Generalized hydrodynamics for randomized box-ball system

Atsuo Kuniba (Univ. Tokyo)

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Commuting time evolutions, Family of conserved quantities,
Solitons obeying factorized scattering,
Equation of motion = discrete soliton equation,
Yang-Baxter equation, Bethe ansatz structure.

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A: **Box-ball system (BBS) and its generalization based on quantum groups**

provide exact quasi-particle picture to Bethe's formula for #{string solutions},
allow interpretation of corner transfer matrices as tau functions,
identify $2\pi i/\log(\text{Bethe eigenvalue})$ = Poincaré cycle, etc.

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Plan of the talk

- Basic features and integrability of BBS (Review)
- Recent progress on statistical aspects of randomized BBS in out of equilibrium

1. Limit shape problem of soliton distributions
   by thermodynamic Bethe ansatz (TBA)

2. Density plateaux emerging from domain wall initial conditions
   by generalized hydrodynamics (GHD)

n-color Box-ball system (BBS)

$n = 3$ example.

$$\cdots 00000000\textcolor{red}{33211}0000000000000000000000000 \cdots$$
$$\cdots 0000000000000\textcolor{red}{33211}00000000000000000000 \cdots$$
$$\cdots 000000000000000000\textcolor{red}{33211}000000000000000 \cdots$$

0 =empty box, $\quad 1, 2, 3 =$ balls with colors

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• time evolution = (move 1) · (move 2) · (move 3)

(move i) · Pick the leftmost ball with color i and move it to the nearest right empty box.

· Do the same for the other color i balls.

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- time evolution = (move 1) $\cdot$ (move 2) $\cdot$ (move 3)

(move i) $\cdot$ Pick the leftmost ball with color i and move it to the nearest right empty box.

$\cdot$ Do the same for the other color i balls.

- soliton=consecutive balls $i_1 \ldots i_a$ with color $i_1 \geq \cdots \geq i_a \geq 1$.
- velocity=amplitude.

- **Collisions of 2 solitons**

$$\cdots 000\color{red}{33211}\color{black}{0000}\color{red}{322}\color{black}{00000000000000000000000} \cdots$$
$$\cdots 00000000\color{red}{33211}\color{black}{00}\color{red}{322}\color{black}{00000000000000000000} \cdots$$
$$\cdots 0000000000000\color{red}{33211322}\color{black}{00000000000000000} \cdots$$
$$\cdots 000000000000000000\color{red}{21133322}\color{black}{000000000000} \cdots$$
$$\cdots 000000000000000000000\color{red}{211}\color{black}{00}\color{red}{33322}\color{black}{0000000} \cdots$$
$$\cdots 000000000000000000000000\color{red}{211}\color{black}{0000}\color{red}{33322}\color{black}{00} \cdots$$

- **Amplitudes are individually conserved.**

- **Collisions of 2 solitons**

$\cdots$ 00033211000032200000000000000000000000 $\cdots$
$\cdots$ 00000000332110032200000000000000000000 $\cdots$
$\cdots$ 00000000000003321132200000000000000000 $\cdots$
$\cdots$ 00000000000000000021133322000000000000 $\cdots$
$\cdots$ 00000000000000000000021100333220000000 $\cdots$
$\cdots$ 00000000000000000000000021100003332200 $\cdots$

- **Amplitudes are individually conserved.**

- **Two body scattering:**

  Exchange of internal labels (colors) like quarks in hadrons

  Phase shift

Collision of 3 solitons

$\cdots 00321003100002000000000000000 \cdots$
$\cdots 00000321031000200000000000000 \cdots$
$\cdots 00000000320311020000000000000 \cdots$
$\cdots 00000000003200312100000000000 \cdots$
$\cdots 00000000000032001032100000000 \cdots$
$\cdots 00000000000000320100032100000 \cdots$
$\cdots 00000000000000003021000032100 \cdots$

$\cdots 00321000031002000000000000000 \cdots$
$\cdots 00000321000310200000000000000 \cdots$
$\cdots 00000000321003120000000000000 \cdots$
$\cdots 00000000000321013200000000000 \cdots$
$\cdots 00000000000000302132100000000 \cdots$
$\cdots 00000000000000030021032100000 \cdots$
$\cdots 00000000000000003000210032100 \cdots$

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

Quantum origin: Solvable lattice model at “ Temperature 0 ”

Quantum origin: Solvable lattice model at " Temperature 0 "

Time evolution pattern

$$\cdots 0\mathbf{31}00\mathbf{2}0000000 \cdots$$
$$\cdots 000\mathbf{31}0\mathbf{2}000000 \cdots$$
$$\cdots 00000\mathbf{312}00000 \cdots$$
$$\cdots 0000000\mathbf{132}000 \cdots$$
$$\cdots 00000000\mathbf{1}0\mathbf{32}0 \cdots$$

Quantum origin: Solvable lattice model at “ Temperature 0 ”

Time evolution pattern

$\cdots$ 03100200000000 $\cdots$

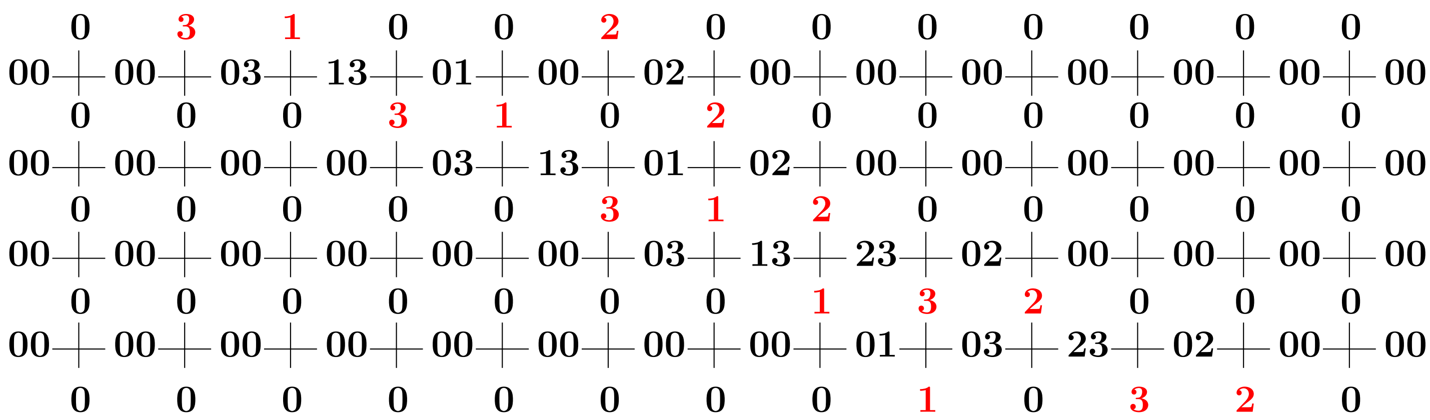
$\cdots$ 00031020000000 $\cdots$

$\cdots$ 00000312000000 $\cdots$

$\cdots$ 00000001320000 $\cdots$

$\cdots$ 00000000103200 $\cdots$

emerges from a configuration of a 2D lattice model in statistical mechanics



by forgetting the hidden variables on the horizontal edges.

- n-color box-ball system

  = 2D solvable vertex model associated with quantum group

$$U_q(\widehat{sl}_{n+1}) \text{ at } q = 0 \quad (q \sim \text{temperature})$$

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  = deterministic map (defined by the unique configuration surviving at $q = 0$)

  = time evolution of box-ball system (forming a commuting family $T_1, T_2, \ldots T_\infty$)

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- **Row transfer matrix at $q = 0$**

  = deterministic map (defined by the unique configuration surviving at $q = 0$)

  = time evolution of box-ball system (forming a commuting family $T_1, T_2, \ldots T_\infty$)

- **Proper formulation uses *crystal base theory* (theory of quantum group at $q = 0$).**

Some outcomes from such insight

- $\exists$ **Integrable cellular automata with quantum group symmetry.**

Example: $\widehat{so}_{10}$-automaton

$$\cdots 000\bar{2}\bar{4}211 0000 \bar{1}\bar{1}\bar{4} 00000000000000000000000000000000 \cdots$$
$$\cdots 00000000\bar{2}\bar{4}211 00 \bar{1}\bar{1}\bar{4} 00000000000000000000000000000 \cdots$$
$$\cdots 0000000000000\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4} 00000000000000000000000000 \cdots$$
$$\cdots 000000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4} 00000000000000000000000 \cdots$$
$$\cdots 00000000000000000000000 3\bar{0}\bar{0}\bar{3}\bar{4}\bar{4} 000000000000000000 \cdots$$
$$\cdots 00000000000000000000000000 311\bar{1}\bar{1}\bar{3}\bar{4}\bar{4} 0000000000000 \cdots$$
$$\cdots 00000000000000000000000000000 311 00 \bar{1}\bar{1}\bar{3}\bar{4}\bar{4} 00000000 \cdots$$
$$\cdots 00000000000000000000000000000000 311 0000 \bar{1}\bar{1}\bar{3}\bar{4}\bar{4} 000 \cdots$$

Some outcomes from such insight

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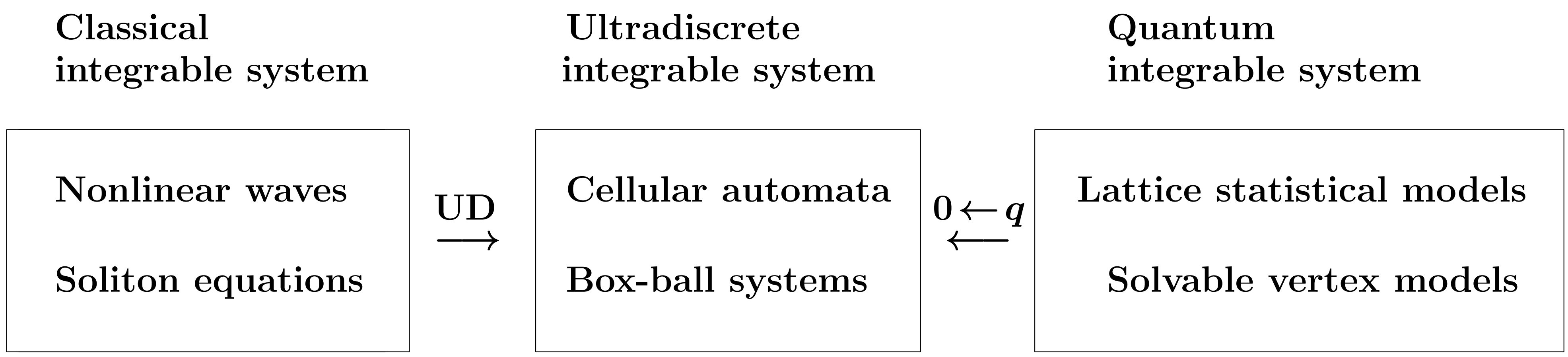
Example: $\widehat{\mathrm{so}}_{10}$-automaton

$$
\begin{array}{l}
\cdots 000\bar{2}\bar{4}2110000\bar{1}\bar{1}\bar{4}00000000000000000000000000000000 \cdots \\
\cdots 00000000\bar{2}\bar{4}21100\bar{1}\bar{1}\bar{4}00000000000000000000000000000 \cdots \\
\cdots 0000000000000\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4}00000000000000000000000000 \cdots \\
\cdots 000000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4}00000000000000000000000 \cdots \\
\cdots 000000000000000000000003\bar{0}\bar{0}\bar{3}\bar{4}\bar{4}000000000000000000 \cdots \\
\cdots 00000000000000000000000000311\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}0000000000000 \cdots \\
\cdots 0000000000000000000000000000031100\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}00000000 \cdots \\
\cdots 000000000000000000000000000000003110000\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}000 \cdots
\end{array}
$$

- Particles and antiparticles undergo pair-creations/annihilations.

