

**Randomized box-ball systems,  
limit shape of soliton distributions and  
thermodynamic Bethe ansatz**

Atsuo Kuniba (Univ. Tokyo)

Workshop on Recent Progress in Mathematical and Statistical Physics

University of Tokyo, 30 October 2019

# Contents

## Box-ball system (short review)

Time evolution, solitons, integrability,  
KKR bijection, action-angle variables.

## Randomized box-ball system

Soliton contents, limit shape problem,  
Induced measure, Fermionic formula,  
Thermodynamic Bethe ansatz, results.





# Collision of 3 solitons

. . . 00**321**00**31**0000**2**0000000000000000 . . .  
 . . . 00000**321**0**31**000**2**0000000000000000 . . .  
 . . . 000000000**3203110**2000000000000000 . . .  
 . . . 00000000000**32003121**000000000000 . . .  
 . . . 0000000000000**320010321**00000000 . . .  
 . . . 000000000000000**3201000321**00000 . . .  
 . . . 00000000000000000**3021000032100** . . .  
  
 . . . 00**321**0000**31**00**2**0000000000000000 . . .  
 . . . 00000**321**000**31**0**2**0000000000000000 . . .  
 . . . 000000000**32100312**0000000000000000 . . .  
 . . . 0000000000000**3210132**000000000000 . . .  
 . . . 0000000000000000**3021321**00000000 . . .  
 . . . 00000000000000000**300210321**00000 . . .  
 . . . 000000000000000000**3000210032100** . . .

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

## Double (classical and quantum) origin of integrability

### (1) Ultra-Discretization (UD) of soliton equations

- Key formula

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \log \left( \exp\left(\frac{a}{\varepsilon}\right) + \exp\left(\frac{b}{\varepsilon}\right) \right) = \mathbf{max}(a, b)$$

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \log \left( \exp\left(\frac{a}{\varepsilon}\right) \times \exp\left(\frac{b}{\varepsilon}\right) \right) = a + b$$

$$(+, \times) \longrightarrow (\mathbf{max}, +)$$

keeps distributive law:

$$AB + AC = A(B + C) \rightarrow \max(a + b, a + c) = a + \max(b, c)$$

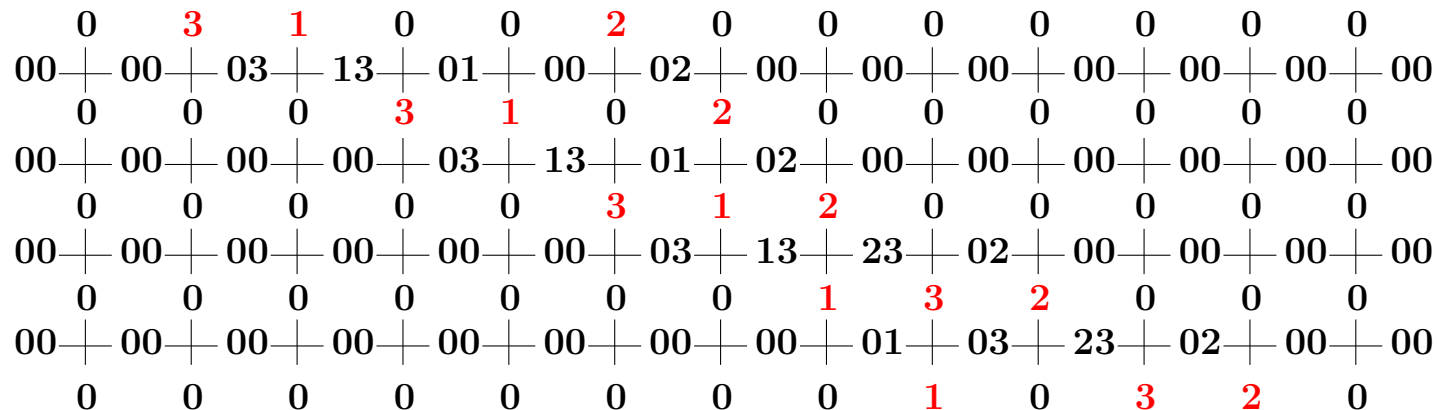
- UD of a discrete KdV equation gives an evolution equation of the  $n = 1$  BBS (1996).

