe vectors"

highest path $= b_1\, b_2\, \ldots\, b_L$

b_i = row shape (λ_i) semistandard tableau.

(+ highest condition)

Example of KKR algorithm from sl_3



Top left rigged configuration $\xrightarrow{\text{KKR}}$ **11232132**

Theorem.([KSY])

Image of the KKR map

$$(\boldsymbol{\lambda}, (\boldsymbol{\mu}^{(1)}, \boldsymbol{r}^{(1)}), \ldots, (\boldsymbol{\mu}^{(n-1)}, \boldsymbol{r}^{(n-1)})) \overset{\mathrm{KKR}}{\longmapsto} b_1 \ldots b_L$$

$$b_k = \underbrace{(\overbrace{1 \ldots 1}^{x_{k,1}}, \ldots, \overbrace{n \ldots n}^{x_{k,n}})}_{\lambda_k} \quad \text{(semistandard tableau)},$$

is given by

$$\boxed{x_{k,i} = \tau_{k,i} - \tau_{k-1,i} - \tau_{k,i-1} + \tau_{k-1,i-1}}$$

We will see that this is an analogue of

$$u = -2\frac{\partial^2 \log \tau}{\partial x^2}$$

for KdV eq.

Crystals for $U_q(\widehat{sl}_n)$

$$B_l = \{ \boxed{i_1, \ldots, i_l} \mid \text{semistandard}\}$$

equipped with crystal structures.

$u_l := \boxed{11\ldots1} \in B_l$ **is the (classically) highest element.**

An element of $B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots$ **is called a *path*.**

Combinatorial R

$$R\ :\ \ B_l \otimes B_m \xrightarrow{\sim} B_m \otimes B_l, \qquad x \otimes y \mapsto \tilde{y} \otimes \tilde{x}$$

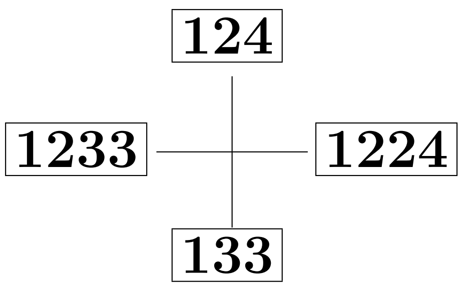
$$\tilde{x}_i - x_i = y_i - \tilde{y}_i = Q_i(x \otimes y) - Q_{i-1}(x \otimes y) \quad (i \bmod n),$$

$x_i = \sharp$ of letter i in tableau x (y_i : similar),

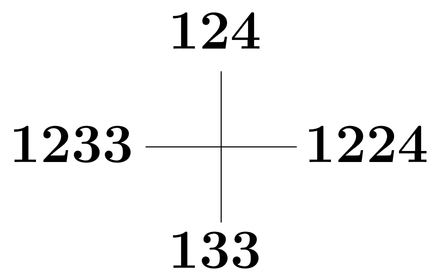
$$Q_i(x \otimes y) = \min_{1 \le k \le n}\{\sum_{j=1}^{k-1} x_{i+j} + \sum_{j=k+1}^{n} y_{i+j}\} \quad \cdots \quad i \text{ th local energy.}$$

Example : $\boxed{1233} \otimes \boxed{124} \simeq \boxed{133} \otimes \boxed{1224}$

will be denoted by

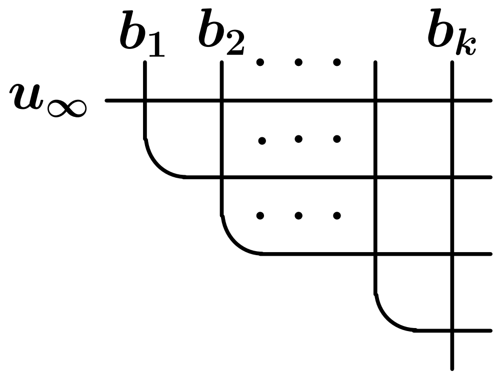


or simply



Energy $\mathcal{E}_i$ of path $b_1 \otimes \cdots \otimes b_k$ (i mod n)

