Aspects of stationary states in multispecies exclusion and zero range processes

Atsuo Kuniba (Univ. Tokyo)

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Contents.

This talk is a survey of the combinatorial aspects of the simple exclusion / zero range processes that become manifest in **multispecies** setting.

Based on joint works with V.V. Mangazeev, M. Okado, S. Maruyama, S. Watanabe in arXiv: 1506.04490, 1509.09018, 1511.09168, 1602.00764, 1602.04574, 1604.08304, 1608.02779, 1610.00531, 1701.07279, 1705.10979.

Contents.

I. n-TASEP and stationary states

master equation, n-line process, Ferrari-Martin algorithm

II. Quantum group formulation of stationary probability

III. Hidden 3D structure related to tetrahedron equation

IV. Remarks

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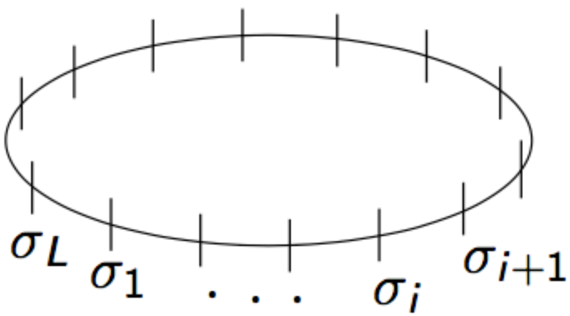
n-species Totally Asymmetric Simple Exclusion Process (n-TASEP)



1D periodic chain with L sites

$$\sigma_i \in \{0, 1, \ldots, n\} \quad (n\text{-TASEP})$$

Stochastic dynamics

$$(\sigma_i, \sigma_{i+1}) \rightarrow (\sigma'_i, \sigma'_{i+1})$$

$$(\alpha, \beta) \rightarrow (\beta, \alpha) \quad \text{if } \alpha > \beta$$

Master equation

$$\frac{d}{dt}|P\rangle = H|P\rangle, \quad |P\rangle = \sum_{\{\sigma_i\}} \mathbb{P}(\sigma_1, \ldots, \sigma_L)|\sigma_1, \ldots, \sigma_L\rangle \in (\mathbb{C}^{n+1})^{\otimes L}$$

$$H = \sum_{i \in \mathbb{Z}_L} h_{i.i+1}, \quad h_{i,i+1} = 1 \otimes \cdots \otimes 1 \otimes \overset{i,i+1}{h} \otimes 1 \otimes \cdots \otimes 1$$

$$h|\alpha, \beta\rangle = \begin{cases} |\beta, \alpha\rangle - |\alpha, \beta\rangle & (\alpha > \beta), \\ 0 & (\alpha \leq \beta). \end{cases}$$

Stationary states

Sectors labeled by multiplicities $\mathbf{m} = (m_0, \ldots, m_n) \in \mathbb{Z}_+^{n+1}$:

$$S(\mathbf{m}) = \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_L) \in \{0, \ldots, n\}^L \mid \#_k(\boldsymbol{\sigma}) = m_k\}.$$

Each sector has a unique stationary state

$$|\bar{P}(\mathbf{m})\rangle \in \sum_{\boldsymbol{\sigma} \in S(\mathbf{m})} \mathbb{Z}_+ |\boldsymbol{\sigma}\rangle \ \text{ s.t. } \ H|\bar{P}(\mathbf{m})\rangle = 0.$$

$$\begin{aligned}
|\bar{P}(1,1,1)\rangle &= 2|012\rangle + |021\rangle + |102\rangle + 2|120\rangle + 2|201\rangle + |210\rangle, \\
|\bar{P}(2,1,1)\rangle &= 3|0012\rangle + |0021\rangle + 2|0102\rangle + 3|0120\rangle + 2|0201\rangle + |0210\rangle \\
&\quad + |1002\rangle + 2|1020\rangle + 3|1200\rangle + 3|2001\rangle + 2|2010\rangle + |2100\rangle, \\
|\bar{P}(1,2,1)\rangle &= 2|0112\rangle + |0121\rangle + |0211\rangle + |1012\rangle + |1021\rangle + |1102\rangle \\
&\quad + 2|1120\rangle + 2|1201\rangle + |1210\rangle + 2|2011\rangle + |2101\rangle + |2110\rangle.
\end{aligned}$$

Steady states are non-trivial for $n \geq 2$.

Results on stationary probability $\mathbb{P}(\boldsymbol{\sigma})$

$$|\bar{P}(\mathbf{m})\rangle = \sum_{\boldsymbol{\sigma} \in S(\mathbf{m})} \mathbb{P}(\boldsymbol{\sigma}) |\boldsymbol{\sigma}\rangle$$

Combinatorial algorithm: Ferrari-Martin (2007)
Matrix product formulas: Evans, Ferrari, Mallick, Prolhac,…(2009-)

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Today's topic：Quantum group symmetry and hidden 3D structure

Ferrari-Martin algorithm = composition of Combinatorial R

New matrix prod. operators = corner transfer mat. of q=0 boson-valued 5V model

$\mathbb{P}(\boldsymbol{\sigma})$ = Partition function of a 3D model associated with Tetrahedron equation

Ferrari-Martin algorithm

Sector $S(\mathbf{m})$ with multiplicity $\mathbf{m} = (m_0, \ldots, m_n)$ with $\forall m_i > 0$

$\mathcal{B}(\mathbf{m}) := B^{s_1} \otimes \cdots \otimes B^{s_n}$ ··· Set of states for ***n-line process***

$$\boxed{B^s := \{\mathbf{b} = (b_1, \ldots, b_L) \in \{0,1\}^L \mid b_1 + \cdots + b_L = s\},}$$

e.g. $B^1 = \{100, 010, 001\}$, $B^2 = \{110, 101, 011\}$ for $L = 3$

$$s_i := m_{n-i+1} + \cdots + m_{n-1} + m_n, \quad 0 < s_1 < \cdots < s_n < L$$

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$$s_i := m_{n-i+1} + \cdots + m_{n-1} + m_n, \quad 0 < s_1 < \cdots < s_n < L$$

Elements of $\mathcal{B}(\mathbf{m})$ are depicted as **dot patterns**

Example. $n = 3$, $\mathbf{m} = (2,1,2,2)$, $\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \in \mathcal{B}(\mathbf{m}) = B^2 \otimes B^4 \otimes B^5$
$L = 7$

$B^5 \ni \mathbf{b}_3 =$		•	•	•		•	•
$B^4 \ni \mathbf{b}_2 =$	•	•			•		•
$B^2 \ni \mathbf{b}_1 =$			•		•		

$\mathbf{b}_1 = (0010100)$, etc.