## (2) Solvable lattice model at “Temperature 0”

Time evolution pattern

... 0**3**100**2**0000000 ...  
 ... 000**3**10**2**0000000 ...  
 ... 00000**3**1**2**000000 ...  
 ... 0000000**1**3**2**000 ...  
 ... 00000000**1**0**3**20 ...

emerges from a configuration of a 2D lattice model in statistical mechanics



- $n$ -color BBS = 2D solvable vertex model associated with quantum group

$$U_q(\widehat{sl}_{n+1}) \text{ at } q = 0. \quad (q \sim \text{temperature})$$

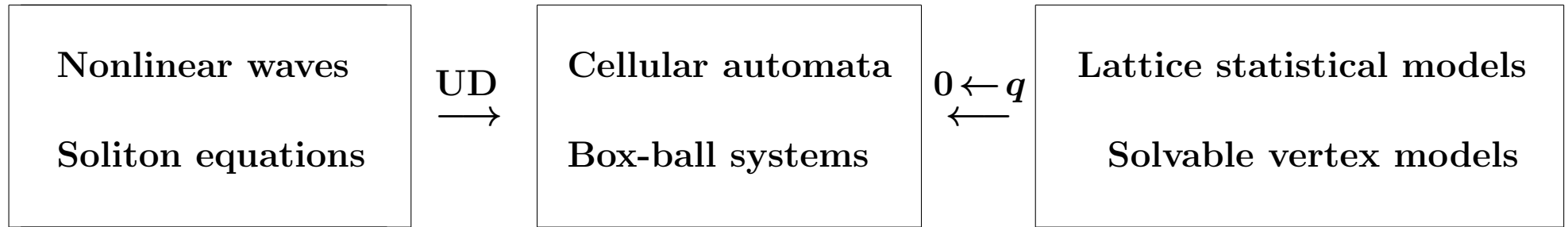




Classical  
integrable system

Ultra-Discrete  
integrable system

Quantum  
integrable system



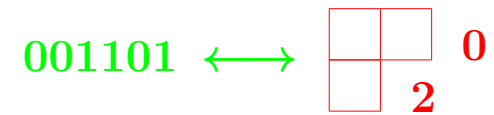
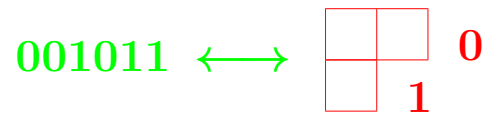
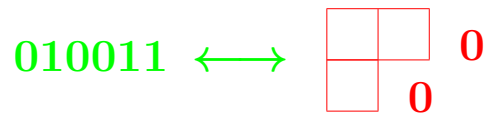
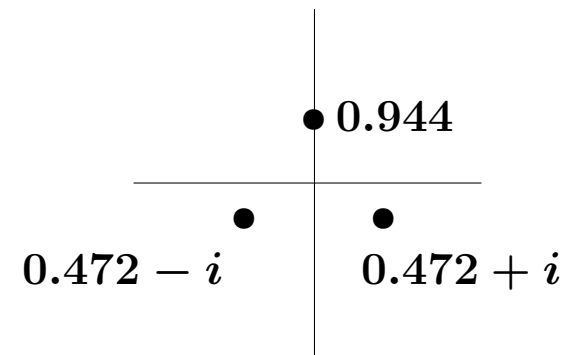
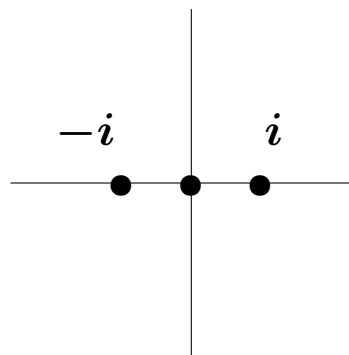
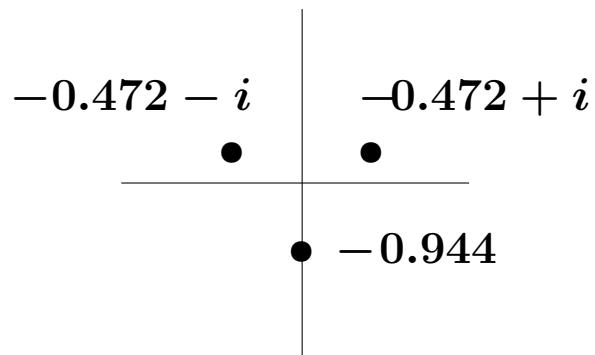
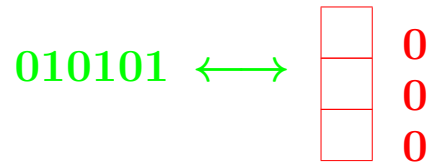
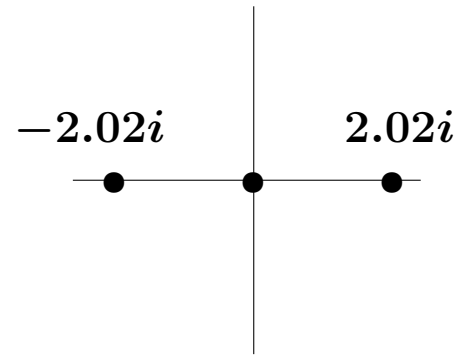
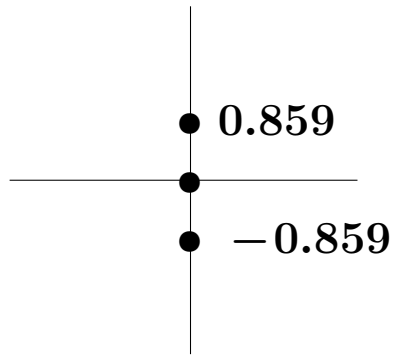
Inverse scattering method

