Multispecies TASEP and the tetrahedron equation

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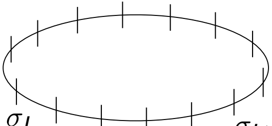
Totally Asymmetric Simple Exclusion Process (TASEP)



1D periodic chain with L sites

$\sigma_i \in \{0, 1, \ldots, n\}$ (n-TASEP)

Stochastic dynamics

$$(\sigma_i, \sigma_{i+1}) \;\to\; (\sigma'_i, \sigma'_{i+1})$$

$$(\alpha, \beta) \;\to\; (\beta, \alpha) \quad \text{if } \alpha > \beta$$

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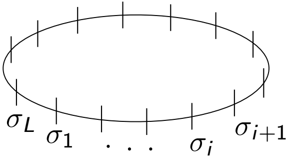

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Master equation

$$\frac{d}{dt}|P\rangle = H|P\rangle, \quad |P\rangle = \sum_{\{\sigma_i\}} \mathbb{P}(\sigma_1, \dots, \sigma_L)|\sigma_1, \dots, \sigma_L\rangle \in (\mathbb{C}^{n+1})^{\otimes L}$$

$$H = \sum_{i \in \mathbb{Z}_L} h_{i.i+1}, \quad h_{i,i+1} = 1 \otimes \cdots \otimes 1 \otimes \overset{i,i+1}{h} \otimes 1 \otimes \cdots \otimes 1$$

$$h|\alpha, \beta\rangle = \begin{cases} |\beta, \alpha\rangle - |\alpha, \beta\rangle & (\alpha > \beta), \\ 0 & (\alpha \leq \beta). \end{cases}$$

Sectors and Steady states

Sectors labeled by multiplicities $\mathbf{m} = (m_0, \ldots, m_n) \in \mathbb{Z}_+^{n+1}$:

$$S(\mathbf{m}) = \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_L) \in \{0, \ldots, n\}^L \mid \#_k(\boldsymbol{\sigma}) = m_k\}.$$

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$\exists$Unique (up to overall) $|\bar{P}(\mathbf{m})\rangle \in \sum_{\boldsymbol{\sigma} \in S(\mathbf{m})} \mathbb{Z}_+ |\boldsymbol{\sigma}\rangle$ s.t. $H|\bar{P}(\mathbf{m})\rangle = 0$.

$$\begin{aligned}
|\bar{P}(1,1,1)\rangle &= 2|012\rangle + |021\rangle + |102\rangle + 2|120\rangle + 2|201\rangle + |210\rangle, \\
|\bar{P}(2,1,1)\rangle &= 3|0012\rangle + |0021\rangle + 2|0102\rangle + 3|0120\rangle + 2|0201\rangle + |0210\rangle \\
&\quad + |1002\rangle + 2|1020\rangle + 3|1200\rangle + 3|2001\rangle + 2|2010\rangle + |2100\rangle, \\
|\bar{P}(1,2,1)\rangle &= 2|0112\rangle + |0121\rangle + |0211\rangle + |1012\rangle + |1021\rangle + |1102\rangle \\
&\quad + 2|1120\rangle + 2|1201\rangle + |1210\rangle + 2|2011\rangle + |2101\rangle + |2110\rangle.
\end{aligned}$$

Steady states are non-trivial for $n \geq 2$.

$$|\bar{P}(\mathbf{m})\rangle = \sum_{\boldsymbol{\sigma} \in S(\mathbf{m})} \mathbb{P}(\boldsymbol{\sigma})|\boldsymbol{\sigma}\rangle$$

Results on steady state probability $\mathbb{P}(\boldsymbol{\sigma})$

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- Matrix product formulas: Evans-Ferrari-Mallick (2009), ...