Ferrari-Martin gave a combinatorial construction of the surjective map

$$\pi : \mathcal{B}(\mathbf{m}) \to S(\mathbf{m}) \quad \text{such that} \quad \mathbb{P}(\boldsymbol{\sigma}) = \#\left(\pi^{-1}(\boldsymbol{\sigma})\right).$$

Uniform measure on $\mathcal{B}(\mathbf{m})$ induces n-TASEP stationary measure via π

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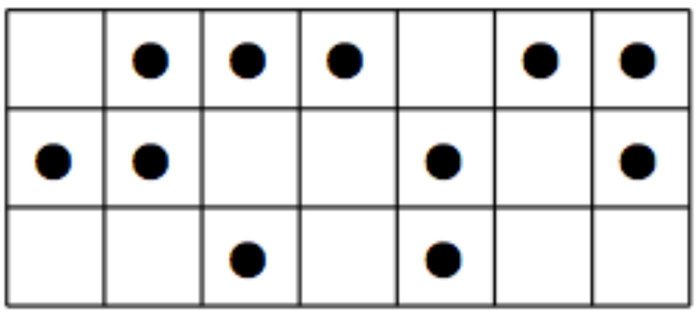

$$\xmapsto{\quad \pi \quad} \quad 0\ 3\ 1\ 3\ 0\ 2\ 2$$

n-line process

n-TASEP

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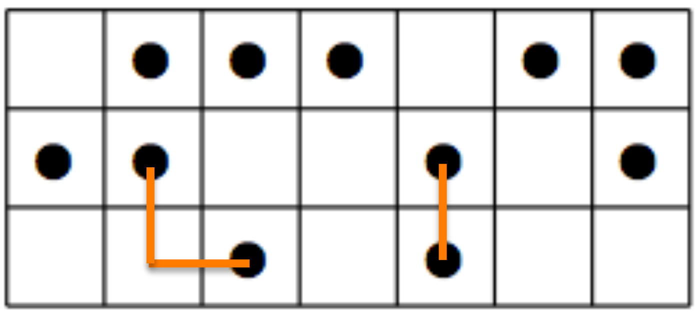

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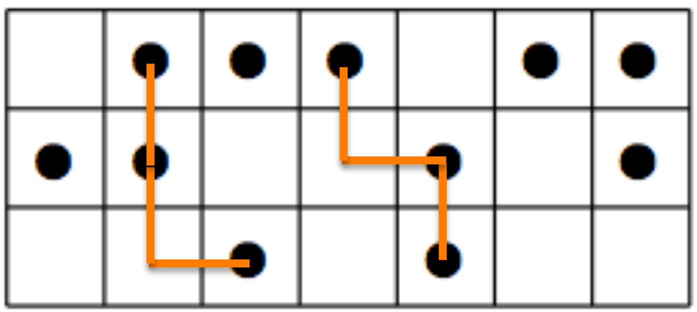

n-line process

$\xrightarrow[\pi]{}$ 0 3 1 3 0 2 2

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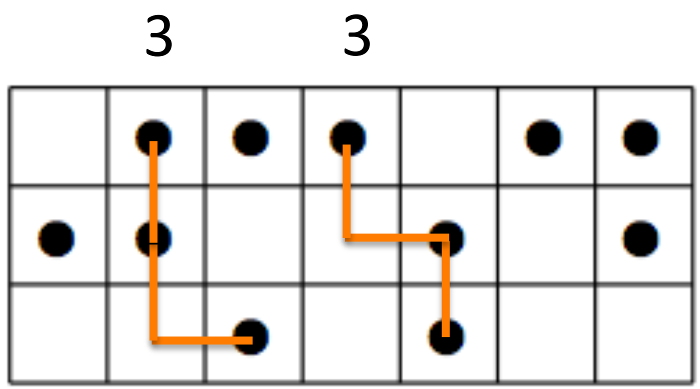

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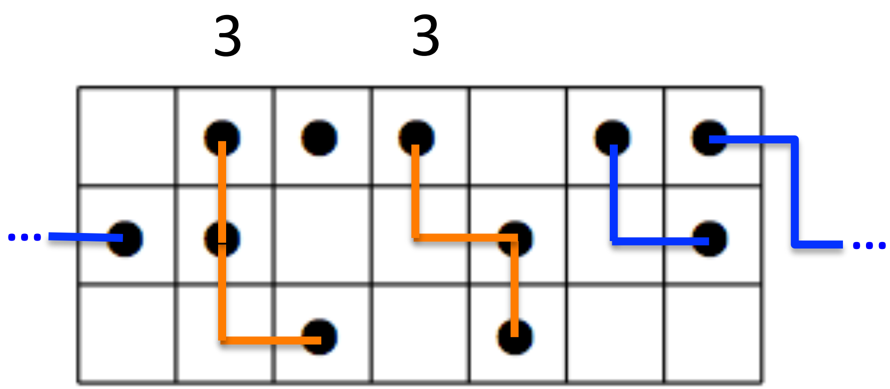

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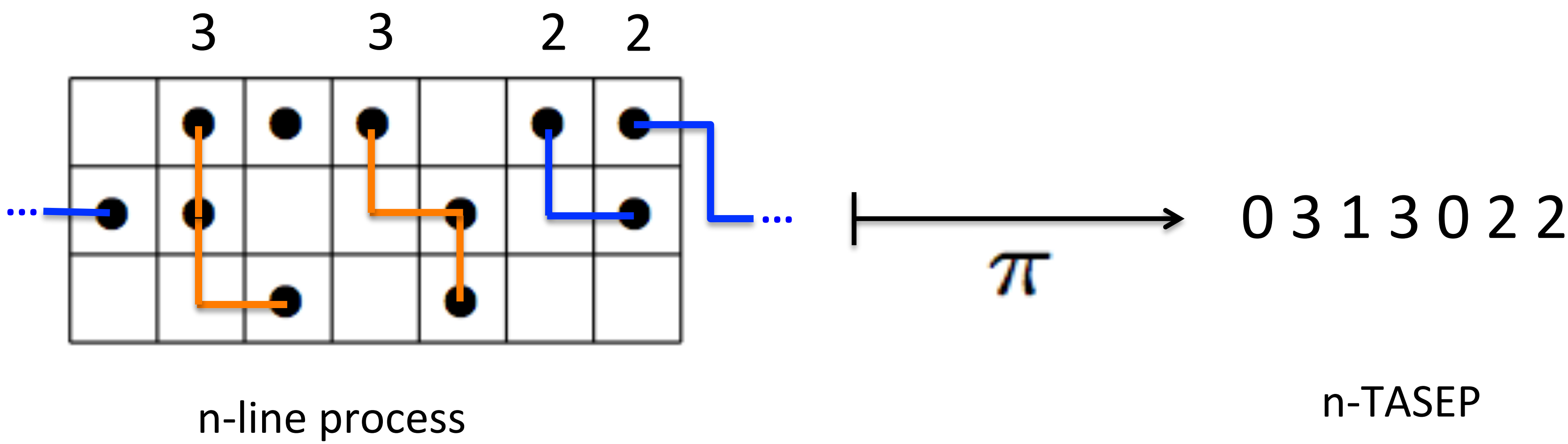

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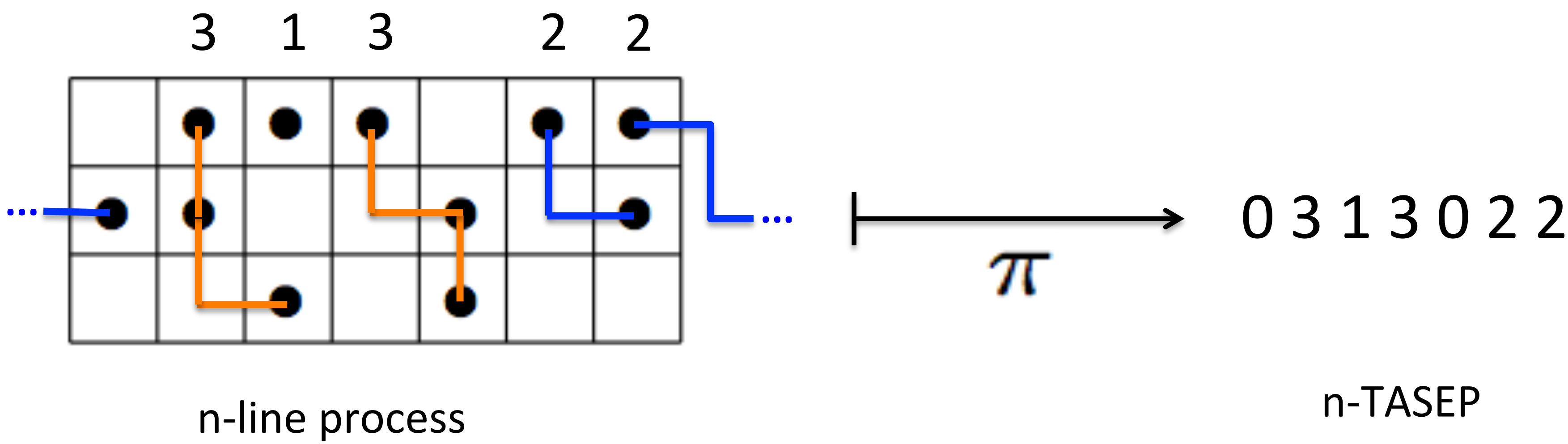