$\mathcal{E}_i(b_1 \otimes \cdots \otimes b_k) :=$ Sum of $Q_i(x \otimes y)$ attached to all vertices in



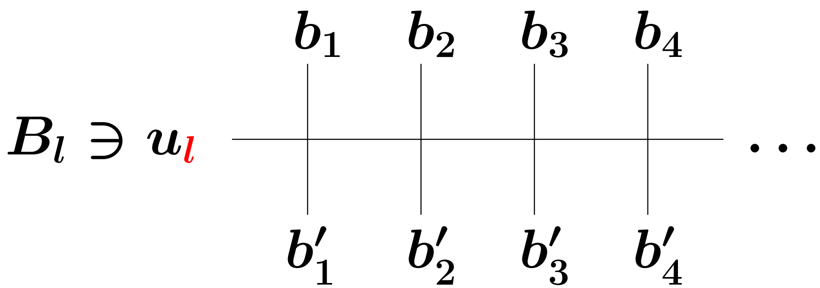
Theorem.([KSY] "Tropical fermionic formula")

Suppose $b_1 \otimes \cdots \otimes b_L \overset{\mathrm{KKR}}{\longleftrightarrow} (\mu, r) \longrightarrow \{\tau_{k,i}\}$. Then,

$$\mathcal{E}_i(b_1 \otimes \cdots \otimes b_k) = \tau_{k,i} \quad (1 \le k \le L) \qquad \blacksquare$$

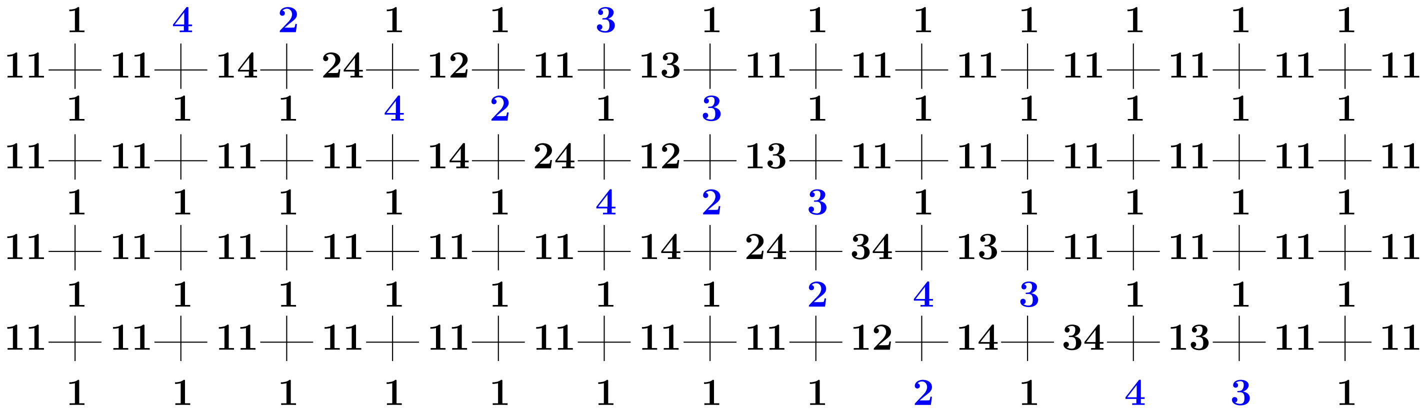
$U_q(\widehat{sl}_n)$ vertex model at $q = 0$

$$T_{l} : \; B_1 \otimes B_1 \otimes B_1 \otimes \cdots \longrightarrow B_1 \otimes B_1 \otimes B_1 \otimes \cdots$$
$$b_1 \otimes b_2 \otimes b_3 \otimes \cdots \longmapsto b'_1 \otimes b'_2 \otimes b'_3 \otimes \cdots$$