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Results on steady state probability $\mathbb{P}(\boldsymbol{\sigma})$

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- Matrix product formulas: Evans-Ferrari-Mallick (2009), ...
- Today's talk: Hidden 3D structure related to the tetrahedron eq.
  - Ferrari-Martin algorithm = composition of **combinatorial R**
  - **Corner transfer matrix** of **q-oscillator valued 5V model** at $q = 0$
  - New matrix product formula:
    $\mathbb{P}(\boldsymbol{\sigma})$ = **Partition function** of a 3D integrable lattice model

Ferrari-Martin algorithm: consists of queuing process

Sector $S(\mathbf{m})$ with multiplicity $\mathbf{m} = (m_0, \dots, m_n)$ with $\forall m_i > 0$.

$$s_i := m_{n-i+1} + \cdots + m_{n-1} + m_n, \quad 0 < s_1 < \cdots < s_n < L,$$

$$B^s := \{\mathbf{b} = (b_1, \dots, b_L) \in \{0,1\}^L \mid b_1 + \cdots + b_L = s\},$$

$$\text{e.g. } B^1 = \{100, 010, 001\}, \ \ B^2 = \{110, 101, 011\} \ \text{ for } L = 3,$$

$$\mathcal{B}(\mathbf{m}) := B^{s_1} \otimes \cdots \otimes B^{s_n}.$$

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$$\pi : \mathcal{B}(\mathbf{m}) \to S(\mathbf{m}) \quad \text{such that} \quad \mathbb{P}(\boldsymbol{\sigma}) = \#\left(\pi^{-1}(\boldsymbol{\sigma})\right).$$

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$L = 7$

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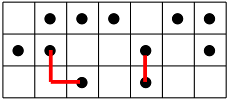

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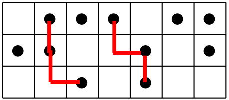

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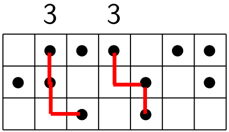

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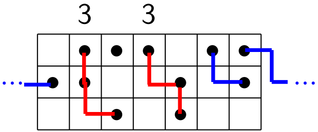

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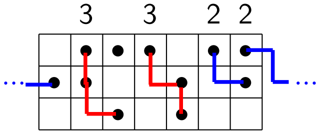

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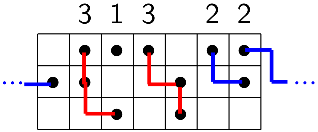

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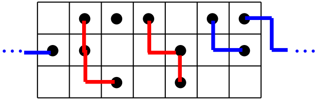

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$U_q(\widehat{sl}_L)$ and quantum R matrix

Degree s anti-symmetric tensor representation:

$$V^s = \bigoplus_{\mathbf{b} \in B^s} \mathbb{C}|\mathbf{b}\rangle, \qquad B^s = \text{crystal of } V^s.$$

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$$\mathcal{R}(z)\big(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle\big) = \sum_{\mathbf{a},\mathbf{b}} \mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} |\mathbf{b}\rangle \otimes |\mathbf{a}\rangle \quad (\mathbf{a} = (a_1,\ldots,a_L) \in B^s, \text{ etc}).$$

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$\exists$ Normalization s.t. $\mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} =$ *polynomials* in q and z.

Example. Nonzero elements of $\mathcal{R}^{1,2}(z)$ for $U_q(\widehat{sl}_3)$ are as follows.

$$\mathcal{R}^{100,110}_{100,110} = \mathcal{R}^{010,110}_{010,110} = \mathcal{R}^{100,101}_{100,101} = \mathcal{R}^{001,101}_{001,101} = \mathcal{R}^{010,011}_{010,011} = \mathcal{R}^{001,011}_{001,011} = 1 + q^3 z,$$
$$\mathcal{R}^{001,110}_{001,110} = \mathcal{R}^{010,101}_{010,101} = \mathcal{R}^{100,011}_{100,011} = q(1+qz),$$
$$\mathcal{R}^{001,110}_{010,101} = \mathcal{R}^{010,101}_{100,011} = z\mathcal{R}^{100,011}_{001,110} = -q(1-q^2)z,$$
$$\mathcal{R}^{001,110}_{100,011} = z\mathcal{R}^{010,101}_{001,110} = z\mathcal{R}^{100,011}_{010,101} = (1-q^2)z.$$