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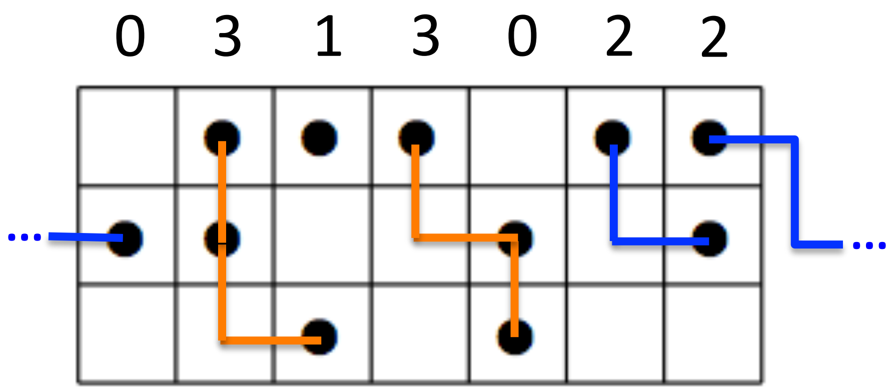

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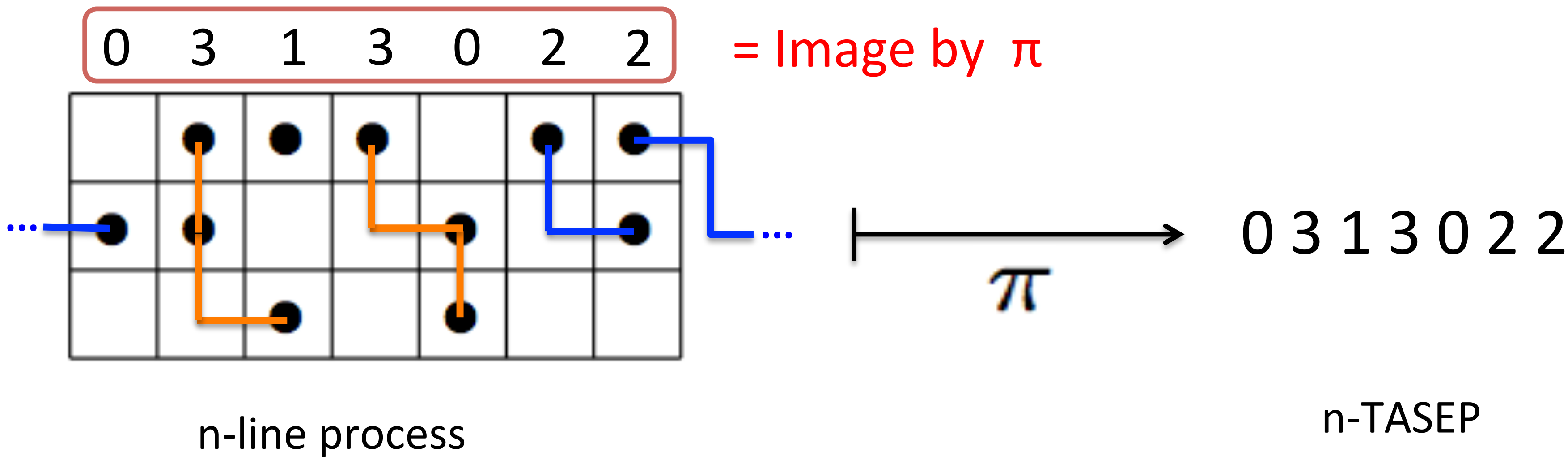

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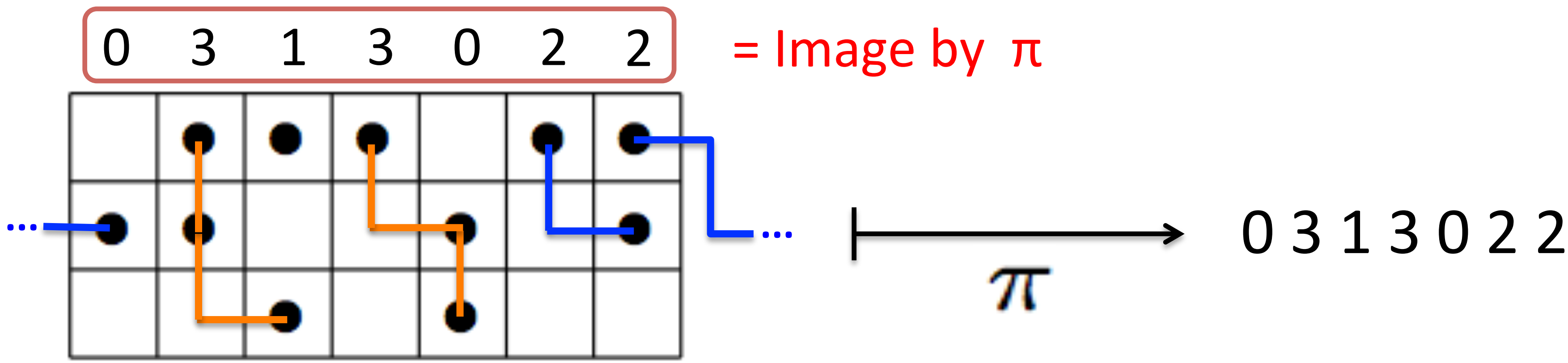

n-line process

n-TASEP

Pairing of dots in adjacent rows is referred to as a **queuing process** of customers and service.

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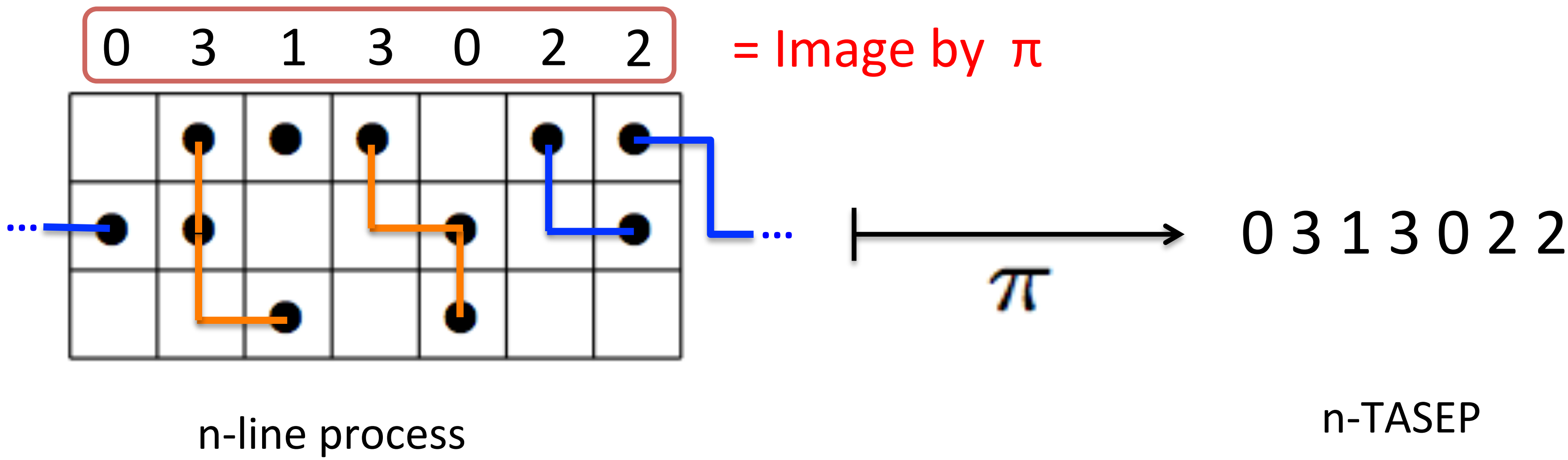