$T_1, T_2, \ldots$: commuting family of time evolutions
(deterministic transfer matrices)

Example of time evolution T_2 :



The dynamics on vertical edges reproduces Box-ball system with carrier (Takahashi, Satsuma, Matsukidaira).

$$\cdots 1\mathbf{42}11\mathbf{3}1111111 \cdots$$
$$\cdots 111\mathbf{42}1\mathbf{3}111111 \cdots$$
$$\cdots 11111\mathbf{423}11111 \cdots$$
$$\cdots 1111111\mathbf{243}111 \cdots$$
$$\cdots 11111111\mathbf{2}1\mathbf{43}1 \cdots$$

1= empty box, 2, 3, 4 = colored balls.

Theorem.([KSY])

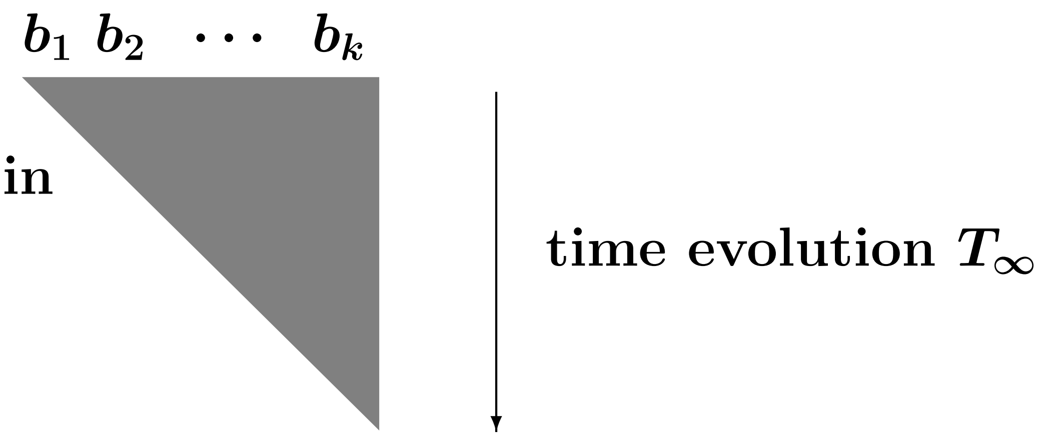
(1) Tropical tau function

= Energy of crystal (math) $\cdots$ previous theorem

= Baxter's corner transfer matrix for box-ball system (phys)

Let $b_1 \, b_2 \cdots b_L \overset{\text{KKR}}{\longleftrightarrow} (\mu, r) \longrightarrow \{\tau_{k,i}\}$. Then,

$\tau_{k,i} = \sharp$ of "balls" in

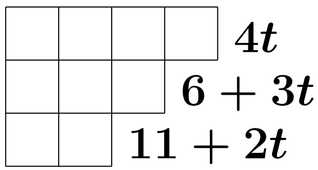
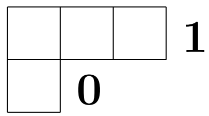
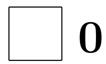


(2) Tropical Hirota equation = eq. of motion of box-ball system.

	Bethe ansatz	Corner transfer matrix
main combinatorial object	rigged configuration	energy (charge) in crystal
role in box-ball system	action-angle variable	tau function
dynamics	linear	bilinear

Dynamics of box-ball system in terms of rigged configuration

$t = 0$: 111122221111133211431111111111111111111111111111
$t = 1$: 111111112222111133214311111111111111111111111111
$t = 2$: 111111111111222211133243111111111111111111111111
$t = 3$: 111111111111111122221132433111111111111111111111
$t = 4$: 111111111111111111112221322433111111111111111111
$t = 5$: 111111111111111111111112211322433211111111111111
$t = 6$: 111111111111111111111111122111322143321111111111
$t = 7$: 111111111111111111111111111221111322114332111111