Matrix product formula for the quantum R

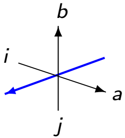
Theorem (K-Okado-Sergeev 2015)

$$\mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \varrho(z)\mathrm{Tr}\left(z^{\mathbf{h}}\mathcal{L}^{a_1,b_1}_{i_1,j_1}\cdots\mathcal{L}^{a_L,b_L}_{i_L,j_L}\right),$$

where $\mathbf{h}|m\rangle = m|m\rangle$, $\mathbf{a} = (a_1,\ldots,a_L) \in B^s$, *etc and* $\varrho(z) =$*known.*

$\mathcal{L} = (\mathcal{L}^{a,b}_{i,j})_{a,b,i,j=0,1} =$ **3D L-operator** (defined in the next page)

$\mathcal{L}^{a,b}_{i,j} =$



= operator on the Fock space

$F = \oplus_m \mathbb{C}|m\rangle$ (blue arrow)

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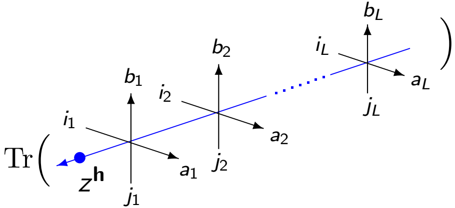
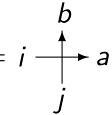
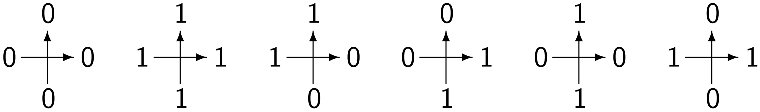


= BBQ stick with X shape sausages

3D L-operator = q-oscillator valued 6V

$$\mathcal{L} = (\mathcal{L}^{a,b}_{i,j}) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F), \quad \mathcal{L}^{a,b}_{i,j} =$$


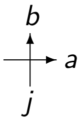
$$\in \mathrm{End}(F)$$



1	1	$\mathbf{a}^+$	$\mathbf{a}^-$	$\mathbf{k}$	$q\mathbf{k}$

3D L-operator $=$ q-oscillator valued 6V

$$\mathcal{L} = (\mathcal{L}^{a,b}_{i,j}) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F), \quad \mathcal{L}^{a,b}_{i,j} = $$



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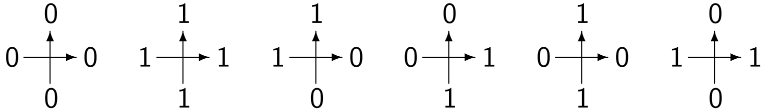

$1 \qquad 1 \qquad \mathbf{a}^+ \qquad \mathbf{a}^- \qquad \mathbf{k} \qquad q\mathbf{k}$

$\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ are q-oscillators acting on the Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = (-q)^m|m\rangle$$

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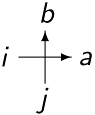

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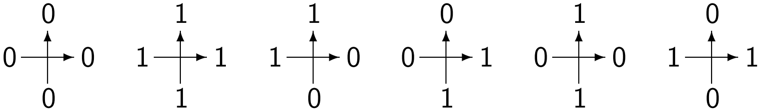

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Origin of q-oscillators and 3D L:

Quantized coordinate ring $A_q(sl_3)$ (Kapranov-Voevodsky 1994)
Quantization of Miquel's theorem (Bazhanov-Mangazeev-Sergeev 2008)