Pairing of dots in adjacent rows is referred to as a **queuing process** of customers and service.

π is most naturally formulated by **Combinatorial R** of the quantum affine algebra $U_q(\widehat{sl}_L)$

Contents.

I. n-TASEP and stationary states

II. Quantum group formulation of stationary probability

  quantum R, 3D L-operator, combinatorial R,
  corner transfer matrix of (q=0)-boson valued 5V,
  a new matrix product formula

III. Hidden 3D structure related to tetrahedron equation

IV. Remarks

Quantum R matrix for $U_q(\widehat{sl}_L)$

Degree s anti-symmetric tensor representation:

$$V^s = \bigoplus_{\mathbf{b} \in B^s} \mathbb{C}|\mathbf{b}\rangle, \quad B^s = \{(x_1, \ldots, x_L) \in \{0,1\}^L \mid x_1 + \cdots + x_L = s\}$$

$$\overset{1:1}{\longleftrightarrow} \text{ Length } s \text{ column shape standard tableaux}$$

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$\overset{1:1}{\longleftrightarrow}$ Length s column shape standard tableaux

Quantum R matrix: $\mathcal{R}(z) = \mathcal{R}^{s,r}(z) : V^s \otimes V^r \to V^r \otimes V^s$

$$\mathcal{R}(z)\big(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle\big) = \sum_{\mathbf{a},\mathbf{b}} \mathcal{R}(z)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}} |\mathbf{b}\rangle \otimes |\mathbf{a}\rangle \quad (\mathbf{a} = (a_1, \ldots, a_L) \in B^s, \text{ etc}).$$

$\exists$ Normalization s.t. $\mathcal{R}(z)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}} = \textit{polynomials}$ in q and z.

Quantum R matrix for $U_q(\widehat{sl}_L)$

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$\exists$ Normalization s.t. $\mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} =$ *polynomials* in q and z.

Example. Nonzero elements of $\mathcal{R}^{1,2}(z)$ for $U_q(\widehat{sl}_3)$ are as follows.

$$\mathcal{R}^{100,110}_{100,110} = \mathcal{R}^{010,110}_{010,110} = \mathcal{R}^{100,101}_{100,101} = \mathcal{R}^{001,101}_{001,101} = \mathcal{R}^{010,011}_{010,011} = \mathcal{R}^{001,011}_{001,011} = 1+q^3z,$$

$$\mathcal{R}^{001,110}_{001,110} = \mathcal{R}^{010,101}_{010,101} = \mathcal{R}^{100,011}_{100,011} = q(1+qz),$$

$$\mathcal{R}^{001,110}_{010,101} = \mathcal{R}^{010,101}_{100,011} = z\mathcal{R}^{100,011}_{001,110} = -q(1-q^2)z,$$

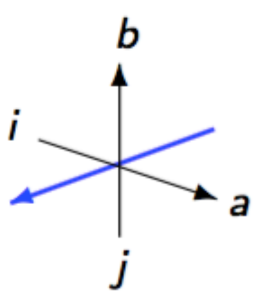
$$\mathcal{R}^{001,110}_{100,011} = z\mathcal{R}^{010,101}_{001,110} = z\mathcal{R}^{100,011}_{010,101} = (1-q^2)z.$$

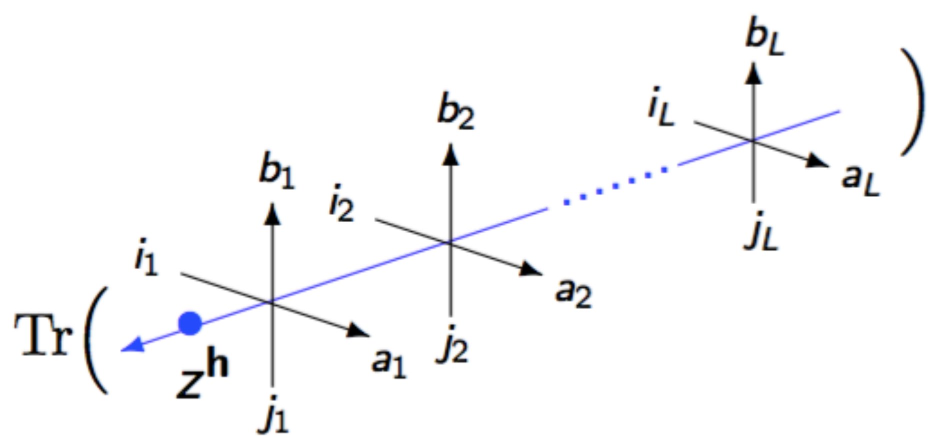
Matrix product (BBQ stick) formula for the Quantum R

$$\boxed{\mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \varrho(z)\mathrm{Tr}\left(z^{\mathbf{h}}\mathcal{L}^{a_1,b_1}_{i_1,j_1}\cdots\mathcal{L}^{a_L,b_L}_{i_L,j_L}\right)}$$

$\mathbf{h}|m\rangle = m|m\rangle$, $\mathbf{a} = (a_1,\ldots,a_L) \in B^s$, etc, $\varrho(z) =$known.

$\mathcal{L} = (\mathcal{L}^{a,b}_{i,j})_{a,b,i,j=0,1} =$ **3D *L*-operator** (defined in the next page)

$\mathcal{L}^{a,b}_{i,j} =$ 

$\mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} =$ 

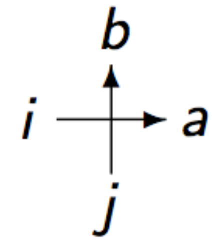
= operator on the Fock space

$F = \oplus_m \mathbb{C}|m\rangle$ (blue arrow)