λ	$\mu^{(1)}$	$\mu^{(2)}$	$\mu^{(3)}$
(1^{48})			

configuration $\cdots$ conserved quantity (action variable)

rigging $\cdots$ linear flow (angle variable)

KKR bijection $\cdots$ direct/inverse scattering map (separation of variables)

Summary so far

classical KP tau

↓ Ultradiscretization (top down)

Tropical tau

↑ Comb. Bethe ansatz (bottom up)

quantum Charge (energy of crystal)

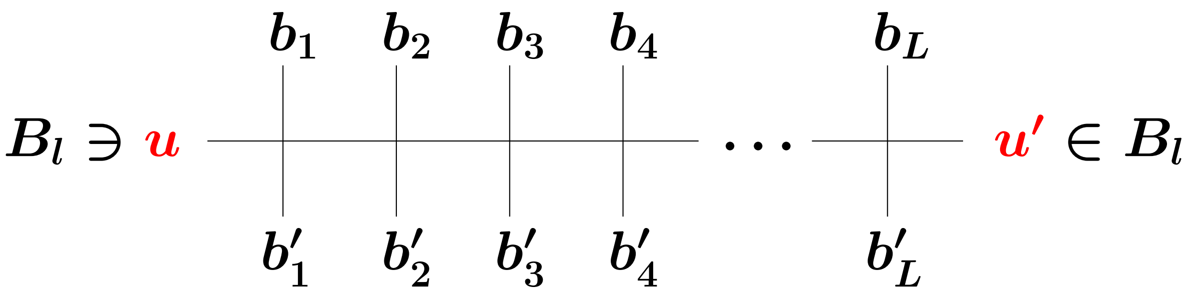
- KKR theory = inverse scattering scheme of box-ball system on ∞ lattice [K-Okado-S-Takagi-Y 2006]

- Initial value problem solved and General N-soliton solution constructed for $\widehat{sl}_n$ symmetric tensor reps. [KSY 2007]

  ($\widehat{sl}_2$ 2-dim. rep. case also by [Mada-Idzumi-Tokihiro 2008])

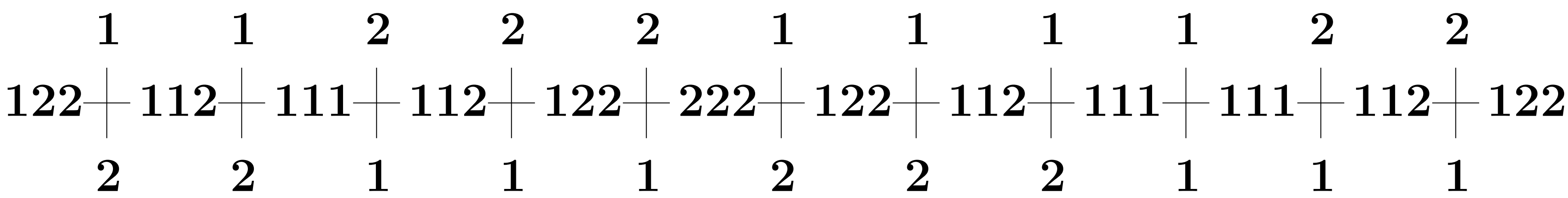
Periodic generalization (sl_2 case)

$$T_l: \; B_1 \otimes B_1 \otimes \cdots \otimes B_1 \longrightarrow B_1 \otimes B_1 \otimes \cdots \otimes B_1$$
$$b_1 \otimes b_2 \otimes \cdots \otimes b_L \longmapsto b'_1 \otimes b'_2 \otimes \cdots \otimes b'_L$$



Choice s.t. $u = u'$: periodic box-ball system (Yura-Tokihiro 2002)

Example of T_3 : ($B_1 = \{ \boxed{1} , \boxed{2} \}$)