$q = 0$: Combinatorial R

Quantum R : $\quad \mathcal{R}(z) : V^s \otimes V^r \to V^r \otimes V^s, \quad V^s = \bigoplus_{\mathbf{b} \in B^s} \mathbb{C}|\mathbf{b}\rangle,$

$|001\rangle \otimes |110\rangle$

$\mapsto q(1+qz)|110\rangle \otimes |001\rangle + (1-q^2)|101\rangle \otimes |010\rangle - q(1-q^2)|011\rangle \otimes |100\rangle$

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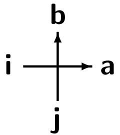
Fact ($\exists$ general theory of *crystal base* by Kashiwara, 1991∼)

Combinatorial R is a bijection $B^s \otimes B^r \to B^r \otimes B^s$ satisfying the YBE.

- Write $R(\mathbf{i} \otimes \mathbf{j}) = \sum_{\mathbf{a},\mathbf{b}} R^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} \mathbf{b} \otimes \mathbf{a}$ for simplicity without $|\ \rangle$
- $R^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = 1$ for $\exists 1$ $(\mathbf{a}, \mathbf{b})$ determined from $(\mathbf{i}, \mathbf{j})$. $R^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = 0$ elsewhere.

Depict $R(\mathbf{i} \otimes \mathbf{j}) = \mathbf{b} \otimes \mathbf{a}$ as
(No sum)



Nontrivial in that $\mathbf{b} \otimes \mathbf{a} \neq \mathbf{j} \otimes \mathbf{i}$ in general.

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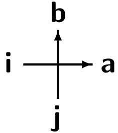
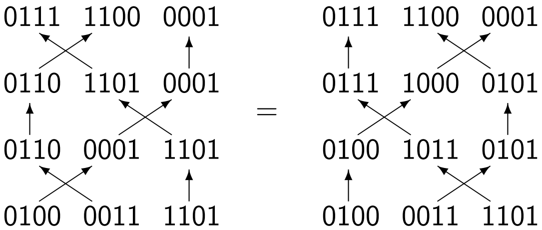


Nontrivial in that $\mathbf{b} \otimes \mathbf{a} \neq \mathbf{j} \otimes \mathbf{i}$ in general.

Example of YBE: $B^1 \otimes B^2 \otimes B^3 \to B^3 \otimes B^2 \otimes B^1$



Combinatorial R: systematic examples of set-theoretical sol. of YBE.

The algorithm for finding the image of the combinatorial R

$$R : B^s \otimes B^r \to B^r \otimes B^s$$
$$\mathbf{i} \otimes \mathbf{j} \quad \mapsto \quad \mathbf{b} \otimes \mathbf{a}$$

is know as the Nakayashiki-Yamada (NY)-rule (1997).

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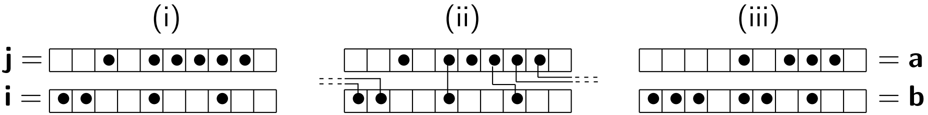
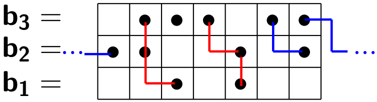
Example.

$$\begin{array}{ccccccc} B^4 & \otimes & B^6 & & B^6 & \otimes & B^4 \\ 1100100100 & \otimes & 0010111110 & \mapsto & 1110110100 & \otimes & 0000101110 \\ \mathbf{i} & & \mathbf{j} & & \mathbf{b} & & \mathbf{a} \end{array}$$



NY-rule = queuing process in the Ferrari-Martin algorithm!

$$\pi(\mathbf{b_1} \otimes \mathbf{b_2} \otimes \mathbf{b_3}) = 0 \;\; 3 \;\; 1 \;\; 3 \;\; 0 \;\; 2 \;\; 2$$