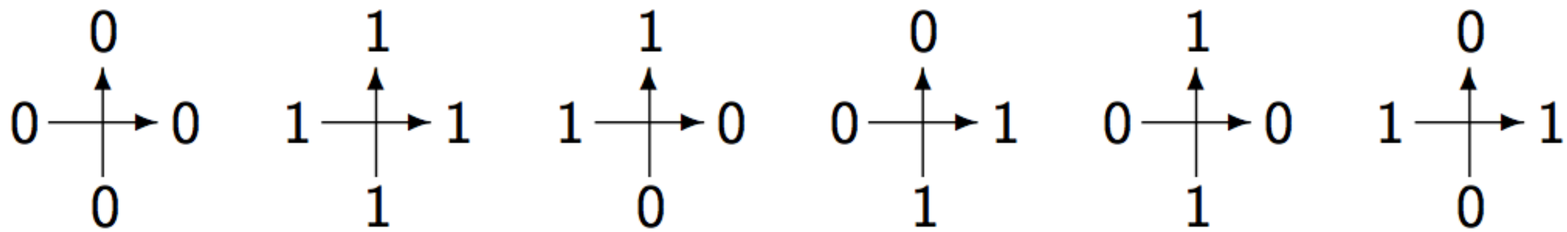
= **BBQ stick with X shape sausages**

3D L-operator (q-boson valued 6V model)

$$\mathcal{L} = (\mathcal{L}^{a,b}_{i,j}) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F), \qquad \mathcal{L}^{a,b}_{i,j} =$$



$$\in \mathrm{End}(F)$$



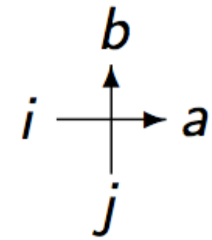
1	1	$\mathbf{a}^+$	$\mathbf{a}^-$	$\mathbf{k}$	$q\mathbf{k}$

$\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ are q-boson operators on the Fock space $F = \bigoplus_{m\geq 0} \mathbb{C}|m\rangle$

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = (-q)^m|m\rangle$$

3D L-operator (q-boson valued 6V model)

$$\mathcal{L} = (\mathcal{L}^{a,b}_{i,j}) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F), \quad \mathcal{L}^{a,b}_{i,j} = $$



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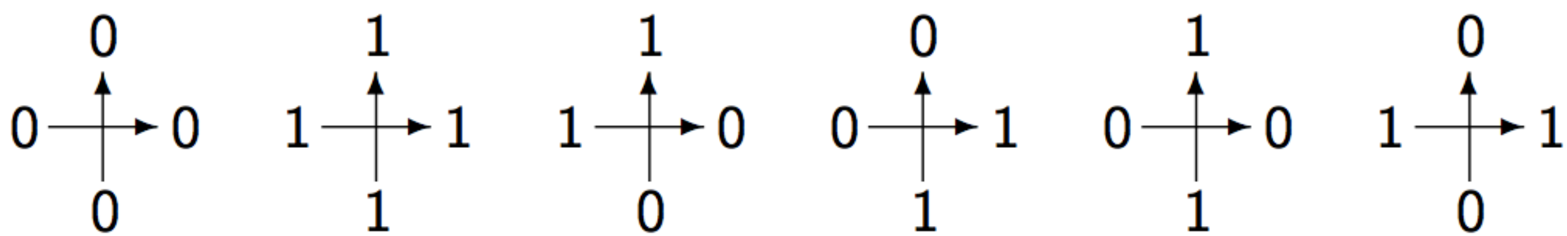

$1 \qquad 1 \qquad \mathbf{a}^+ \qquad \mathbf{a}^- \qquad \mathbf{k} \qquad q\mathbf{k}$

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Origin of q-boson valued L-operator

Quantized coordinate ring $A_q(sl_3)$ [Kapranov-Voevodsky, 1994]

Quantization of Miquel's theorem [Bazhanov-Mangazeev-Sergeev, 2008]

q=0: Combinatorial R

Quantum R : $\quad \mathcal{R}(z) : V^s \otimes V^r \to V^r \otimes V^s, \quad V^s = \bigoplus_{\mathbf{b} \in B^s} \mathbb{C}|\mathbf{b}\rangle,$

$|001\rangle \otimes |110\rangle$

$\mapsto q(1+qz)|110\rangle \otimes |001\rangle + (1-q^2)|101\rangle \otimes |010\rangle - q(1-q^2)|011\rangle \otimes |100\rangle$

q=0: Combinatorial R

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Combinatorial R: $R := \mathcal{R}(z=1)|_{q=0}$

(A more intrinsic definition exists, but this suffices for this talk.)

In the above example, R: $|001\rangle \otimes |110\rangle \mapsto |101\rangle \otimes |010\rangle$

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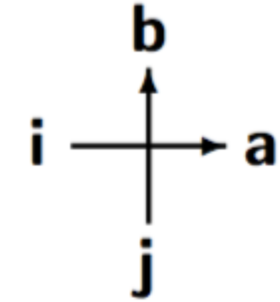
In the above example, R: $|001\rangle \otimes |110\rangle \mapsto |101\rangle \otimes |010\rangle$

Fact (from general theory of *crystal base* by [Kashiwara, 1991-])

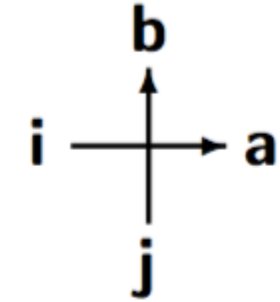
Combinatorial R is a bijection $B^r \otimes B^s \to B^s \otimes B^r$ satisfying YBE.

Depict $R(\mathbf{i} \otimes \mathbf{j}) = \mathbf{b} \otimes \mathbf{a}$ as



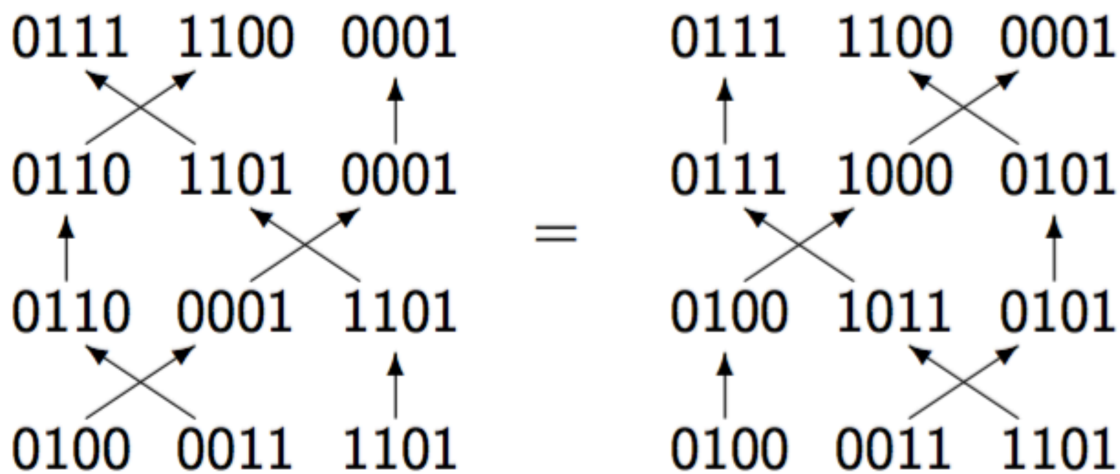
Nontrivial in that $\mathbf{b} \otimes \mathbf{a} \neq \mathbf{j} \otimes \mathbf{i}$ in general.

Depict $R(\mathbf{i} \otimes \mathbf{j}) = \mathbf{b} \otimes \mathbf{a}$ as



Nontrivial in that $\mathbf{b} \otimes \mathbf{a} \neq \mathbf{j} \otimes \mathbf{i}$ in general.

Example of YBE: $B^1 \otimes B^2 \otimes B^3 \to B^3 \otimes B^2 \otimes B^1$



Combinatorial R: systematic examples of set-theoretical sol. of YBE.

The algorithm for finding the image of the combinatorial R

$$R : B^s \otimes B^r \to B^r \otimes B^s$$
$$\mathbf{i} \otimes \mathbf{j} \quad \mapsto \quad \mathbf{b} \otimes \mathbf{a}$$

is known as the Nakayashiki-Yamada (NY) rule (1997).