- n-color BBS $= \widehat{sl}_{n+1}$-automaton $= \widehat{\mathrm{so}}_{2n+2}$-automaton in antipaticle-free sector.

- Soliton & scattering most naturally captured in quantum group framework.


Classical
integrable system
Ultradiscrete
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Nonlinear waves
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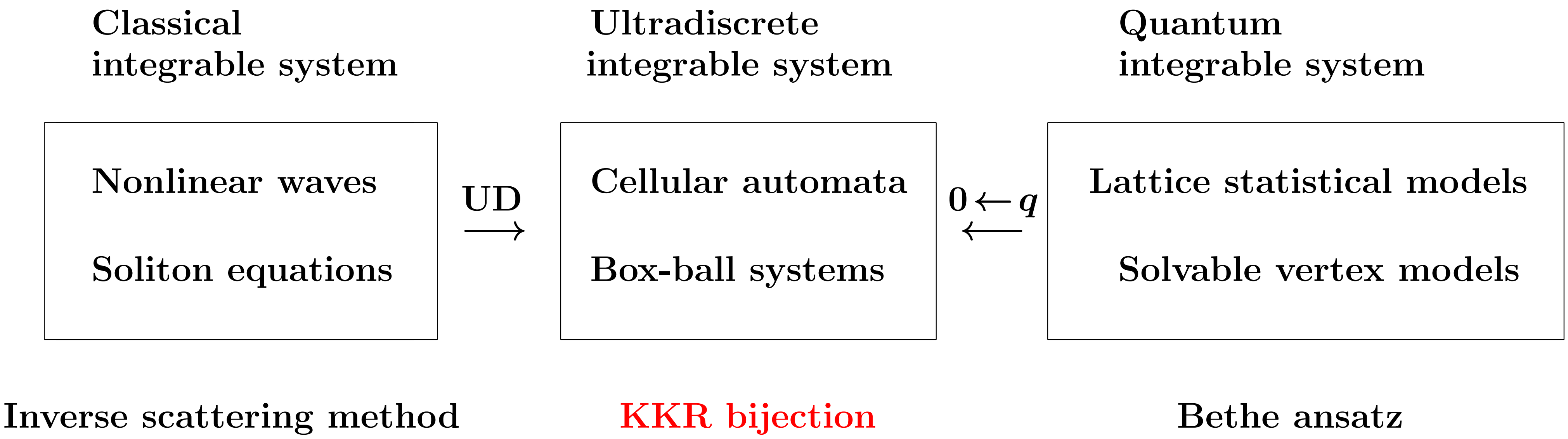
Cellular automata
Box-ball systems
0 ← q
Lattice statistical models
Solvable vertex models



Classical integrable system
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Inverse scattering method
KKR bijection
Bethe ansatz

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	$\overset{\text{UD}}{\longrightarrow}$	Cellular automata Box-ball systems	$\overset{0 \leftarrow q}{\longleftarrow}$	Lattice statistical models Solvable vertex models
Inverse scattering method		KKR bijection		Bethe ansatz

- **Kerov-Kirillov-Reshetikhin (KKR) bijection** (1986) asserts "formal completeness" of the hypothetical string solutions to the Bethe equation at combinatorial level.

- Its remarkable connection to BBS was discovered in 2002.

• **Example. Spin $\frac{1}{2}$ periodic Heisenberg chain**

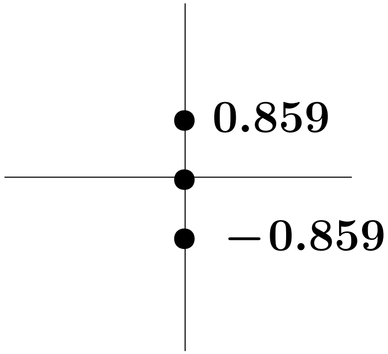
$$H = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)$$

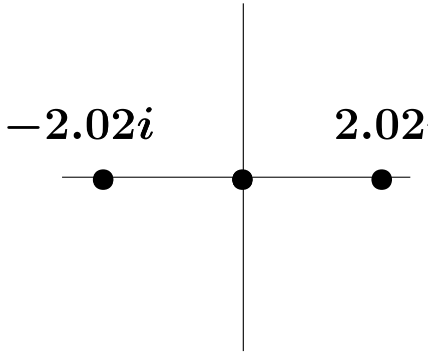
For $L = 6$ sites in 3 down-spin sector, the Bethe equation reads

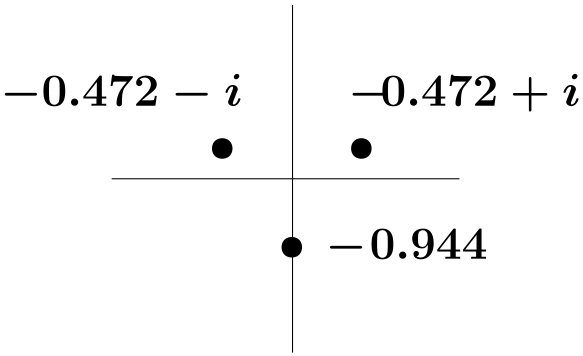
$$\left(\frac{u_1+i}{u_1-i}\right)^6 = \frac{(u_1-u_2+2i)(u_1-u_3+2i)}{(u_1-u_2-2i)(u_1-u_3-2i)},$$
$$\left(\frac{u_2+i}{u_2-i}\right)^6 = \frac{(u_2-u_1+2i)(u_2-u_3+2i)}{(u_2-u_1-2i)(u_2-u_3-2i)},$$
$$\left(\frac{u_3+i}{u_3-i}\right)^6 = \frac{(u_3-u_1+2i)(u_3-u_2+2i)}{(u_3-u_1-2i)(u_3-u_2-2i)}.$$

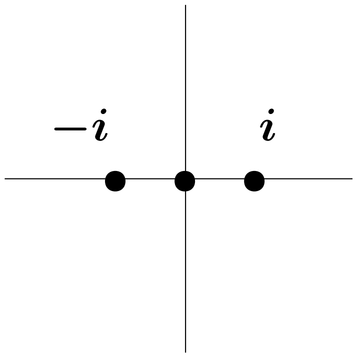
KKR bijection is a combinatorial analogue of

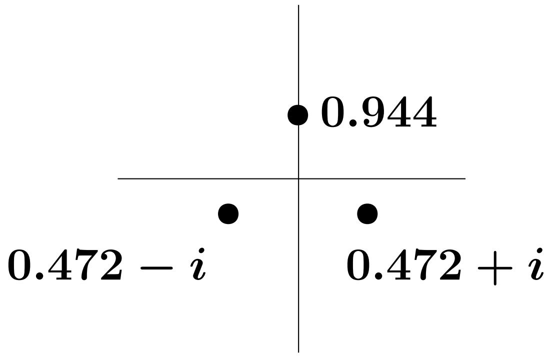
Bethe states $\leftrightsquigarrow$ Bethe roots