Action-angle variables

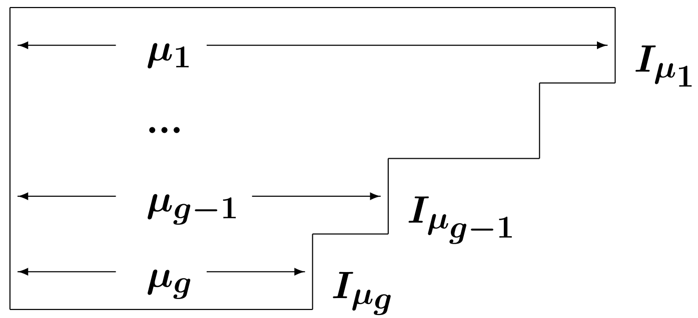
any path highest path rigged conf.

$$b_1 \ldots\ldots b_L \overset{\text{cyclic shift}}{\longmapsto} b_{d+1} \ldots b_L b_1 \ldots b_d \overset{\text{KKR}}{\longmapsto} (\mu, I) \qquad \text{(not unique)}$$

$$\mu = (\mu_1, \ldots, \mu_g)$$

$$\mathrm{I} = (I_{\mu_1}, \ldots, I_{\mu_g})$$

$$p_i := L - 2 \sum_{j \in \mu} \min(i, j)$$



<u>Lemma</u>. (For simplicity assume $\mu_1 > \cdots > \mu_g$)

- μ is unique and invariant under $\{T_l\}$ (action variable)
- $(\mathrm{I} + d\,\mathrm{h}_1)/A\mathbb{Z}^g$ is unique (angle variable), where

$$\mathrm{h}_l = (\min(l, i))_{i \in \mu} \in \mathbb{Z}^g, \quad A = \big(\delta_{ij} p_i + 2 \min(i, j)\big)_{i,j \in \mu}$$

$\mathcal{P}(\mu) := \{\text{paths whose action variable} = \mu\}$ iso-level set

$\mathcal{J}(\mu) := \mathbb{Z}^g / A\mathbb{Z}^g$ set of angle variables

$\Phi : \mathcal{P}(\mu) \longrightarrow \mathcal{J}(\mu)$ by $\Phi(b_1 \dots b_L) := (\mathrm{I} + d\,\mathrm{h}_1)/A\mathbb{Z}^g$

Theorem. ([KT-Takenouchi 2006] "Tropical Abel-Jacobi" map)

Φ is a bijection and
$$\begin{array}{ccc} \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \\ {\scriptstyle T_l}\downarrow & & \downarrow{\scriptstyle T_l} \\ \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \end{array}$$
is commutative.