The algorithm for finding the image of the combinatorial R

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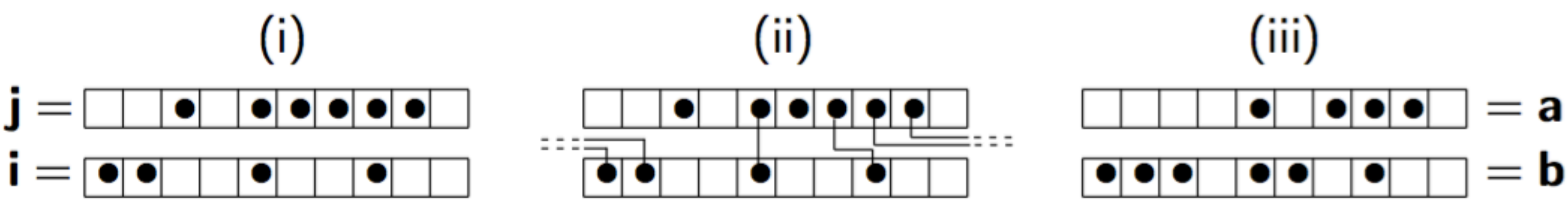
is known as the Nakayashiki-Yamada (NY) rule (1997).

Example.

$$\begin{array}{ccccccc} B^4 & \otimes & B^6 & & B^6 & \otimes & B^4 \\ 1100100100 & \otimes & 0010111110 & \mapsto & 1110110100 & \otimes & 0000101110 \\ \mathbf{i} & & \mathbf{j} & & \mathbf{b} & & \mathbf{a} \end{array}$$



NY-rule = queuing process in the Ferrari-Martin algorithm!

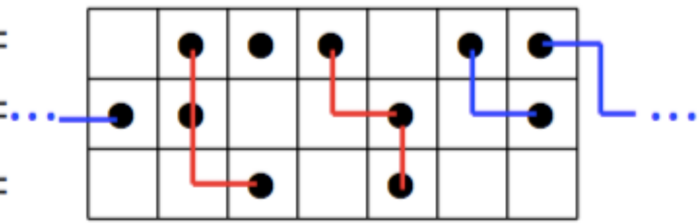
Ferrari-Martin map π as ``Corner transfer matrix (CTM)”

$$\pi(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = 0\ 3\ 1\ 3\ 0\ 2\ 2$$

$\mathbf{b}_3 =$

$\mathbf{b}_2 =$

$\mathbf{b}_1 =$



Elementary relabeling of n-TASEP $\boldsymbol{\sigma}$ by $\varphi_k(\boldsymbol{\sigma}) \in B^{s_k}$ $(k = 1, \dots, n)$ defined as

$\boldsymbol{\sigma}$	0 3 1 3 0 2 2
$\varphi_1(\boldsymbol{\sigma})$	0 1 0 1 0 0 0
$\varphi_2(\boldsymbol{\sigma})$	0 1 0 1 0 1 1
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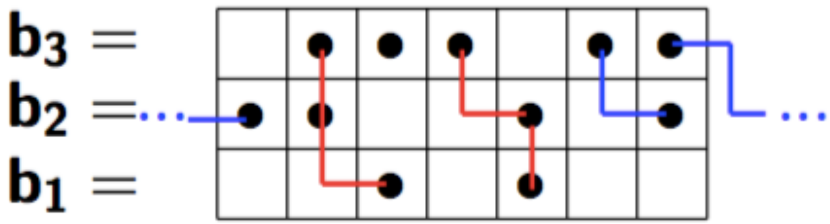

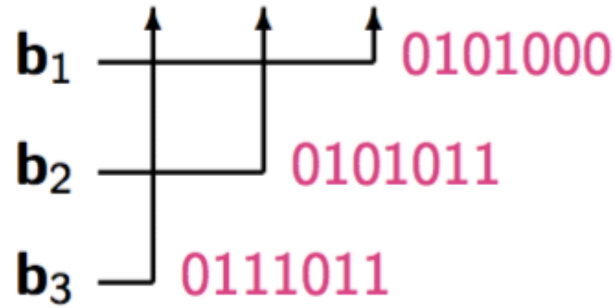

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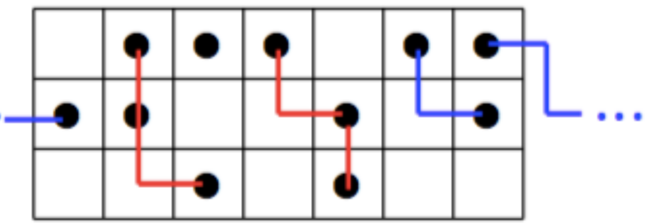

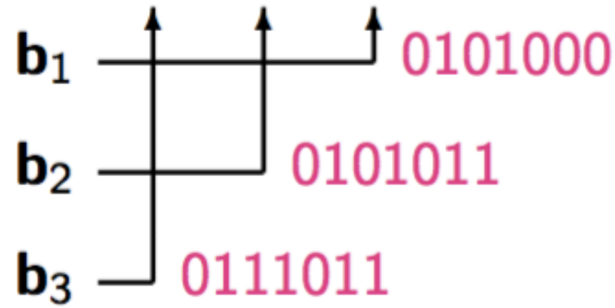

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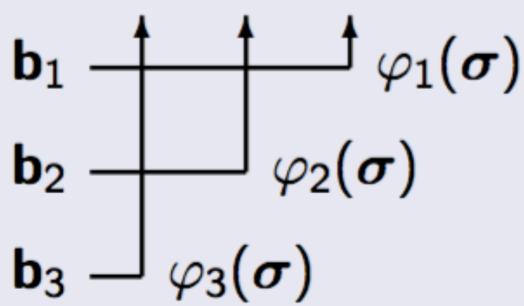
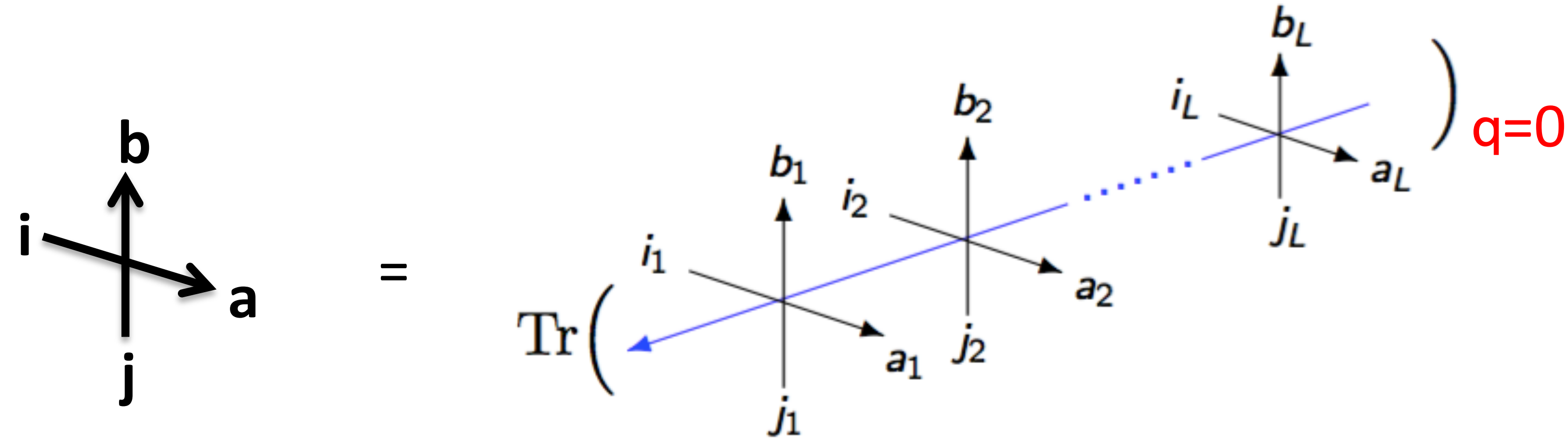
Stationary probability $\mathbb{P}(\boldsymbol{\sigma}) = \#(\pi^{-1}(\boldsymbol{\sigma}))$ is expressed as CTM

$$\mathbb{P}(\boldsymbol{\sigma}) = \sum_{\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \in \mathcal{B}(\mathbf{m})}$$