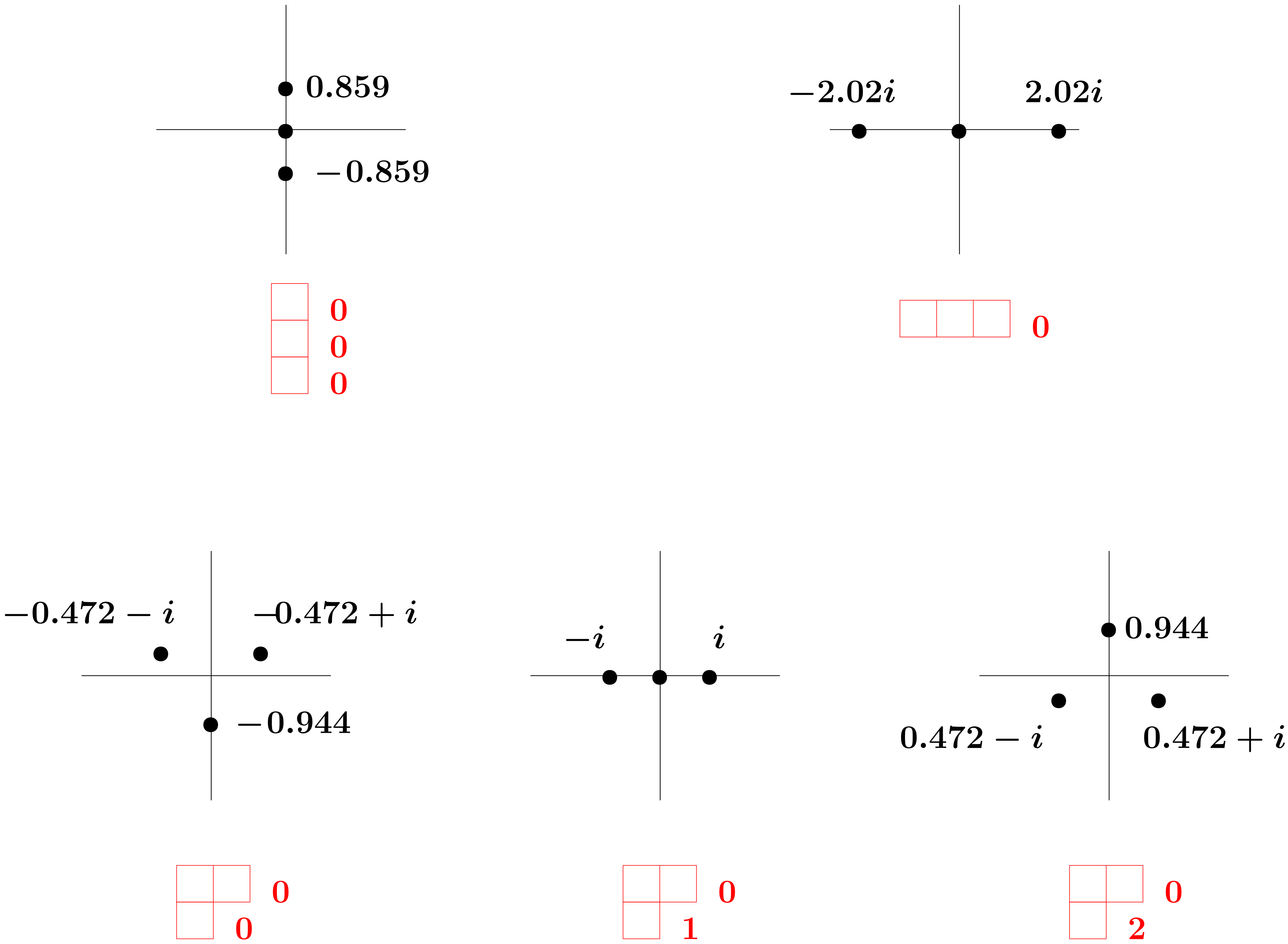

0.859
−0.859


−2.02i
2.02i


−0.472 − i
−0.472 + i
−0.944


−i
i


0.944
0.472 − i
0.472 + i



0.859
−0.859
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−2.02i
2.02i
0
−0.472 − i
−0.472 + i
−0.944
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−i
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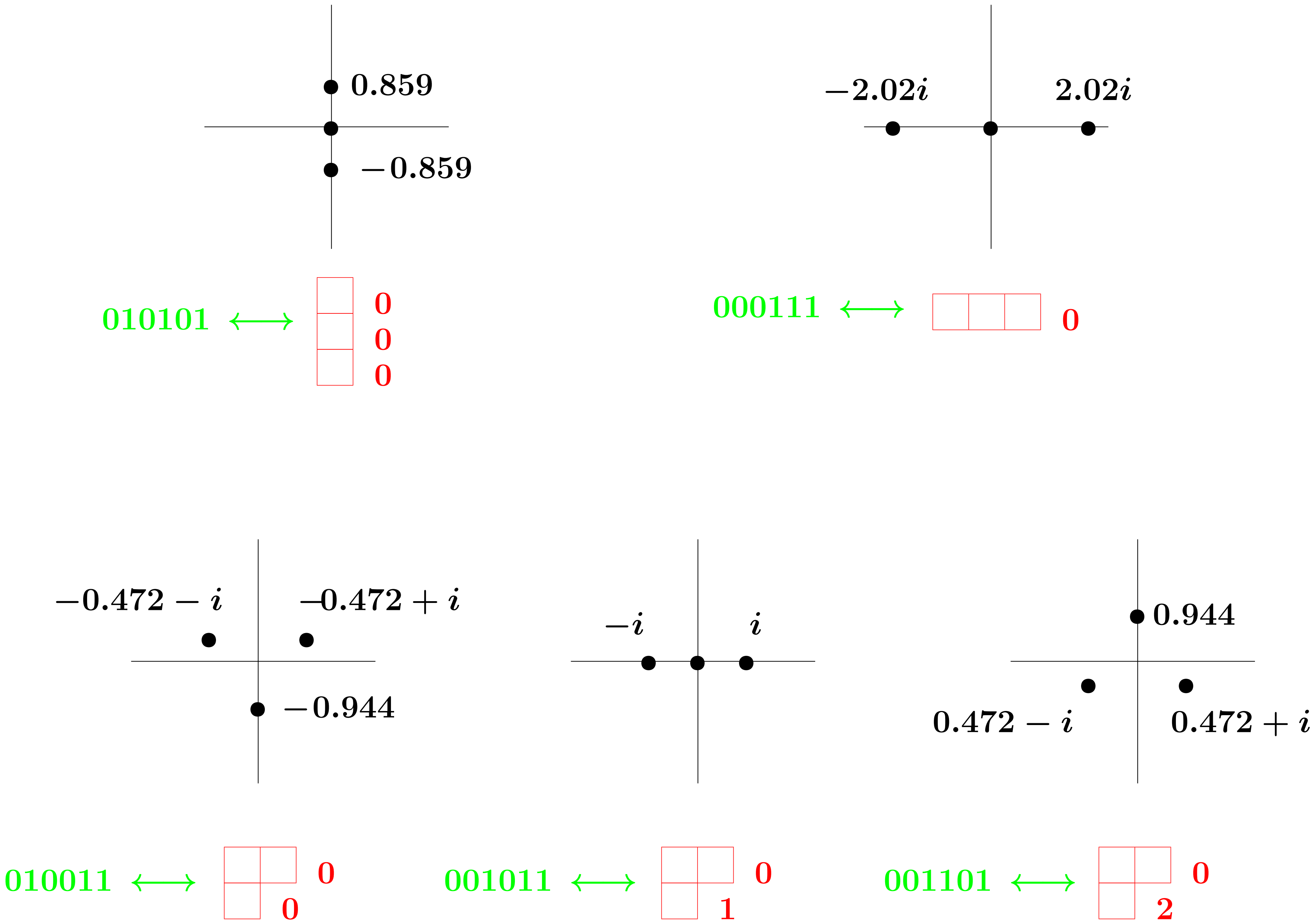
0.944
0.472 − i
0.472 + i
0
2



0.859
−0.859
010101 ⟷
0
0
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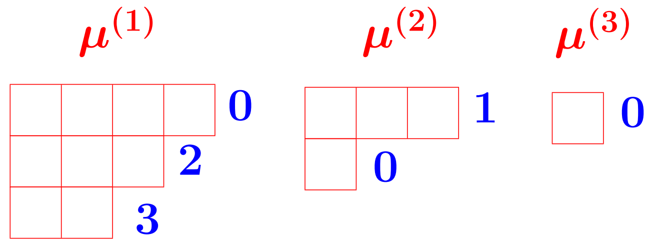
−2.02i
2.02i
000111 ⟷
0
−0.472 − i
−0.472 + i
−0.944
010011 ⟷
0
0
−i
i
001011 ⟷
0
1
0.944
0.472 − i
0.472 + i
001101 ⟷
0
2

KKR bijection for sl_{n+1}

$$\{\text{highest states}\} \overset{1:1}{\longleftrightarrow} \{\text{rigged configurations}\}$$

$n = 3$ example

$$00001110211322000 \quad \longleftrightarrow$$



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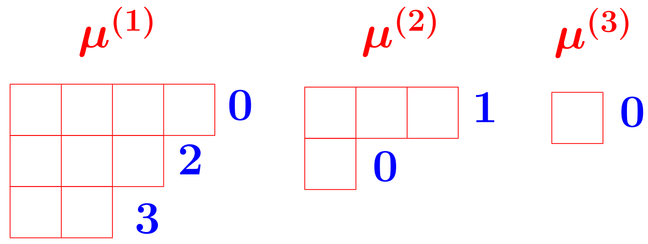

“Bethe vectors” — “Bethe roots”

Solitons — Strings (bound states of magnons)

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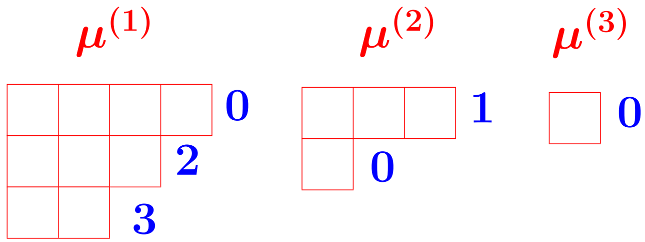

“Bethe vectors” — “Bethe roots”

Solitons — Strings (bound states of magnons)

- highest states $= i_1 i_2 \dots i_L$ $(0 \le i_k \le n)$ satisfying the highest condition:

$$\#_0\{i_1, \dots, i_k\} \ge \#_1\{i_1, \dots, i_k\} \ge \cdots \ge \#_n\{i_1, \dots, i_k\} \quad (\forall k)$$

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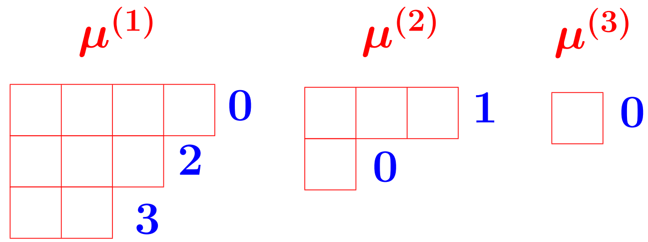

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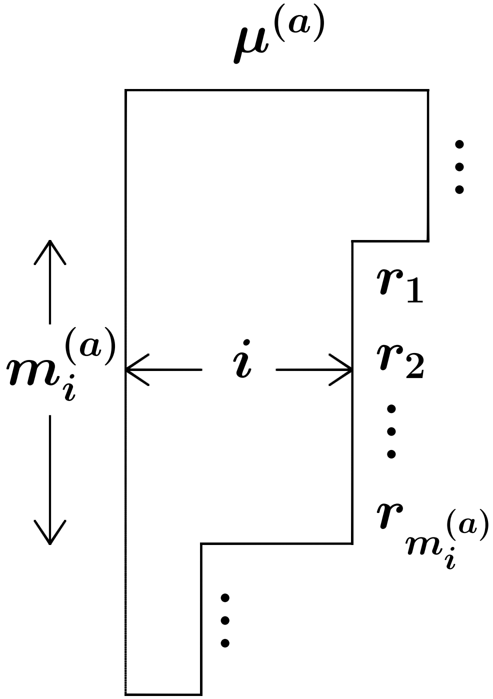
- rigged configuration: $((\mu^{(1)}, r^{(1)}), \ldots, (\mu^{(n)}, r^{(n)}))$

$$\left.\begin{array}{l} \mu^{(1)}, \ldots, \mu^{(n)} : \text{configuration} = n\text{-tuple of Young diagrams} \\ r^{(1)}, \ldots, r^{(n)} : \text{rigging} = \text{integers assigned to each row} \end{array}\right\} + \text{selection rule (next page)}$$

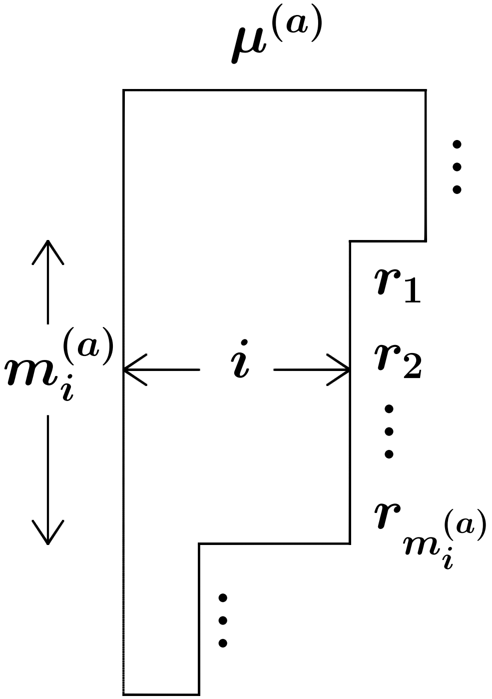


$$m_i^{(a)} = \#(\text{length } i \text{ rows in } \mu^{(a)}), \ \sum_{i\ge 1} i m_i^{(a)} = |\mu^{(a)}|$$

$$0 \le r_1 \le \cdots \le r_{m_i^{(a)}} \le h_i^{(a)}$$

$\cdots$ “Fermionic” selection rule

$$\#\text{ of rigging choices for a fixed configuration} = \prod_{a=1}^{n} \prod_{i\ge 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}$$