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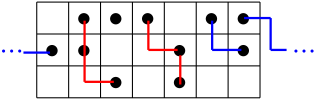
$\mathbf{b_3} =$

$\mathbf{b_2} =$

$\mathbf{b_1} =$



Elementary relabeling of n-TASEP $\boldsymbol{\sigma}$ by $\varphi_k(\boldsymbol{\sigma}) \in B^{s_k}$ $(k = 1, \dots, n)$ defined as

$\boldsymbol{\sigma}$	0 3 1 3 0 2 2
$\varphi_1(\boldsymbol{\sigma})$	0 1 0 1 0 0 0
$\varphi_2(\boldsymbol{\sigma})$	0 1 0 1 0 1 1
$\varphi_3(\boldsymbol{\sigma})$	0 1 1 1 0 1 1

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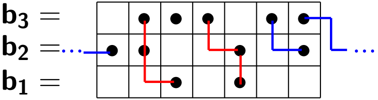

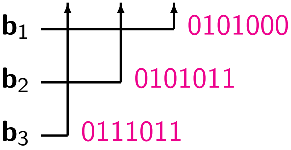

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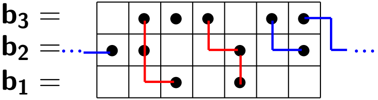
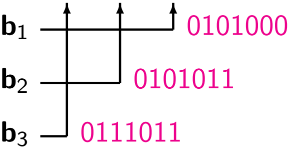
Ferrari-Martin map π as Corner transfer matrix

$\pi(\mathbf{b_1}\otimes\mathbf{b_2}\otimes\mathbf{b_3}) = 0\ \ 3\ \ 1\ \ 3\ \ 0\ \ 2\ \ 2$





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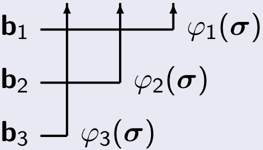
$\varphi_3(\boldsymbol{\sigma})$	0 1 1 1 0 1 1

Steady state probability $\mathbb{P}(\boldsymbol{\sigma}) = \#\left(\pi^{-1}(\boldsymbol{\sigma})\right)$ is expressed as CTM

$$\mathbb{P}(\boldsymbol{\sigma}) = \sum_{\mathbf{b}_1\otimes\mathbf{b}_2\otimes\mathbf{b}_3\in\mathcal{B}(\mathbf{m})}$$



$(n = 3$ case$)$

Into this CTM formula, we are further to substitute :

Matrix product form of the combinatorial R

$$R^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \mathrm{Tr}\big(L^{a_1,b_1}_{i_1,j_1} \cdots L^{a_L,b_L}_{i_L,j_L}\big).$$

- $(L^{a,b}_{i,j})$ = 3D L-operator at $q = 0$.

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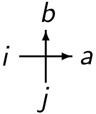
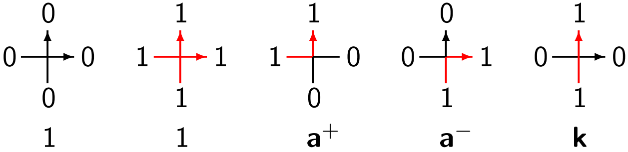
$L^{a,b}_{i,j} =$  $\in \mathrm{End}(F)$ $\quad$ ($q = 0$)-oscillator valued 5V

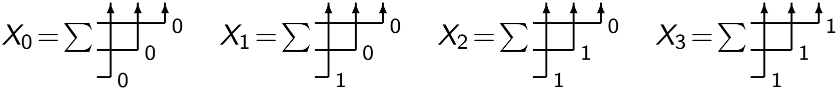


- ($q = 0$)-oscillator acts on $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ as

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-\delta_{m,0})|m-1\rangle, \quad \mathbf{k}|m\rangle = \delta_{m,0}|m\rangle$$