**KKR bijection**

Bethe ansatz

- **Kerov-Kirillov-Reshetikhin (KKR) bijection** (1986) asserts “formal completeness” of the hypothetical string solutions to the Bethe equation at combinatorial level.
- Its unexpected connection to BBS was discovered in 2002.
- Example. Spin  $\frac{1}{2}$  periodic Heisenberg chain with 6 sites in 3 down-spin sector.

$$\begin{aligned} \left(\frac{u_1 + i}{u_1 - i}\right)^6 &= \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)}, \\ \left(\frac{u_2 + i}{u_2 - i}\right)^6 &= \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)}, \\ \left(\frac{u_3 + i}{u_3 - i}\right)^6 &= \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}. \end{aligned}$$



# KKR bijection for $sl_{n+1}$

$$\{\text{highest states}\} \xleftrightarrow{1:1} \{\text{rigged configurations}\}$$

$n = 3$  example

$$000011102113220000 \longleftrightarrow \begin{array}{c} \mu^{(1)} \\ \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} 0 \\ \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} 2 \\ \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array} 3 \end{array} \quad \begin{array}{c} \mu^{(2)} \\ \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} 1 \\ \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} 0 \end{array} \quad \begin{array}{c} \mu^{(3)} \\ \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} 0 \end{array}$$

“Bethe vectors”

Solitons

“Bethe roots”

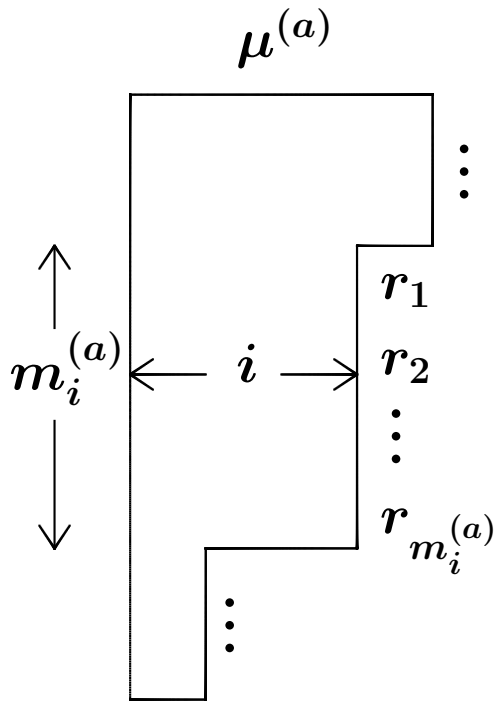
Strings (bound states of magnons)