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$$h_i^{(a)} = L\delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j\ge 1} \min(i,j) m_j^{(b)}$$

$\cdots$ vacancy for *holes*

$$C_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1}$$

(C_{ab}) $\cdots$ Cartan matrix of sl_{n+1}

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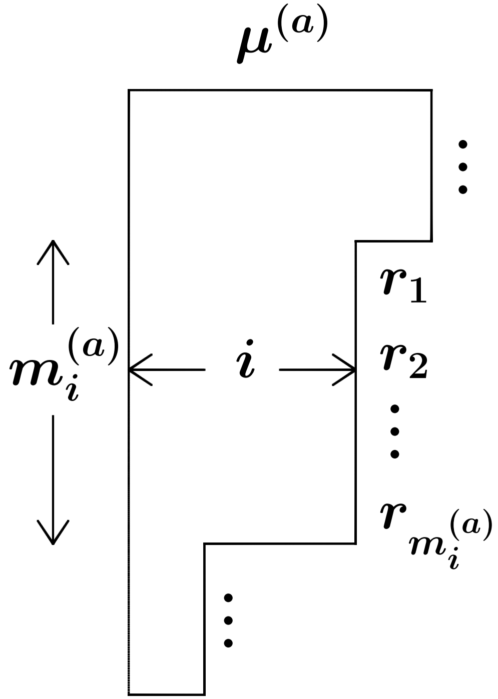

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This is an sl_{n+1} generalization of Bethe's formula for # of string solutions (1931).

hat also eine Möglichkeit weniger, die des letzten Komplexes von n Wellen, λ_{q_n}, kann schließlich nur noch

$$Q'_n - (q_n - 1) = Q_n + 1$$

verschiedene Werte annehmen, wo

$$Q_n(N, q_1 q_2 \ldots) = N - 2 \sum_{p<n} p\, q_p - 2 \sum_{p \geqq n} n\, q_p. \tag{44}$$

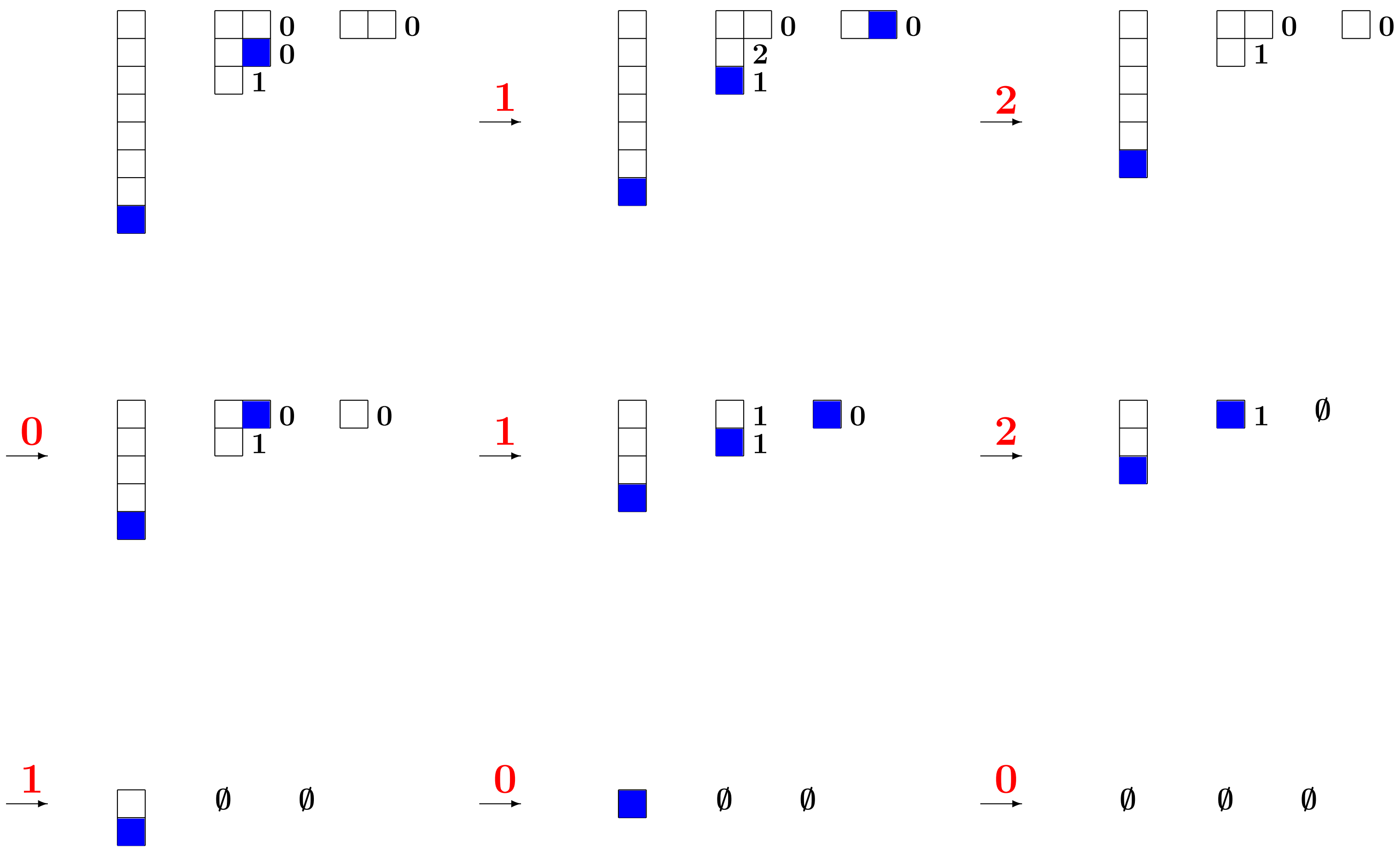
Schließlich ist zu berücksichtigen, daß Vertauschung der λ der verschiedenen Wellenkomplexe mit gleicher Anzahl n von Wellen nicht zu neuen Lösungen führt. Die gesamte Zahl unserer Lösungen wird somit

$$z(N, q_1 q_2 \ldots) = \prod_{n=1}^{\infty} \frac{(Q_n + q_n) \ldots (Q_n + 1)}{q_n!} = \prod_n \binom{Q_n + q_n}{q_n}, \tag{45}$$

wo die Q_n durch (44) definiert sind.

8. Wir werden nun nachweisen, daß wir die richtige Anzahl Lösungen erhalten haben.

Example of KKR algorithm



Top left rigged configuration $\xrightarrow{\text{KKR}}$ **00121021**

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000

$t = 1$: 000000001111000022103200000000000000000000000000

$t = 2$: 000000000000111100022132000000000000000000000000

$t = 3$: 000000000000000011110021322000000000000000000000

$t = 4$: 000000000000000000001110211322000000000000000000

$t = 5$: 000000000000000000000001100211322100000000000000

$t = 6$: 000000000000000000000000011000211032210000000000

$t = 7$: 000000000000000000000000000110000211003221000000

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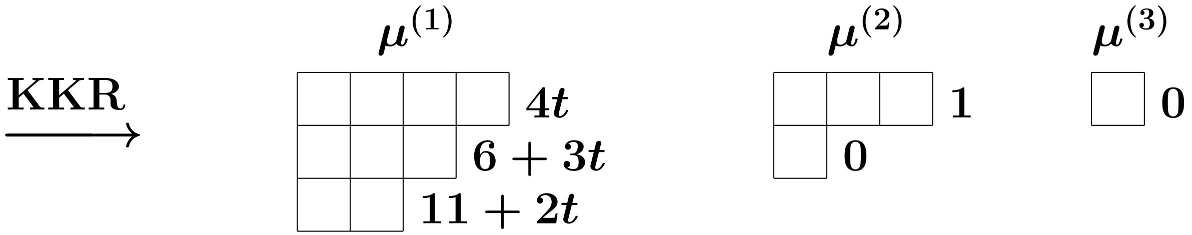

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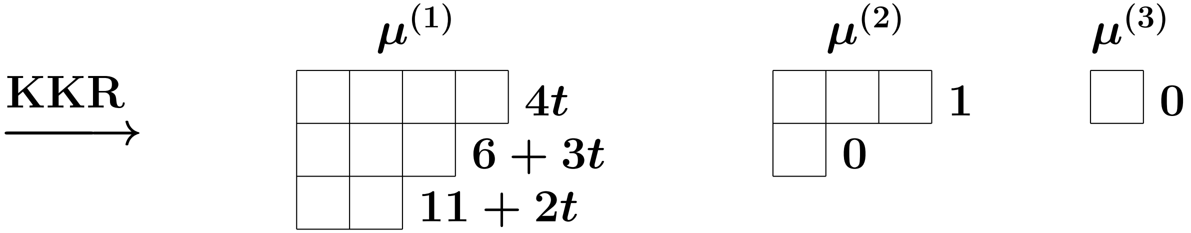

- **Configuration is conserved (action variable)**
- **Rigging flows linearly (angle variable)**
- **KKR bijection linearizes the dynamics (direct/inverse scattering map)**

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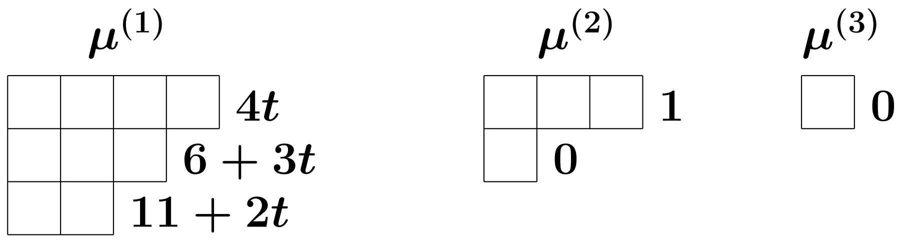
$t = 6$: 000000000000000000000000011000211032210000000000

$t = 7$: 000000000000000000000000000110000211003221000000

$\xrightarrow{\text{KKR}}$

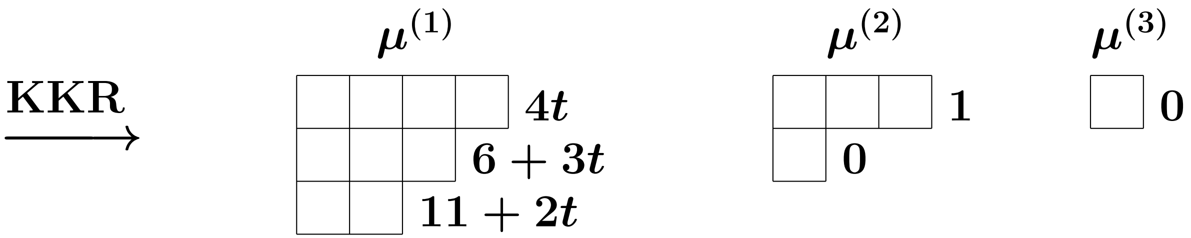


- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

Rigged configuration = action angle variable of BBS!