Result : $\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}_{F^{\otimes 3}}\left(X_{\sigma_1} \cdots X_{\sigma_L}\right)$ ($n = 3$ case)

where X_i's are CTMs of the 5V-model with the boundary conditions:



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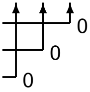
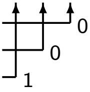
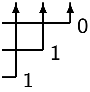
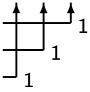
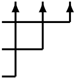
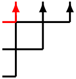
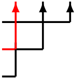
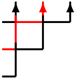
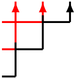
$X_0 = \sum$ 

$X_1 = \sum$ 

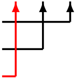
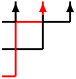
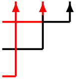
$X_2 = \sum$ 

$X_3 = \sum$ 

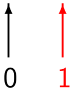
Example.

$X_0 =$  $+$  $+$  $+$  $+$ 

$$= \; 1\otimes 1\otimes 1 + \mathbf{a}^+\otimes 1\otimes 1 + \mathbf{k}\otimes\mathbf{a}^+\otimes 1 + \mathbf{a}^-\otimes\mathbf{a}^+\otimes\mathbf{a}^+ + 1\otimes\mathbf{a}^+\otimes\mathbf{a}^+$$

$X_1 =$  $+$  $+$ 

$$= \; \mathbf{k}\otimes\mathbf{k}\otimes 1 + \mathbf{a}^-\otimes\mathbf{k}\otimes\mathbf{a}^+ + 1\otimes\mathbf{k}\otimes\mathbf{a}^+$$

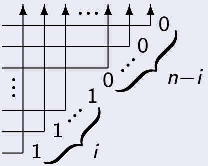


Theorem (Matrix product formula for steady state prob. of n-TASEP)

$$\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}\left(X_{\sigma_1}\cdots X_{\sigma_L}\right)$$

$X_0, \ldots, X_n$ are CTMs of $(q=0)$-oscillator valued 5V model:

$X_i = \sum$



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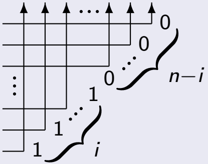
$X_i = \sum$ 

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- X_i = layer-to-layer transfer matrix with  shape
- $\mathbb{P}(\boldsymbol{\sigma}) =$ Partition function of a 3D system with prism shape

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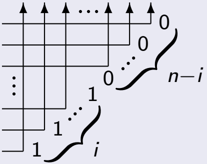
$X_i = \sum$  $\in \mathrm{End}(F^{\otimes n(n-1)/2})$

- X_i = layer-to-layer transfer matrix with ◸ shape
- $\mathbb{P}(\boldsymbol{\sigma})$ = Partition function of a 3D system with prism shape

	initial setup	cross channel (we are here!)
Physical space	$\mathbb{Z}_L$ ring	size n ◸
Internal degree	$\{0, \ldots, n\}$	$U_q(\widehat{sl}_L)$ at $q=0$

Hat relations: 3D integrability

TASEP Hamiltonian: $H = \sum_i h_{i,i+1}$, $h =$ local Hamiltonian

$h|\alpha, \beta\rangle = |\min(\alpha, \beta), \max(\alpha, \beta)\rangle - |\alpha, \beta\rangle =: \sum_{0 \le \gamma, \delta \le n} h^{\gamma,\delta}_{\alpha,\beta} |\gamma, \delta\rangle.$

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A sufficient condition for $\mathrm{Tr}\big(X_{\sigma_1}\cdots X_{\sigma_L}\big) =$ steady state probability

$$\exists\,\hat{X}_0,\ldots,\hat{X}_n \quad \text{satisfying} \quad X_\alpha\hat{X}_\beta - \hat{X}_\alpha X_\beta = \sum_{0\le\gamma,\delta\le n} h^{\alpha,\beta}_{\gamma,\delta}\,X_\gamma X_\delta.$$