- highest states =  $i_1 i_2 \dots i_L$  ( $0 \leq i_k \leq n$ ) satisfying the highest condition:

$$\#_0\{i_1, \dots, i_k\} \geq \#_1\{i_1, \dots, i_k\} \geq \dots \geq \#_n\{i_1, \dots, i_k\} \quad (\forall k)$$

- rigged configuration:  $((\mu^{(1)}, r^{(1)}), \dots, (\mu^{(n)}, r^{(n)}))$

$$\left. \begin{array}{l} \mu^{(1)}, \dots, \mu^{(n)} : \text{configuration} = n\text{-tuple of Young diagrams} \\ r^{(1)}, \dots, r^{(n)} : \text{rigging} = \text{integers assigned to each row} \end{array} \right\} + \text{selection rule (next page)}$$



$$m_i^{(a)} = \#(\text{length } i \text{ rows in } \mu^{(a)}), \quad \sum_{i \geq 1} i m_i^{(a)} = |\mu^{(a)}|$$

$$0 \leq r_1 \leq \dots \leq r_{m_i^{(a)}} \leq h_i^{(a)}$$

... “Fermionic” selection rule

$$h_i^{(a)} = L\delta_{a,1} - \sum_{b=1}^n C_{ab} \sum_{j \geq 1} \min(i, j) m_j^{(b)}$$

... vacancy for holes

$$C_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1}$$

$(C_{ab}) \dots$  Cartan matrix of  $sl_{n+1}$

$$\# \text{ of rigging choices for a fixed configuration} = \prod_{a=1}^n \prod_{i \geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}$$

This is an  $sl_{n+1}$  generalization of Bethe’s formula for # of string solutions (1931).

hat also eine Möglichkeit weniger, die des letzten Komplexes von  $n$  Wellen,  $\lambda_{q_n}$ , kann schließlich nur noch

$$Q'_n - (q_n - 1) = Q_n + 1$$

verschiedene Werte annehmen, wo

$$Q_n(N, q_1 q_2 \dots) = N - 2 \sum_{p < n} p q_p - 2 \sum_{p \geq n} n q_p. \quad (44)$$

Schließlich ist zu berücksichtigen, daß Vertauschung der  $\lambda$  der verschiedenen Wellenkomplexe mit gleicher Anzahl  $n$  von Wellen nicht zu neuen Lösungen führt. Die gesamte Zahl unserer Lösungen wird somit

$$z(N, q_1 q_2 \dots) = \prod_{n=1}^{\infty} \frac{(Q_n + q_n) \dots (Q_n + 1)}{q_n!} = \prod_n \binom{Q_n + q_n}{q_n}, \quad (45)$$

wo die  $Q_n$  durch (44) definiert sind.

8. Wir werden nun nachweisen, daß wir die richtige Anzahl Lösungen erhalten haben.





# Randomized box-ball system

$$\begin{array}{ccc} \text{BBS state} & & \text{Soliton content} \\ i_1 i_2 \dots i_L 00000 \dots & \xrightarrow{\text{KKR}} & (\mu^{(1)}, \dots, \mu^{(n)}) \end{array}$$

Randomize  $i_1 i_2 \dots i_L$  by introducing the i.i.d. measure on the set of states:

$$\{0, 1, \dots, n\} \rightarrow (0, 1); \quad i \mapsto p_i \quad (p_0 + \dots + p_n = 1).$$

## Problems

- (1) Find the **induced measure**  $\text{Prob}(\mu^{(1)}, \dots, \mu^{(n)})$  on soliton contents.
- (2) Determine the **scaling form** of the most probable  $(\mu^{(1)}, \dots, \mu^{(n)})$  when  $L \rightarrow \infty$ .