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 00001111000002210032000000000000000000000000000000
$t = 1$: 0000000011110000221032000000000000000000000000000
$t = 2$: 000000000000111100022132000000000000000000000000
$t = 3$: 000000000000000011110021322000000000000000000000
$t = 4$: 000000000000000000001110211322000000000000000000
$t = 5$: 000000000000000000000001100211322100000000000000
$t = 6$: 000000000000000000000000011000211032210000000000
$t = 7$: 000000000000000000000000000110000211003221000000



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Rigged configuration = action angle variable of BBS!

In particular, $\mu^{(1)}$ = list of amplitude of solitons.

$(\mu^{(1)}, \ldots, \mu^{(n)})$ will be called a soliton content (= generalized amplitude).

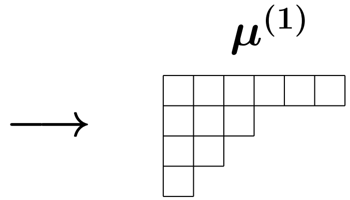
Quiz: What are the soliton contents or at least their amplitude?

11011110011100011100000000000000000000000000000

21022100122001221100000000000000000000000000000

Quiz: What are the soliton contents or at least their amplitude?

110111100111000111000000000000000000000000000000
001000011000111000111111000000000000000000000000
000100000110000111000000111111000000000000000000
000010000001100000111000000000111111000000000000



210221001220012211000000000000000000000000000000
002100220112201000221100000000000000000000000000
000021002001120221000022110000000000000000000000
000000210200012100221000002211000000000000000000
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000000000012200100210000221000000022110000000000
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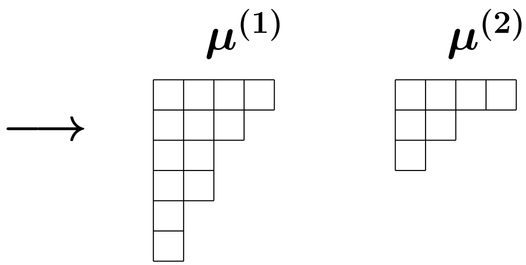

Randomized box-ball system

BBS state **Soliton content**

$$i_1 i_2 \ldots i_L 00000 \ldots \quad \overset{\text{KKR}}{\longmapsto} \quad (\mu^{(1)}, \ldots, \mu^{(n)})$$

Randomize $i_1 i_2 \ldots i_L$ by introducing the i.i.d. measure on the set of states:

$$\{0, 1, \ldots, n\} \to (0, 1); \qquad i \mapsto p_i \qquad (p_0 + \cdots + p_n = 1).$$

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Determine the scaling form of the most probable $(\mu^{(1)}, \ldots, \mu^{(n)})$ when $L \to \infty$.

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Limit shape Problem

Determine the scaling form of the most probable $(\mu^{(1)}, \ldots, \mu^{(n)})$ when $L \to \infty$.

This can be done by TBA minimizing the free energy F associated with

$$\mathrm{Prob}(\mu^{(1)}, \ldots, \mu^{(n)}) \simeq Z_L^{-1} \mathrm{e}^{-\beta_1 |\mu^{(1)}| - \cdots - \beta_n |\mu^{(n)}|} \prod_{a=1}^{n} \prod_{i \geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}},$$

$$\mathrm{e}^{\beta_a} := p_{a-1}/p_a, \qquad Z_L = \text{normalization const (partition function)}.$$

Introduce the scaled string and hole densities $\rho_i^{(a)}, \sigma_i^{(a)}$ by

$$m_i^{(a)} \simeq L\rho_i^{(a)}, \quad h_i^{(a)} \simeq L\sigma_i^{(a)}, \quad \sigma_i^{(a)} = \delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i, j) \rho_j^{(b)},$$

Assume $p_0 \geq \cdots \geq p_n$ in the rest.

The condition $\frac{\delta F}{\delta \rho_i^{(a)}} = 0$ leads to the TBA equation

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \log(1 + (Y_j^{(b)})^{-1})$$

in terms of $Y_i^{(a)} = \frac{\sigma_i^{(a)}}{\rho_i^{(a)}}$ with the boundary condition $\lim_{i\to\infty} \frac{1 + Y_{i+1}^{(a)}}{1 + Y_i^{(a)}} = \mathrm{e}^{\beta_a}$.

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This is equivalent to the (constant) Y-system

$$\left(Y_i^{(a)}\right)^2 = \frac{(1 + Y_{i-1}^{(a)})(1 + Y_{i+1}^{(a)})}{\left(1 + (Y_i^{(a-1)})^{-1}\right)\left(1 + (Y_i^{(a+1)})^{-1}\right)}$$

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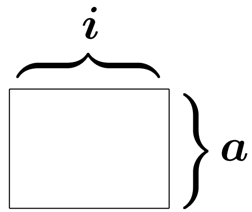
Solution (Rare case for which an exact formula can be given)

$$Y_i^{(a)} = \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{Q_i^{(a-1)} Q_i^{(a+1)}},$$

$$Q_i^{(a)} = Q_i^{(a)}(p_0, \ldots, p_n) = \frac{\det(p_k^{\lambda_j + n - j})_{j,k=0}^{n}}{\det(p_k^{n-j})_{j,k=0}^{n}} \qquad \big((\lambda_0, \ldots, \lambda_n) = (\overbrace{i \ldots i}^{a}\ \overbrace{0, \ldots, 0}^{n+1-a}) \big)$$

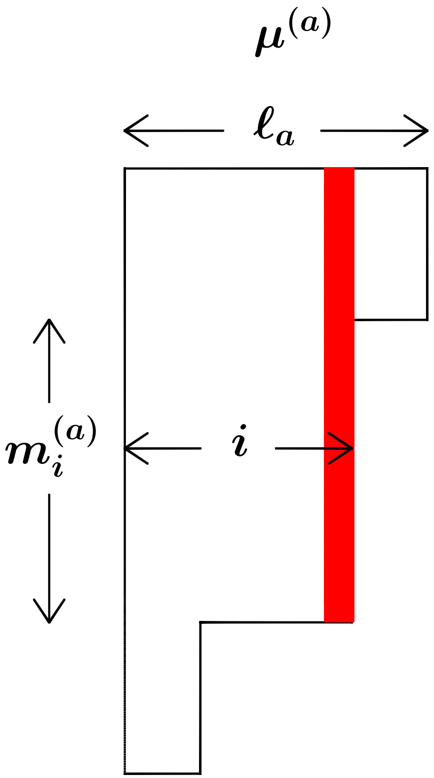
$=$ Schur function for $a \times i$ rectangular Young diagram



Result. The limit shape of soliton content $(\mu^{(1)}, \ldots, \mu^{(n)})$ is given by

$$\eta_i^{(a)} := \lim_{L\to\infty} \frac{1}{L}(\text{Length of the } i \text{ th column of } \mu^{(a)}) = \frac{Q_{i-1}^{(a-1)} Q_i^{(a+1)}}{Q_i^{(a)} Q_{i-1}^{(a)} Q_1^{(1)}}$$

$$\text{width } \ell_a \text{ of } \mu^{(a)} \simeq \frac{\log L}{\log \frac{p_{a-1}}{p_a}} \quad (L \to \infty \text{ if } p_0 > \cdots > p_n)$$



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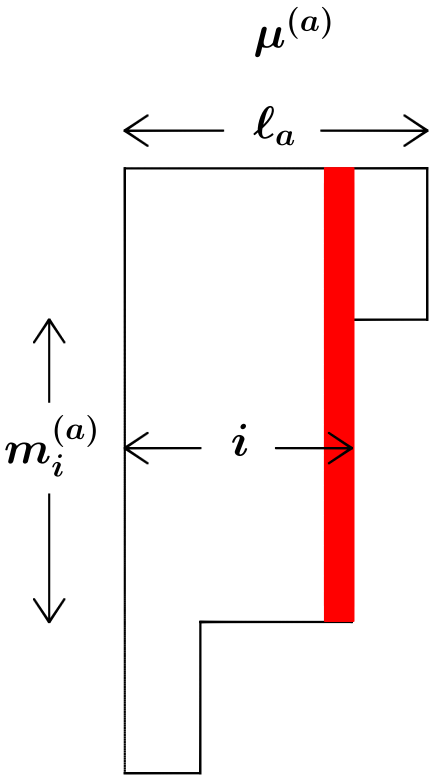
Special case $p_a = \frac{q^a}{1+q+\cdots+q^n}$ $(0 < q \leq 1)$.



Scaled column length of $\mu^{(a)}$

$$\eta_i^{(a)} = \frac{q^{i+a-1}(1-q)(1-q^a)(1-q^{n+1-a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})}$$

Strings

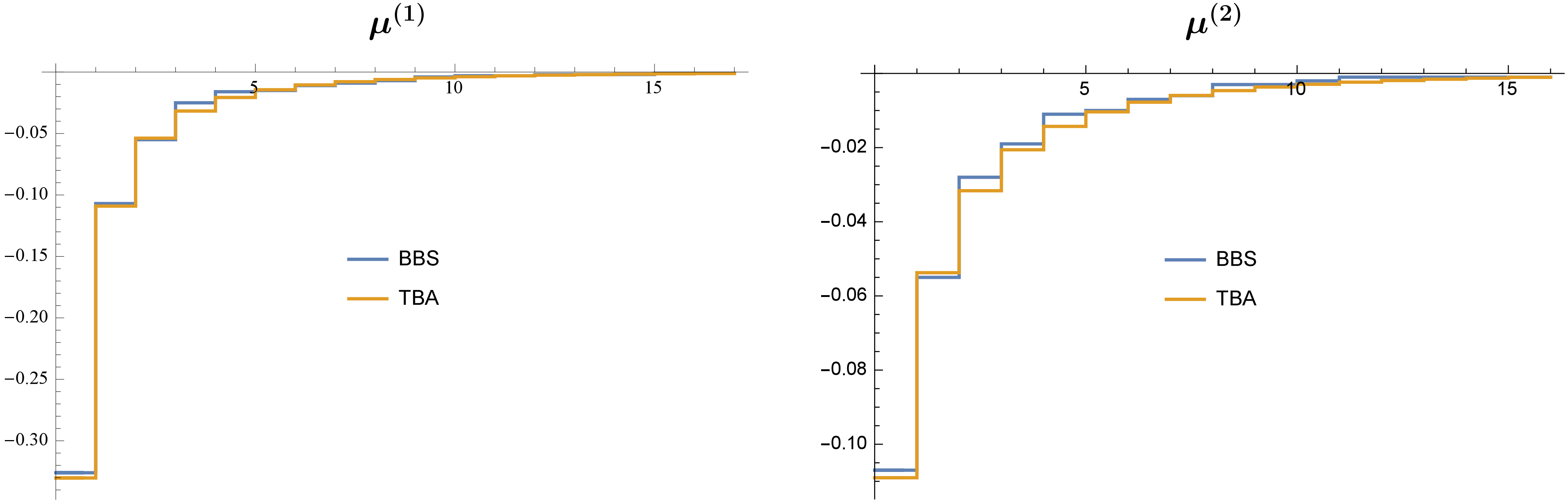
$$\rho_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} m_i^{(a)} = \frac{q^{i+a-1}(1-q)^2(1-q^a)(1-q^{n+1-a})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

Holes

$$\sigma_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} h_i^{(a)} = \frac{q^{a-1}(1-q)^2(1-q^i)(1-q^{n+i+1})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

2-color BBS with $L = 1000$ sites with distribution $(p_0, p_1, p_2) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$.