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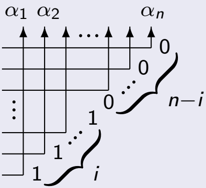
Prop.0 (Proof ultimately upgrades to the tetrahedron equation)

$$\hat{X}_i = \sum (\alpha_1 + \cdots + \alpha_n)$$



$\cdots$ weighted CTM

Upgrade 1: Spectral parameter & Difference analogue

Prop.0' $\simeq$ Prop.0 (Explcit form of the hat relations to be shown)

$$X_iX_j = \hat{X}_iX_j - X_i\hat{X}_j \ \ (i > j), \qquad [X_i, \hat{X}_j] = [\hat{X}_i, X_j] \ \ (\forall i, j).$$

Upgrade 1: Spectral parameter & Difference analogue

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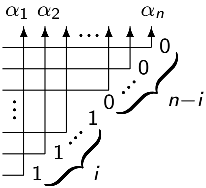
Introduce

$$X_i({\color{red}z}) := \sum {\color{red}z^{\alpha_1+\cdots+\alpha_n}}$$



$$X_i = X_i(z=1)$$

$$\hat{X}_i = \frac{dX_i(z)}{dz}\Big|_{z=1}$$

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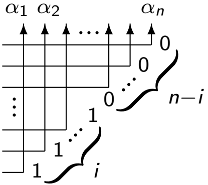

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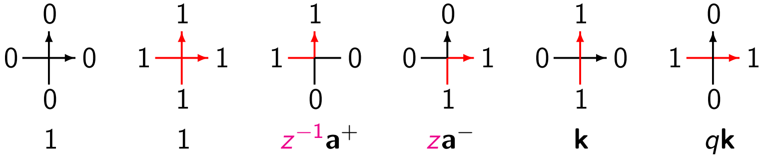
Prop.0' can be upgraded to a difference analogue including spectral parameters:

Prop.1 (CTM commutation relations)

$$xX_i(y)X_j(x) = yX_i(x)X_j(y) \ (i > j), \quad [X_i(x), X_j(y)] = [X_i(y), X_j(x)] \ (\forall i, j).$$

Upgrade 2: q-Melting : 5V $\rightarrow$ 6V & ◸ $\rightarrow$ □

Introduce the spectral parameter dependent 3D L-operator $\mathcal{L}(z)$ by



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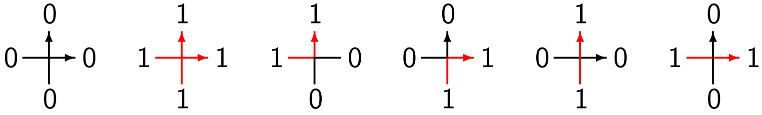
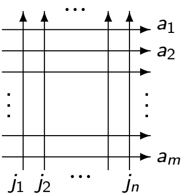


1	1	$z^{-1}\mathbf{a}^+$	$z\mathbf{a}^-$	$\mathbf{k}$	$q\mathbf{k}$

Define $T(z)^{\mathbf{a}}_{\mathbf{j}} \in \mathrm{End}(F^{\otimes mn})$ by

$T(z)^{\mathbf{a}}_{\mathbf{j}} := \sum$



$\mathbf{a} = (a_1, \ldots, a_m)$, $\mathbf{j} = (j_1, \ldots, j_n)$

All vertices are $\mathcal{L}(z)$

Sum over all edge spins except $\mathbf{a}, \mathbf{j}$

Layer-to-layer transfer mat. with SE-fixed/NW-free boundary cond.