## Technical Lemma

$$\text{Prob}(\mu^{(1)}, \dots, \mu^{(n)}) \simeq \text{Prob}_+(\mu^{(1)}, \dots, \mu^{(n)}) \quad \text{if } p_0 \geq p_1 \geq \dots \geq p_n$$

as far as the leading scaling behavior is concerned, where

$$\text{Prob}_+(\mu^{(1)}, \dots, \mu^{(n)}) := \text{induced measure for the } \textit{highest} \text{ BBS states.}$$



## Theorem (“Fermionic” measure for the soliton contents)

$$\text{Prob}_+(\mu^{(1)}, \dots, \mu^{(n)}) = Z_L^{-1} e^{-\beta_1 |\mu^{(1)}| - \dots - \beta_n |\mu^{(n)}|} \prod_{a=1}^n \prod_{i \geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}},$$

$$e^{\beta_a} := \frac{p_{a-1}}{p_a}, \quad Z_L = \text{normalization const. (partition function)}.$$

The above Lemma and Theorem yield a solution to Problem (1) for  $p_0 \geq \dots \geq p_n$ .

Problem (2) is handled by [Thermodynamic Bethe Ansatz \(TBA\)](#).

Introduce the scaled string and hole densities  $\rho_i^{(a)}, \sigma_i^{(a)}$  by

$$m_i^{(a)} \simeq L \rho_i^{(a)}, \quad h_i^{(a)} \simeq L \sigma_i^{(a)}, \quad \sigma_i^{(a)} = \delta_{a,1} - \sum_{b=1}^n C_{ab} \sum_{j \geq 1} \min(i, j) \rho_j^{(b)},$$

The most probable  $(\mu^{(1)}, \dots, \mu^{(n)})$  is given by  $\{\rho_i^{(a)}\}$  achieving the minimum of

$$F = \sum_{a=1}^n \sum_{i \geq 1} \left( \beta_a i \rho_i^{(a)} - (\rho_i^{(a)} + \sigma_i^{(a)}) \log(\rho_i^{(a)} + \sigma_i^{(a)}) - \rho_i^{(a)} \log \rho_i^{(a)} - \sigma_i^{(a)} \log \sigma_i^{(a)} \right)$$

The condition  $\frac{\delta F}{\delta \rho_i^{(a)}} = 0$  leads to the **TBA equation**

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^n C_{ab} \sum_{j \geq 1} \min(i, j) \log(1 + (Y_j^{(b)})^{-1})$$

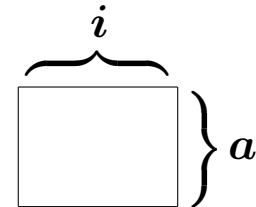
in terms of  $Y_i^{(a)} = \frac{\sigma_i^{(a)}}{\rho_i^{(a)}}$  with the boundary condition  $\lim_{i \rightarrow \infty} \frac{1 + Y_{i+1}^{(a)}}{1 + Y_i^{(a)}} = e^{\beta_a}$ .

**Solution** (Rare case for which an exact formula can be given)

$$Y_i^{(a)} = \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{Q_i^{(a-1)} Q_i^{(a+1)}}$$

$$Q_i^{(a)} = Q_i^{(a)}(p_0, \dots, p_n) = \frac{\det(p_k^{\lambda_j + n - j})_{j,k=0}^n}{\det(p_k^{n-j})_{j,k=0}^n} \quad ( (\lambda_0, \dots, \lambda_n) = (\overbrace{i \dots i}^a, \overbrace{0, \dots, 0}^{n+1-a}) )$$