Vertically L^{-1} scaled soliton contents.



Generalized hydrodynamics, GHD (from here 1-color BBS only)

[Castro-Alvaredo, Doyon, Yoshimura, Bertini, Collura, De Nardis, Fagotti, ... 2016∼]

Densities: ρ_i (amplitude i-solitons), σ_i (hole)

Bethe equation: $\sigma_i = 1 - 2\sum_j \min(i,j)\rho_j$

Speed equation: $v_i = i + \sum_j 2\min(i,j)(v_i - v_j)\rho_j$ $\left(\begin{array}{c} v_i = \text{effective speed} \\ 2\min(i,j) = \text{phase shift} \end{array}\right)$

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Solution for homogeneous BBS:

$$v_i = i\frac{1+q}{1-q} - \frac{2q(1+q)(1-q^i)}{(1-q)^2(1+q^{i+1})}, \qquad (\rho_i, \sigma_i) = (\rho_i^{(1)}, \sigma_i^{(1)}) \text{ in previous pages}$$

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One of the central ideas in GHD:

TBA Y-function $Y_i := \frac{\sigma_i}{\rho_i}$ satisfies the separated equation $\frac{\partial Y_i}{\partial t} + v_i \frac{\partial Y_i}{\partial x} = 0$, and plays the role of normal mode in the Euler-scale hydrodynamics of an "integrable fluid".

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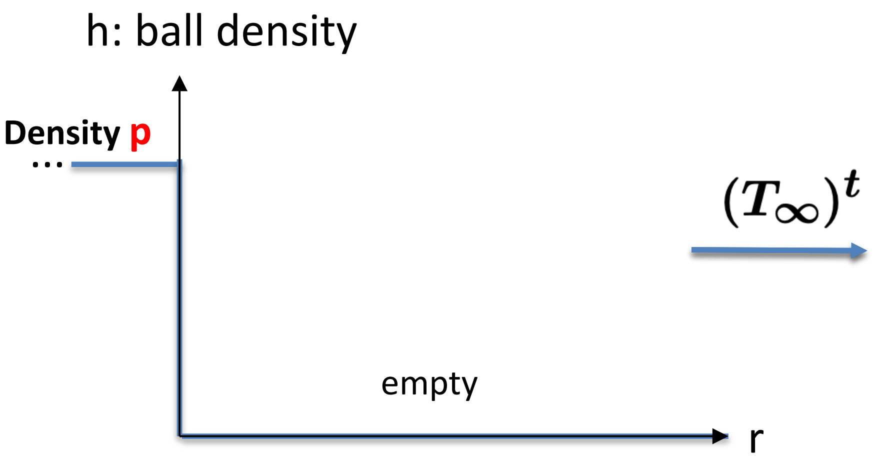
We apply it to the Riemann problem of the BBS soliton gas:

"Describe the profile of the fluid starting from the *homogeneous* initial state except a *single discontinuity* at some point."

Density Plateaux emerging from domain wall initial condition



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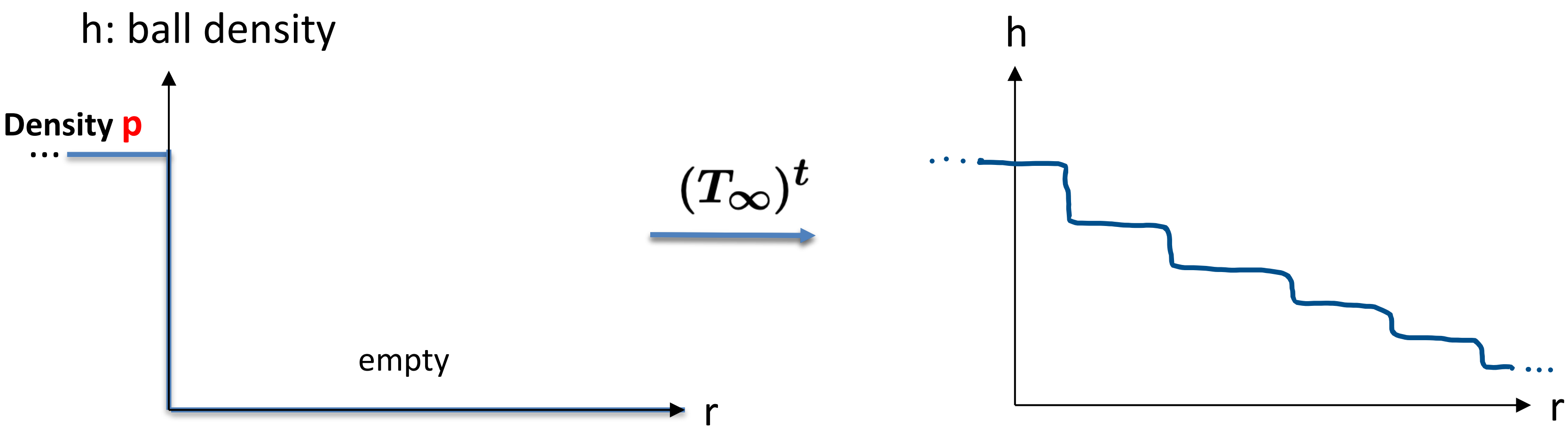

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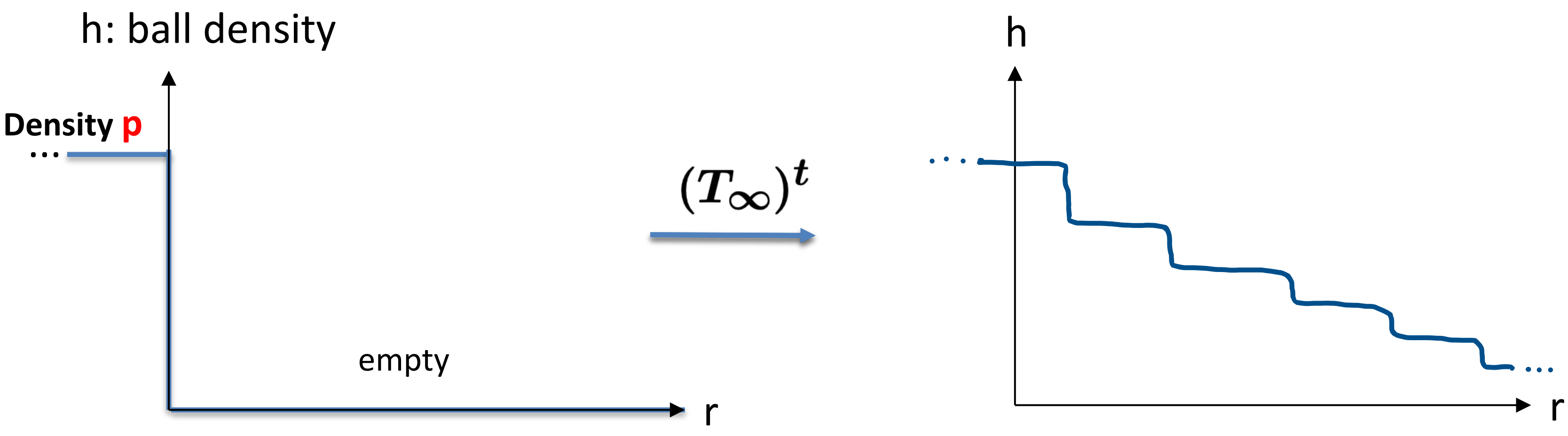

Plateaux broaden linearly in time t. The plot against $\zeta = r/t$ collapses into a single curve.

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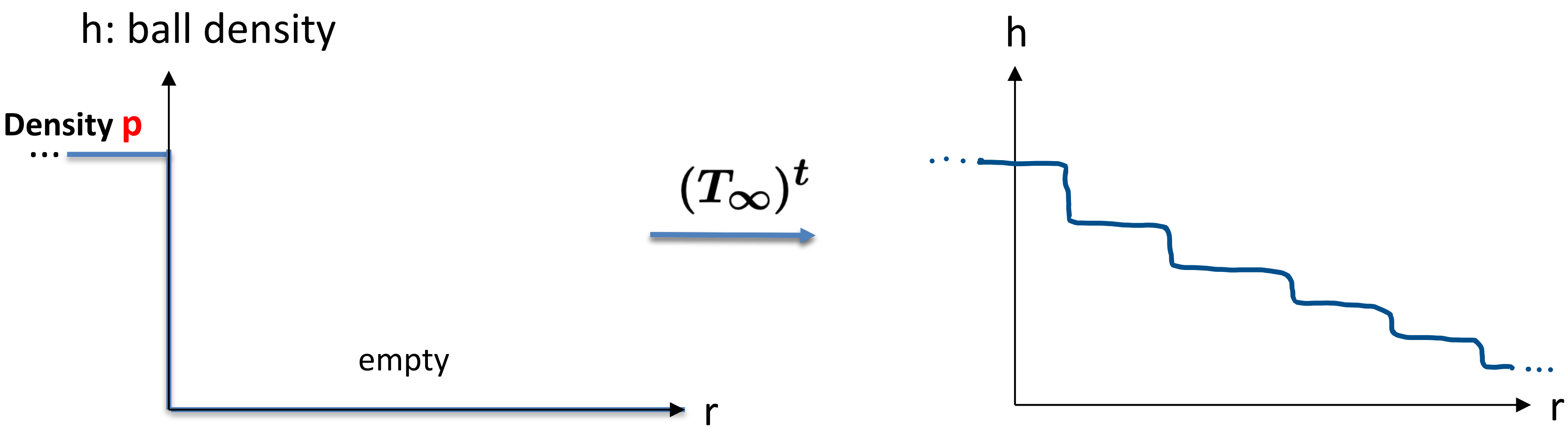

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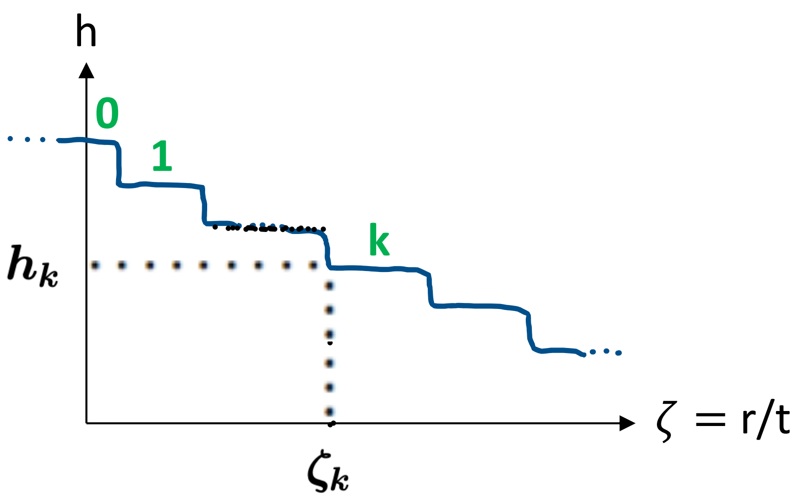