Prop.2 (Bilinear relations of the layer-to-layer transfer matrices)

$$\sum x^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} y^{|\overline{\boldsymbol{\alpha}}|+|\overline{\boldsymbol{\beta}}|} T(x)^{..,\alpha_1,....,\alpha_r,..}_{..,\beta_1,....,\beta_s,..} T(y)^{..,\bar{\alpha}_1,....,\bar{\alpha}_r,..}_{..,\bar{\beta}_1,....,\bar{\beta}_s,..} = (x \leftrightarrow y).$$

- Sum over $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \{0,1\}^r$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s) \in \{0,1\}^s$.
- $0 \le r \le m$, $0 \le s \le n$ are arbitrary, $\bar{\alpha}_i = 1 - \alpha_i$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_r$, etc.
- Arrays "...." are arbitrary, but to be taken common for $T(x)$ and $T(y)$.

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Example.

(1) $r = s = 0$: $T(x)^{\mathbf{a}}_{\mathbf{j}} T(y)^{\mathbf{a}}_{\mathbf{j}} = T(y)^{\mathbf{a}}_{\mathbf{j}} T(x)^{\mathbf{a}}_{\mathbf{j}}$ $\cdots$ Commutativity

(2) $r = s = 1$:

$$y^2 T(x)^{0,\dots}_{0,\dots} T(y)^{1,\dots}_{1,\dots} + xy T(x)^{0,\dots}_{1,\dots} T(y)^{1,\dots}_{0,\dots} + xy T(x)^{1,\dots}_{0,\dots} T(y)^{0,\dots}_{1,\dots} + x^2 T(x)^{1,\dots}_{1,\dots} T(y)^{0,\dots}_{0,\dots} = (x \leftrightarrow y).$$

Prop.2 (Bilinear relations of the layer-to-layer transfer matrices)

$$\sum x^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} y^{|\overline{\boldsymbol{\alpha}}|+|\overline{\boldsymbol{\beta}}|} T(x)^{..,\alpha_1,\ldots,\alpha_r,..}_{..,\beta_1,\ldots,\beta_s,..} T(y)^{..,\bar{\alpha}_1,\ldots,\bar{\alpha}_r,..}_{..,\bar{\beta}_1,\ldots,\bar{\beta}_s,..} = (x \leftrightarrow y).$$

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Fact: CTM commutation relations are included(!) in (1) & (2) at $q = 0$.

Final task: Proof of the bilinear relations for $T(z)^{\mathbf{a}}_{\mathbf{j}}$

Introduce a variant of the 3D L-operator: $\mathcal{M}(z) := \mathcal{L}(z)|_{q \to -q}$.

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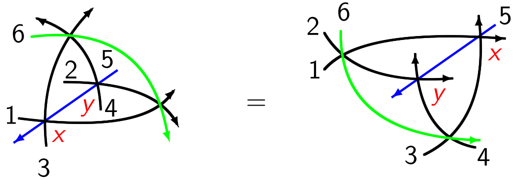
Theorem ($\mathcal{L}(z)$ and $\mathcal{M}(z)$ satisfy the tetrahedron equation)

$$\mathcal{M}(\tfrac{ux}{y})_{126}\mathcal{M}(u)_{346}\mathcal{L}(x)_{135}\mathcal{L}(y)_{245} = \mathcal{L}(y)_{245}\mathcal{L}(x)_{135}\mathcal{M}(u)_{346}\mathcal{M}(\tfrac{ux}{y})_{126}$$



Final task: Proof of the bilinear relations for $T(z)^{\mathbf{a}}_{\mathbf{j}}$

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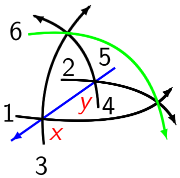

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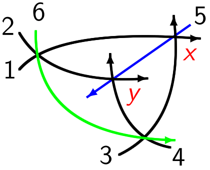
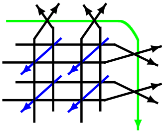


The bilinear relations are proved by evaluating the following identity between various left and right eigenvectors of $\mathcal{M}(\cdot)$ running along the green arrow.



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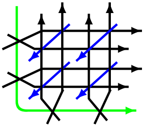

QED

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