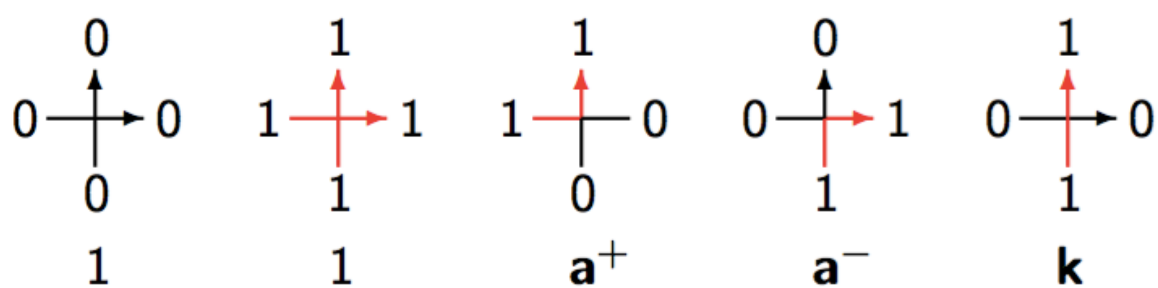
$(n = 3$ case$)$

To each vertex here (= Combinatorial R)
we are to substitute **the BBQ stick formula** **at q=0****.**



Each ***Layer*** forms a configuration of

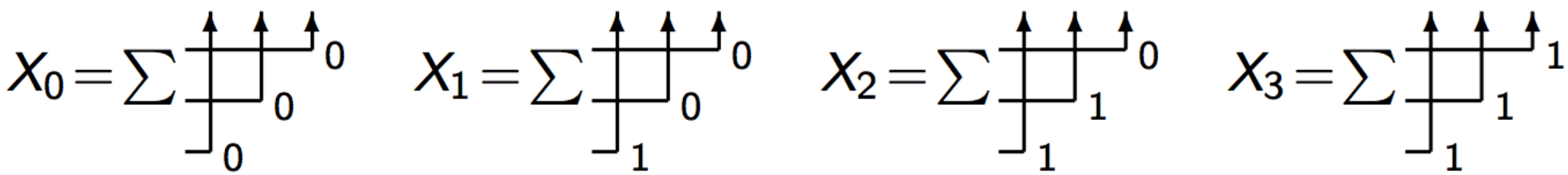
q-boson valued 6V model $\xrightarrow{q \to 0}$ q=0-boson valued 5V model



$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-\delta_{m,0})|m-1\rangle, \quad \mathbf{k}|m\rangle = \delta_{m,0}|m\rangle$$

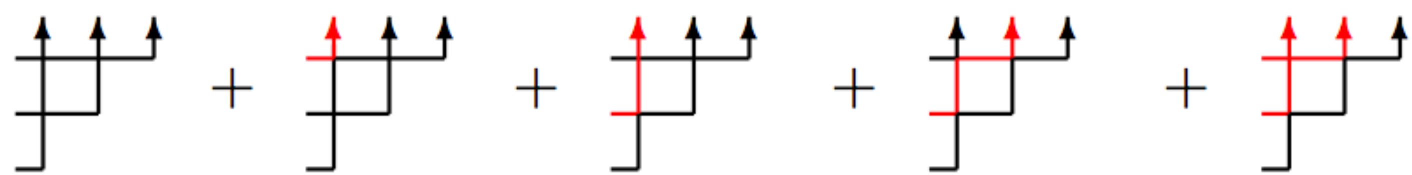
Result : $\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}_{F^{\otimes 3}}\left(X_{\sigma_1}\cdots X_{\sigma_L}\right)$ ($n = 3$ case)

where X_i's are CTMs of the 5V-model with the boundary conditions:



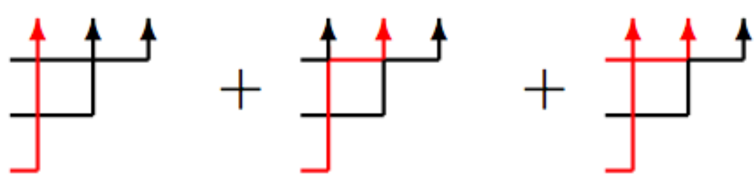
Example.

$X_0 =$

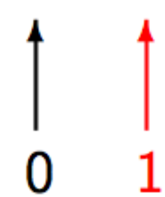


$$= \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1} + \mathbf{a}^{+}\otimes\mathbf{1}\otimes\mathbf{1} + \mathbf{k}\otimes\mathbf{a}^{+}\otimes\mathbf{1} + \mathbf{a}^{-}\otimes\mathbf{a}^{+}\otimes\mathbf{a}^{+} + \mathbf{1}\otimes\mathbf{a}^{+}\otimes\mathbf{a}^{+}$$

$X_1 =$



$$= \mathbf{k}\otimes\mathbf{k}\otimes\mathbf{1} + \mathbf{a}^{-}\otimes\mathbf{k}\otimes\mathbf{a}^{+} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{a}^{+}$$

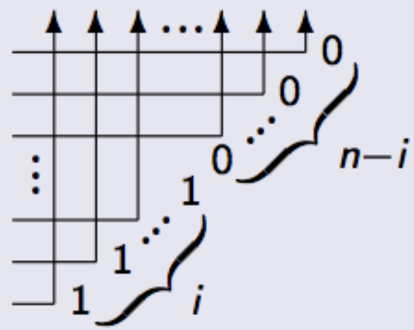


Theorem (Matrix product formula for stationary prob. of n-TASEP)

$$\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}\left(X_{\sigma_1} \cdots X_{\sigma_L}\right)$$

$X_0, \ldots, X_n$ are CTMs of $(q = 0)$-boson valued 5 vertex model:

$X_i = \sum$



$\in \mathrm{End}(F^{\otimes n(n-1)/2})$

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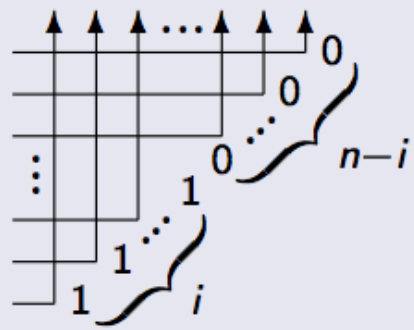

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- $X_i =$ Layer transfer matrix with shape
- $\mathbb{P}(\boldsymbol{\sigma}) =$ Partition function of a 3D system with prism shape

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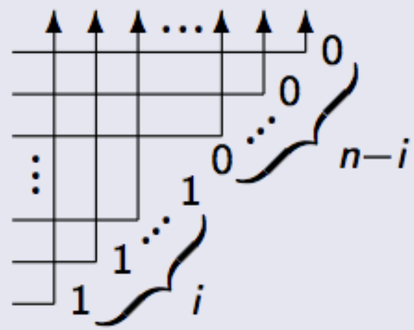
$X_i = \sum$ 

$\in \mathrm{End}(F^{\otimes n(n-1)/2})$

- X_i = Layer transfer matrix with ◸ shape
- $\mathbb{P}(\boldsymbol{\sigma})$ = Partition function of a 3D system with prism shape

	initial setup	cross channel (we are here!)
Physical space	$\mathbb{Z}_L$ ring	size n ◸
Internal degree	$\{0, \ldots, n\}$	$U_q(\widehat{sl}_L)$ at $q=0$

Contents.