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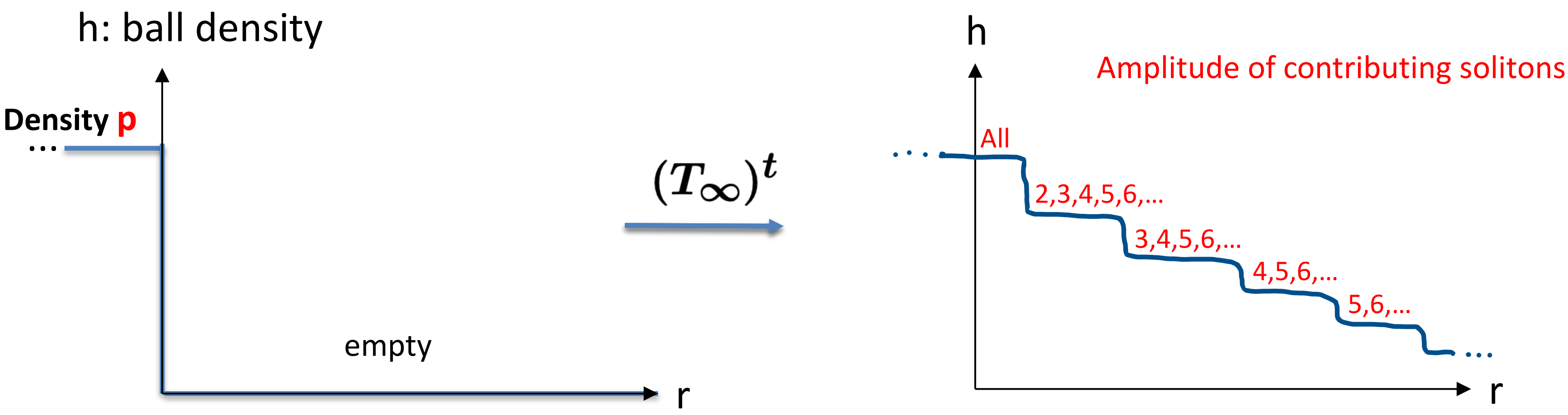

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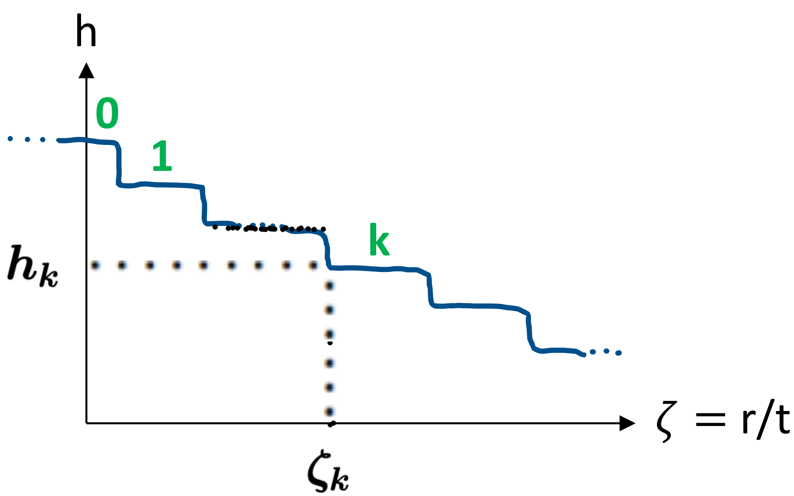

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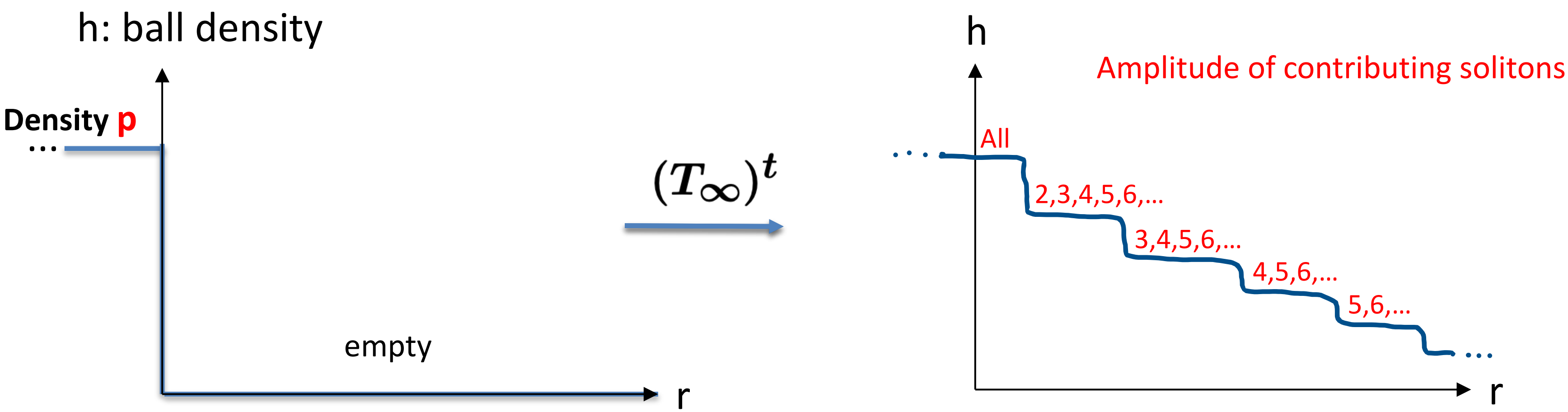

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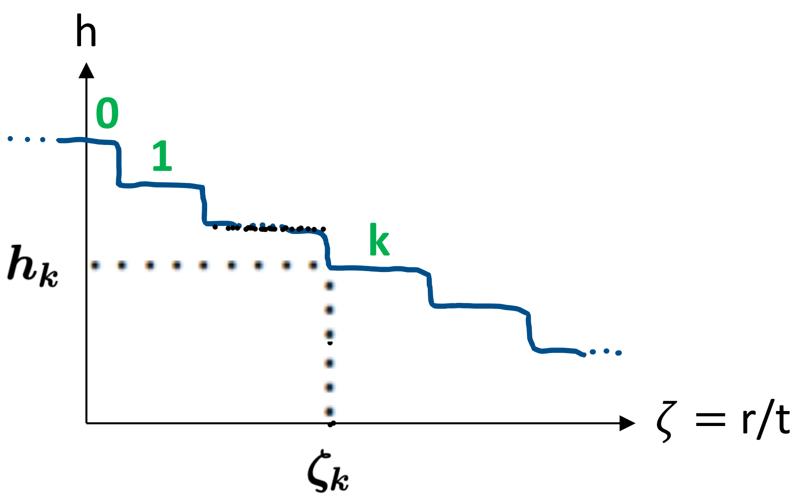


Generalized hydrodynamics (GHD) predicts

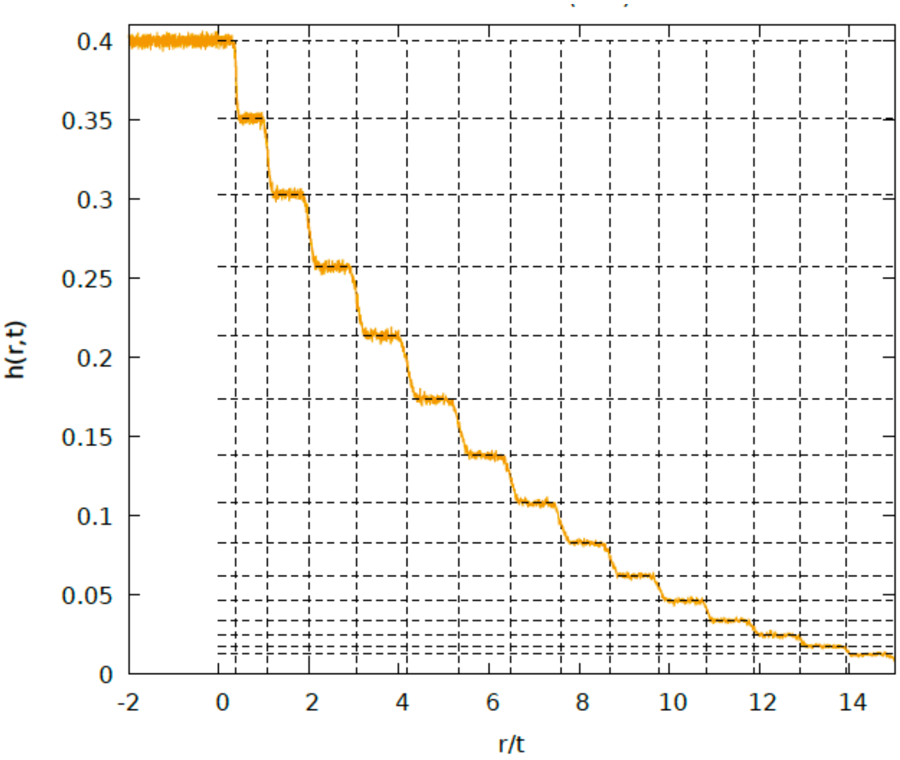
$$h_k = \frac{q^{k+1}(1 - q^{k+2} + k(1-q))}{1 - q^{2k+3} + (2k+1)(1-q)q^{k+1}}$$

$$\zeta_k = \frac{k(1-q^{k+1})}{1+q^{k+1}} \qquad \left(p = \frac{q}{1+q}\right)$$

Simulation with $N_{samples}$ = 50000

(Plots of ball density vs $\zeta = r/t$. Dotted lines are GHD predictions)

p=0.4, q=0.666…, t=500.



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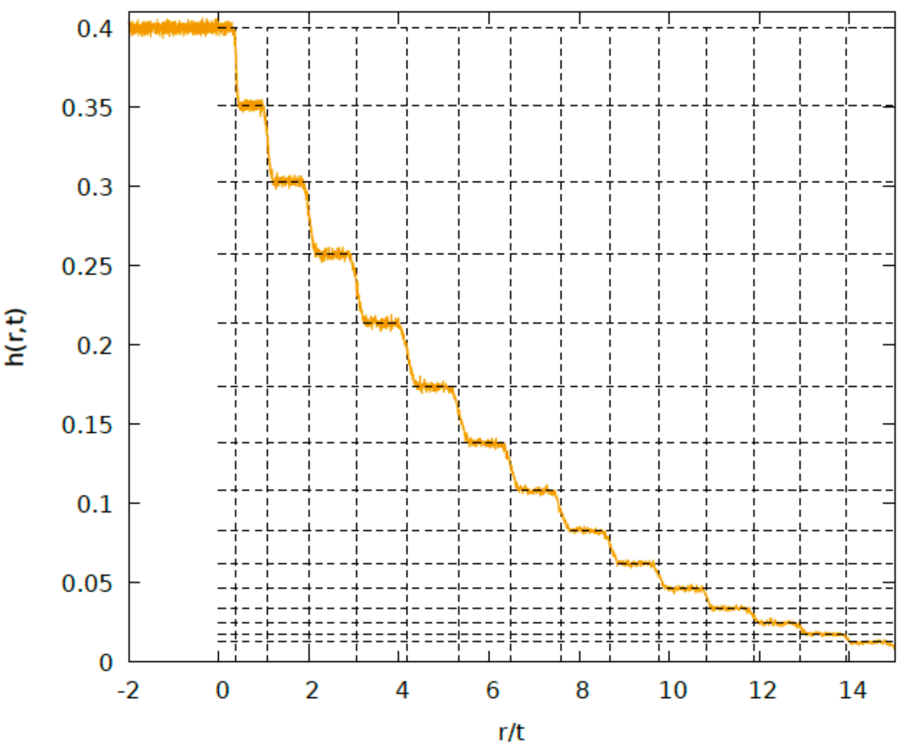


Actual plateau edges are not strict and exhibit some **broadening**.