= **Schur function** for  $a \times i$  rectangular Young diagram

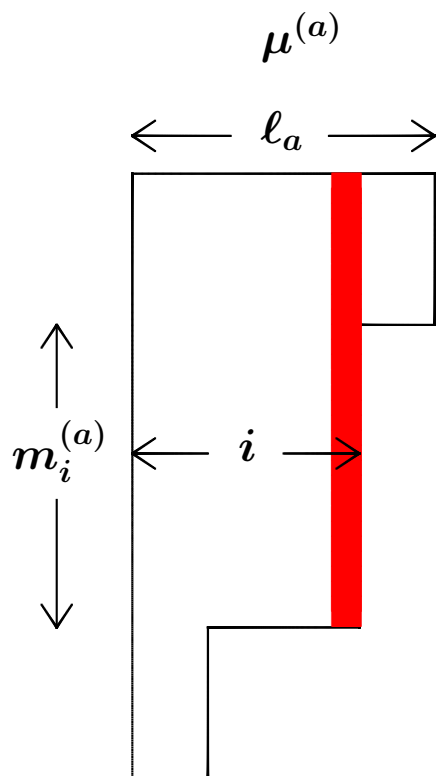


**Result.** The limit shape of soliton content  $(\mu^{(1)}, \dots, \mu^{(n)})$  is given by

$$\eta_i^{(a)} := \lim_{L \rightarrow \infty} \frac{1}{L} (\text{Length of the } i \text{th column of } \mu^{(a)}) = \frac{Q_{i-1}^{(a-1)} Q_i^{(a+1)}}{Q_i^{(a)} Q_{i-1}^{(a)} Q_1^{(1)}}$$

$$\text{width } \ell_a \text{ of } \mu^{(a)} \simeq \frac{\log L}{\log \frac{p_{a-1}}{p_a}} \quad (L \rightarrow \infty \text{ if } p_0 > \dots > p_n)$$

**Special case**  $p_a = \frac{q^a}{1+q+\dots+q^n} \quad (0 < q \leq 1).$



Scaled column length of  $\mu^{(a)}$

$$\eta_i^{(a)} = \frac{q^{i+a-1}(1-q)(1-q^a)(1-q^{n+1-a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})}$$

Strings

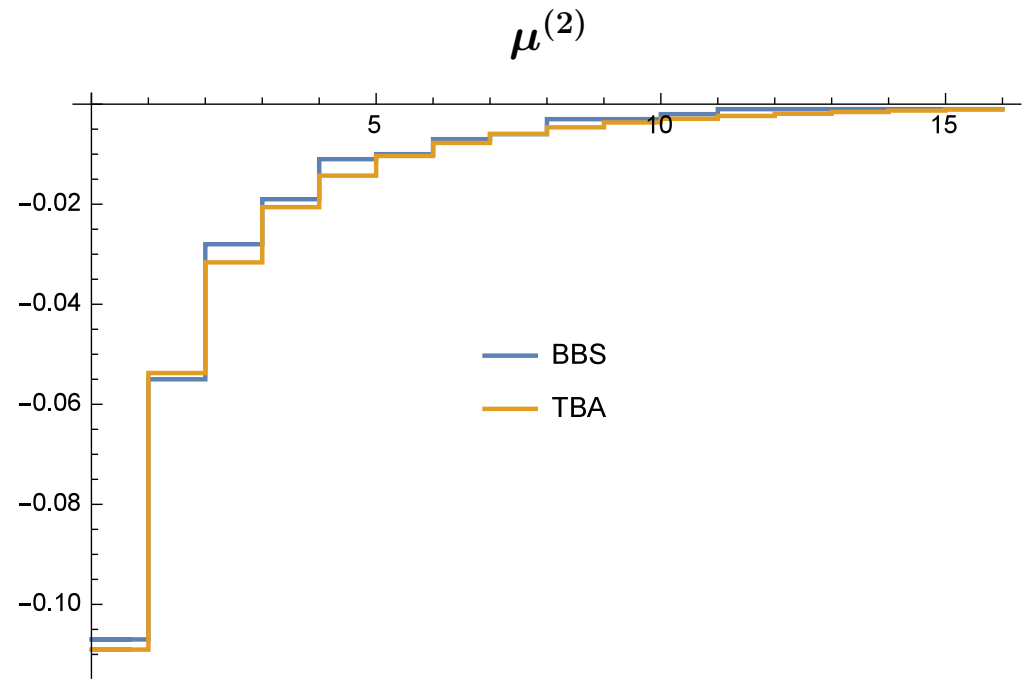
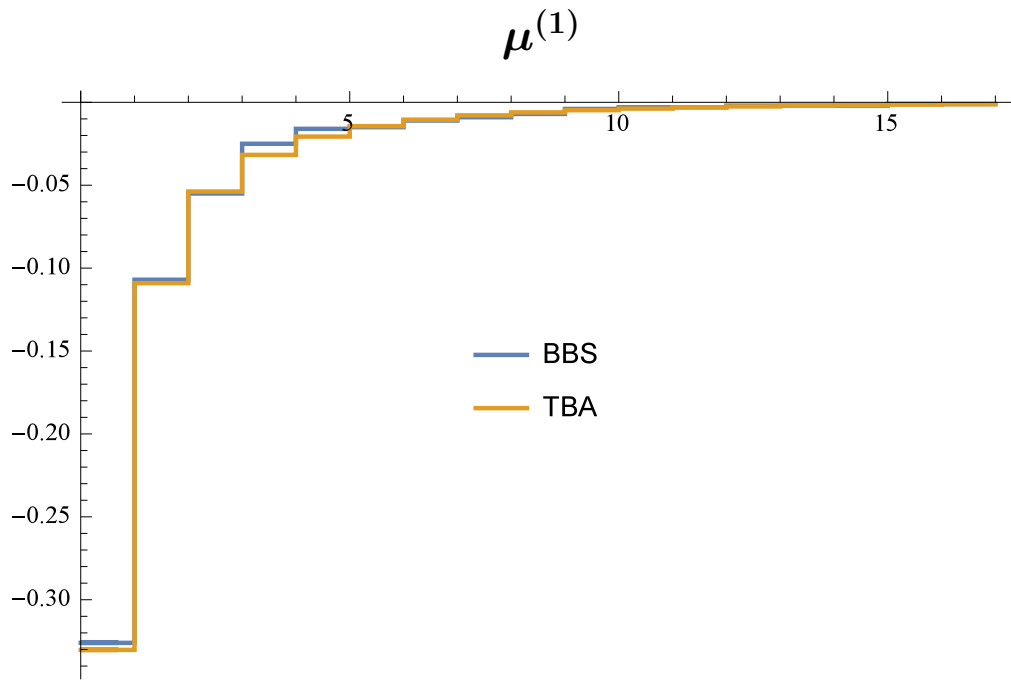
$$\rho_i^{(a)} = \lim_{L \rightarrow \infty} \frac{1}{L} m_i^{(a)} = \frac{q^{i+a-1}(1-q)^2(1-q^a)(1-q^{n+1-a})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