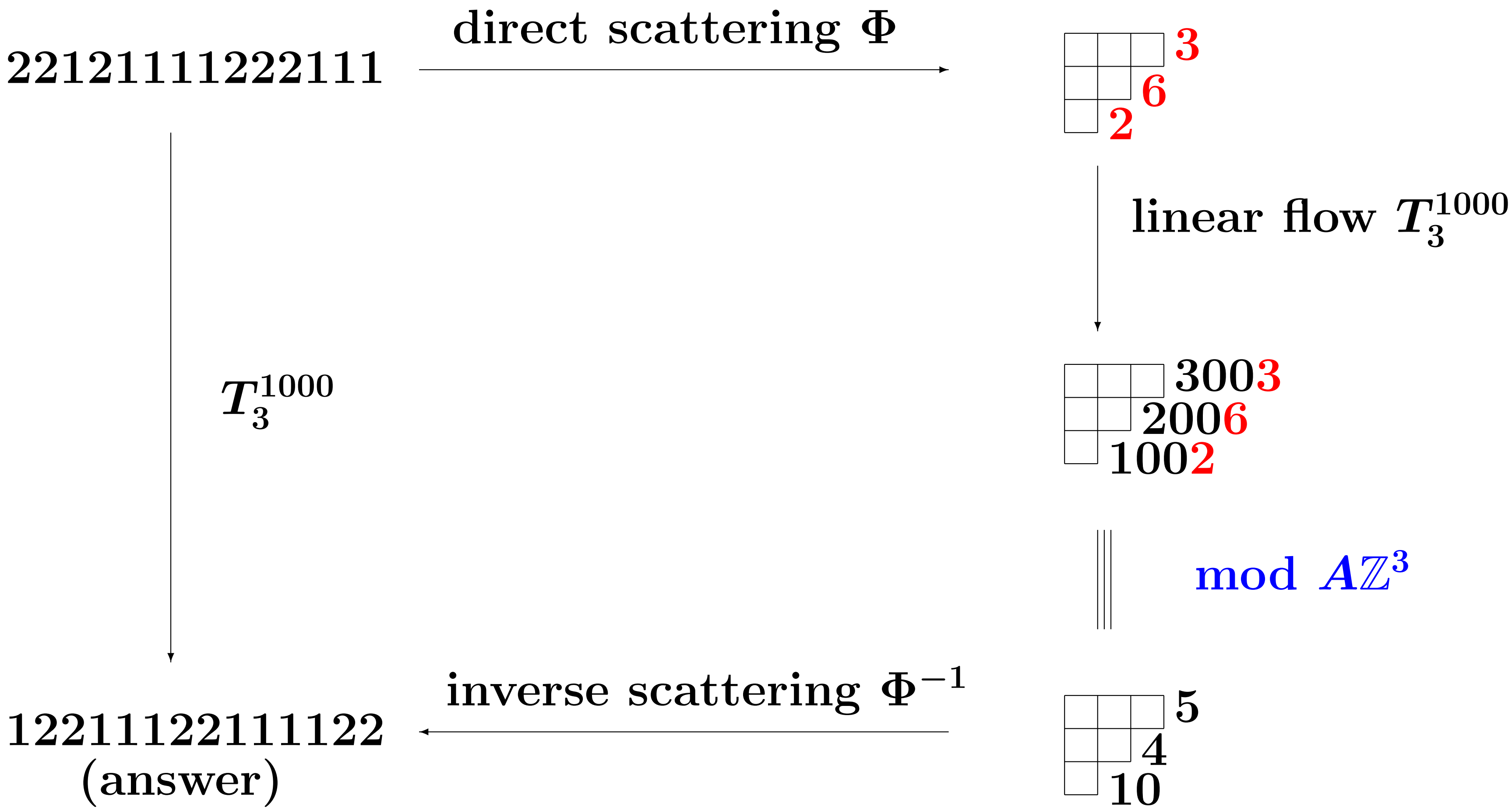
where $T_l(\mathrm{J}) = \mathrm{J} + \mathrm{h}_l$ on $\mathcal{J}(\mu)$. ■

Nonlinear dynamics becomes straight motion in

$$\mathcal{J}(\mu) = \mathbb{Z}^g / A\mathbb{Z}^g,$$

which is an tropical analogue of Jacobi variety.

Solution of initial value problem (inverse method)



Tropical Riemann theta ($\mathrm{z} \in \mathbb{R}^g$):

$$\Theta(\mathrm{z}) := - \min_{\mathrm{n} \in \mathbb{Z}^g} \{ {}^t\mathrm{n} A \mathrm{n}/2 + {}^t\mathrm{n}\mathrm{z} \}$$

Theorem. ([K-Sakamoto 2006] "Tropical Jacobi inversion")

$$\begin{array}{ccc} \mathcal{J}(\mu) & \longrightarrow & \mathcal{P}(\mu) \\ (\mu, \mathrm{I}) & \longmapsto & b_1\, b_2\, \ldots\, b_L \;\; (\in \{1,2\}^L) \end{array}$$

is given by

$$\begin{aligned} b_k = 1 &+ \Theta\big(\mathrm{J} - k\mathrm{h}_1\big) - \Theta\big(\mathrm{J} - (k-1)\mathrm{h}_1\big) \\ &- \Theta\big(\mathrm{J} - k\mathrm{h}_1 + \mathrm{h}_\infty\big) + \Theta\big(\mathrm{J} - (k-1)\mathrm{h}_1 + \mathrm{h}_\infty\big), \end{aligned}$$

with $\mathrm{J}_i = \mathrm{I}_i - \frac{1}{2} p_{\mu_i}$.