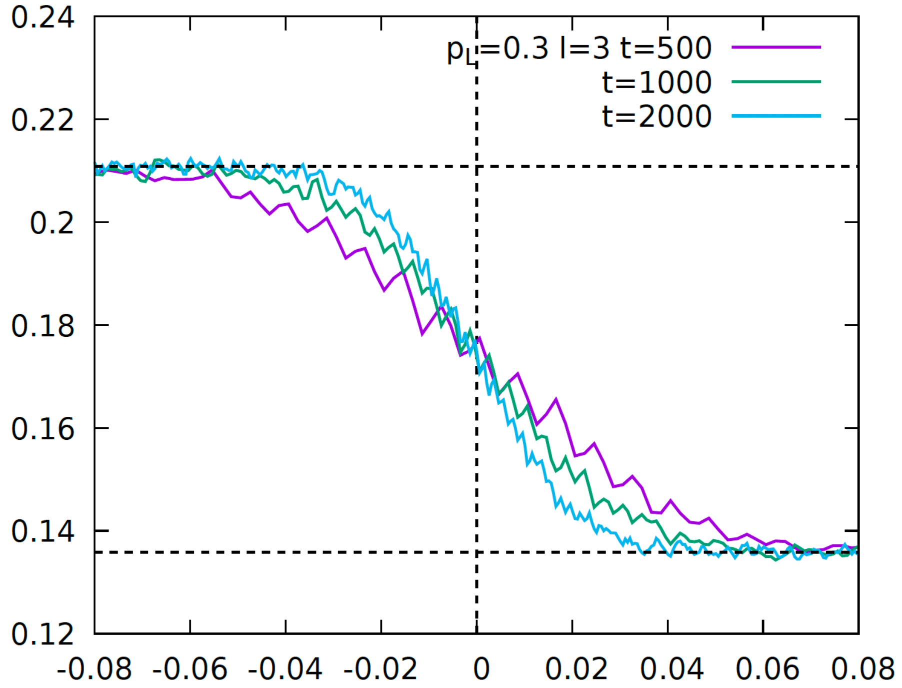
This is due to **diffusive** correction to the **ballistic** picture,
which may be viewed as a finite t effect.

Analytical description of the **diffusive broadening** of plateau edges

Position of plateau edge - ζ

fluctuates over the scale

$$\frac{1}{\sqrt{(\textbf{Diffusion const})t}}$$



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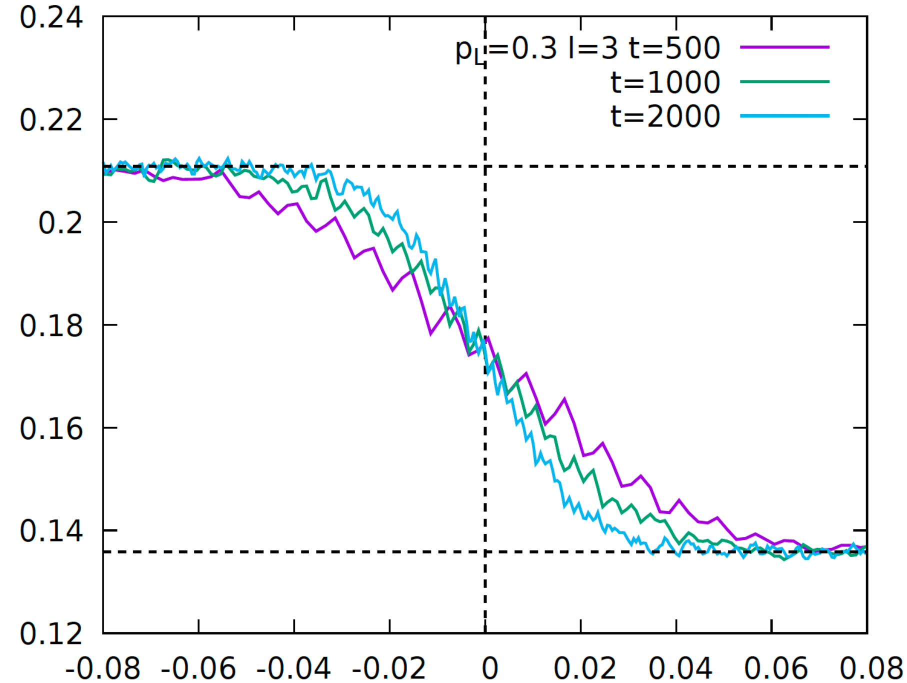

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \mathrm{e}^{-s^2} ds$$

$$\langle \rho_j(r,t) \rangle = \frac{1}{2}\left(\rho_j(k-1) - \rho_j(k)\right) \mathrm{erfc}\left(\frac{r - \zeta(k)t}{\sqrt{2t}\,\Sigma_k^{(l)}}\right) + \rho_j(k)$$

amplitude j-soliton density around the k th plateau edge $r = \zeta(k)t$ under the time evolution T_l.

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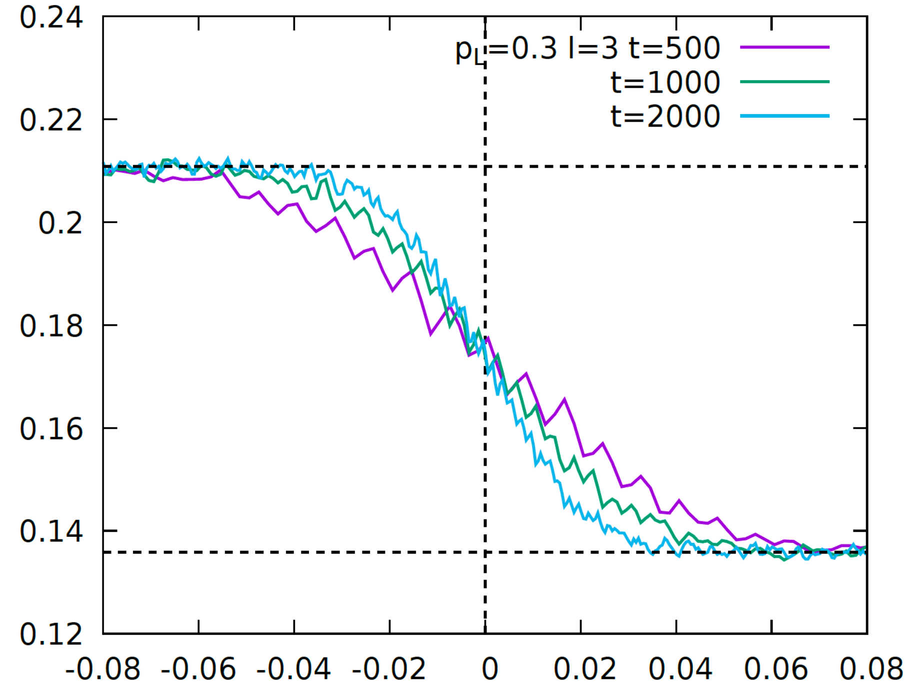

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← GHD (Bethe ansatz)

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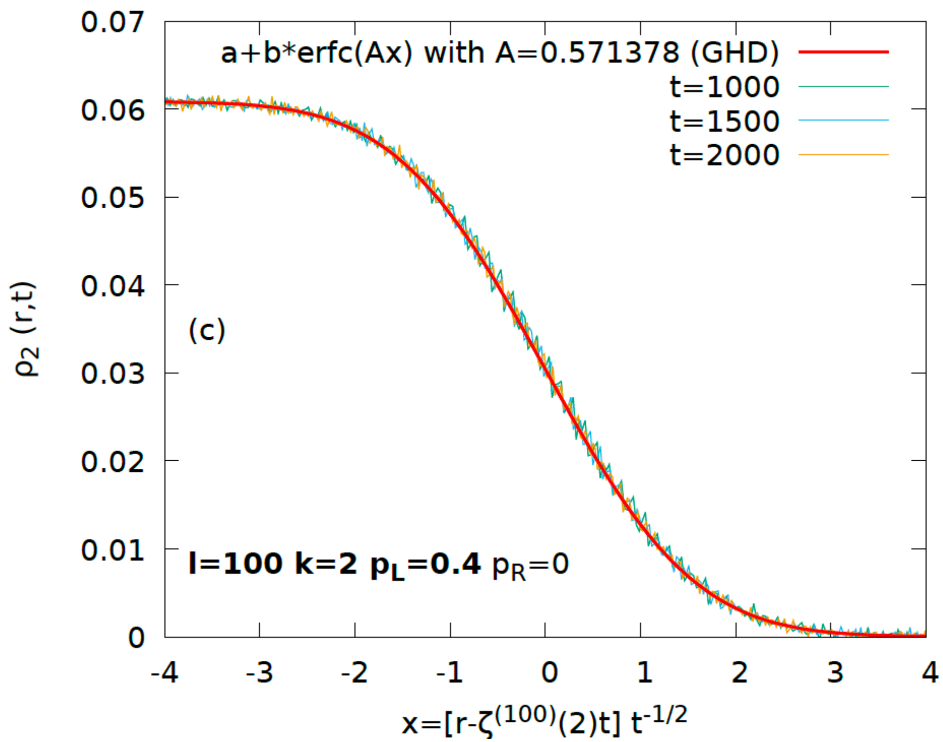

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← GHD (Bethe ansatz)

Summary

1. BBS is a Yang-Baxter integrable cellular automaton with explicit action-angle variables originating in Bethe strings.

2. Limit shape of soliton content in randomized BBS is determined by TBA.

3. Density plateaux emerging from domain wall initial condition is analytically described by GHD.

Reference

Review part:

R.Inoue, AK and T.Takagi
"Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry", JPA Topical Review (2012), arXiv:1109.5349.

Limit shape problem:

AK, H.Lyu and M.Okado
"Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz", NPB(2018), arXiv:1808.02626.

Generalized hydrodynamics of BBS:

AK, G.Misguich and V.Pasquier
"Generalized hydrodynamics in box-ball system", JPA(2020), arXiv:2004.01569.

Complete BBS for higher rank case, in preparation.