Holes

$$\sigma_i^{(a)} = \lim_{L \rightarrow \infty} \frac{1}{L} h_i^{(a)} = \frac{q^{a-1}(1-q)^2(1-q^i)(1-q^{n+i+1})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

2-color BBS with  $L = 1000$  sites with distribution  $(p_0, p_1, p_2) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$ .

Vertically  $L^{-1}$  scaled soliton contents.



- $0 < q < 1$ : Subcritical

$\mu^{(a)}$  scales with depth  $O(L)$ , width  $O(\log L)$

- $q = 1$ : Critical

$\mu^{(1)}$  scales with depth  $O(L)$ , width  $O(\sqrt{L})$  ([Levine-Lyu-Pike] for  $n = 1$ )

# Summary

1. Rigged configurations are action-angle variables of  $n$ -color BBS.
2. In the context of BBS, a configuration  $(\mu^{(1)}, \dots, \mu^{(n)})$  is a soliton content generalizing the list of amplitude  $\mu^{(1)}$ .
3. Randomizing local states by the probabilities  $p_0, \dots, p_n$  induces the measure for  $(\mu^{(1)}, \dots, \mu^{(n)})$  expressed by the Bethe type “Fermionic” formula.
4. By TBA, exact leading scaling form of the soliton content is determined in terms of Schur functions involving  $p_0, \dots, p_n$ .
5. Renormalized velocity and currents in BBS soliton gas (work in progress).

Based on

Review part:

R. Inoue, AK and T.Takagi

“Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry”, JPA(2012), arXiv:1109.5349.

Randomized BBS:

AK, H. Lyu and M.Okado

“Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz”, NPB(2018), arXiv:1808.02626.

AK and H. Lyu

“Large deviations and one-sided scaling limit of randomized multicolor box-ball system”, JSP(to appear) arXiv:1808.08074.