- Also obtained in [Mada-Idzumi-Tokihiro 2008].
- Higher spin generalization, Θ-formula for Carrier [KS2008].

- Tropical period matrix A originates in Bethe ansatz at $q = 0$. (KKR is $q = 1$.)

$U_q(\widehat{sl}_2)$ Bethe equation at $q = 0$ under string hypothesis:

$$A\mathrm{x} \equiv \text{constant vector} \mod A\mathbb{Z}^g \quad \cdots \text{linear!}$$
$$\mathrm{x} \in (\mathbb{R}/\mathbb{Z})^g : \text{Bethe root}$$

$$\text{Bethe root } \mathrm{x} \overset{1:1}{\longleftrightarrow} J \in \mathcal{J}(\mu) = \mathbb{Z}^g / A\mathbb{Z}^g \quad \text{via} \quad A\mathrm{x} = J.$$

- Generic dynamical period is the minimum N s.t.

$$N \times (\text{velocity vector}) \in A\mathbb{Z}^g \; \rightarrow \; N = \mathrm{LCM}\Big(1, \bigcup_j \frac{\det A}{\det A[j]}\Big).$$

A is known for $\forall$affine $\mathfrak{g}$ [K-Nakanishi 2002].

$\exists$Conjectures for generic dyamical period, etc.

$A_2^{(1)}$ path $= 121121213322111133211 \in B_1^{\otimes 21}$

l	LCM of	$=$ period under T_l
1	$1,\ 21,\ 21,\ 21,\ 21$	21
2	$1,\ \frac{822}{29},\ \frac{822}{95},\ \frac{411}{46},\ \frac{411}{37}$	822
3	$1,\ \frac{959}{22},\ \frac{959}{176},\ \frac{959}{169},\ \frac{959}{127}$	959
4	$1,\ \frac{2877}{50},\ \frac{2877}{400},\ \frac{2877}{820},\ \frac{2877}{463}$	2877

$A_3^{(1)}$ path $= 134 \cdot 34 \cdot 1 \cdot 134 \cdot 23 \cdot 1 \cdot 13 \in B_3 \otimes B_2 \otimes B_1 \otimes B_3 \otimes B_2 \otimes B_1 \otimes B_2$

(r,l)	LCM of	$=$ period under $T_l^{(r)}$
(1,1)	$1,\ \frac{380}{39},\ \frac{95}{6},\ \frac{95}{6},\ \frac{380}{31},\ \frac{380}{27},\ \frac{380}{29}$	380
(1,2)	$1,\ \frac{190}{39},\ \frac{95}{12},\ \frac{95}{12},\ \frac{190}{31},\ \frac{190}{27},\ \frac{190}{29}$	190
(2,1)	$1,\ \frac{190}{13},\ \frac{95}{4},\ \frac{95}{4},\ \frac{190}{137},\ \frac{190}{9},\ \frac{190}{73}$	190
(2,2)	$1,\ \frac{76}{5},\ \frac{38}{3},\ \frac{38}{3},\ \frac{76}{41},\ \frac{76}{21},\ \frac{76}{31}$	76
(2,3)	$1,\ \frac{95}{6},\ \frac{95}{11},\ \frac{95}{11},\ \frac{95}{34},\ \frac{95}{48},\ \frac{95}{41}$	95
(3,1)	$1,\ \frac{380}{13},\ \frac{95}{2},\ \frac{95}{2},\ \frac{380}{137},\ \frac{380}{9},\ \frac{380}{263}$	380

"I haven't a slightest idea of what people did with it."

· · · H. Bethe