I. n-TASEP and stationary states

II. Quantum group formulation of stationary probability

III. Hidden 3D structure related to tetrahedron equation

Zamolodchikov-Faddeev algebra,

bilinear relations among layer transfer matrices,

tetrahedron equation

IV. Remarks

Local form of stationarity

If the operators $X = (X_\alpha)_{0 \le \alpha \le n}$, $\hat{X} = (\hat{X}_\alpha)_{0 \le \alpha \le n}$ satisfy

hat relation: $h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$ (h: local Markov matrix),

stationary states under P.B.C. is Tr(X X .... X) due to the cancellation

$$H\,\mathrm{Tr}(XX\cdots X) = \mathrm{Tr}\Big((X\hat{X} - \hat{X}X)XX\cdots X\Big) + \mathrm{Tr}\Big(X(X\hat{X} - \hat{X}X)X\cdots X\Big) + \cdots = 0$$

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The hat relation is an infinitesimal version of the **Zamolodchikov-Faddeev (ZF) algebra**

$$S(x/y)\,(X(x) \otimes X(y)) = X(y) \otimes X(x)$$

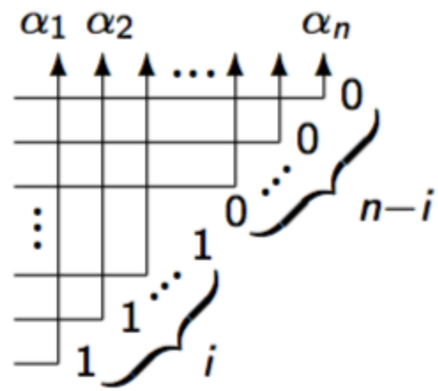
with *spectral parameters* x, y via the correspondence

$$h = S'(1), \qquad X = X(1), \qquad \hat{X} = X'(1).$$

(S(z) = a stochastic R matrix at q=0 given in the next page effectively.)

In our n-TASEP, $X(z) = (X_0(z),\ldots,X_n(z))$ and the ZF relation are given by

$$X_i(z) = \sum z^{\alpha_1+\cdots+\alpha_n}$$



$$xX_i(y)X_j(x) = yX_i(x)X_j(y)\ (i > j), \quad [X_i(x), X_j(y)] = [X_i(y), X_j(x)]\ (\forall i, j)$$

highly non-local

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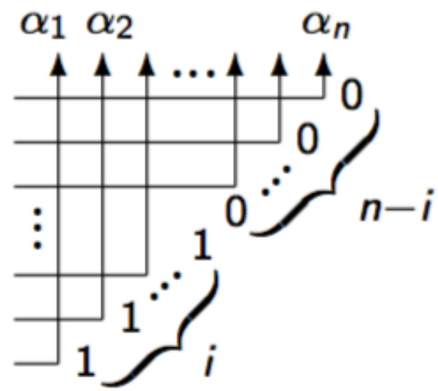

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$q \longrightarrow 0$

Strategy

Layer transfer matrices and their **bilinear relations** (q-melting)

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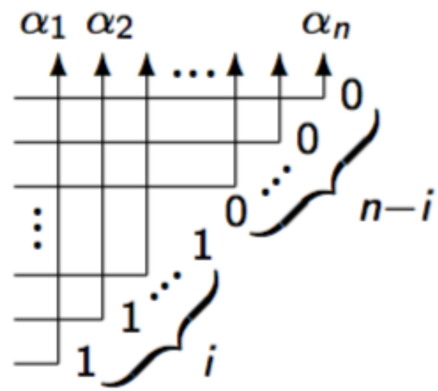

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highly non-local

Strategy

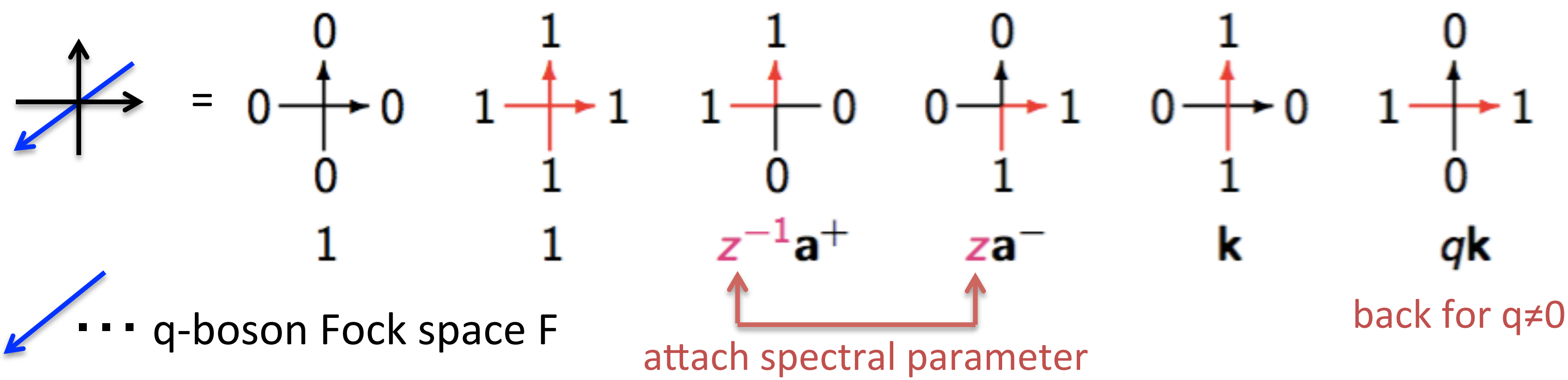
$q \longrightarrow 0$

Layer transfer matrices and their **bilinear relations** (q-melting)

Tetrahedron equation (*Single* local relation)

q-Melting: 5V $\longrightarrow$ 6V and Corner $\longrightarrow$ Square Layer

Introduce the 3D L-operator $\mathcal{L}(z)$ involving q, z by



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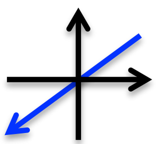
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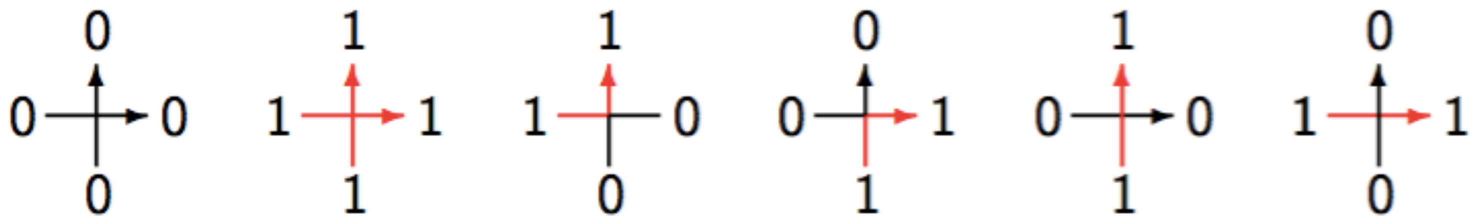
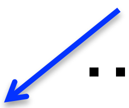


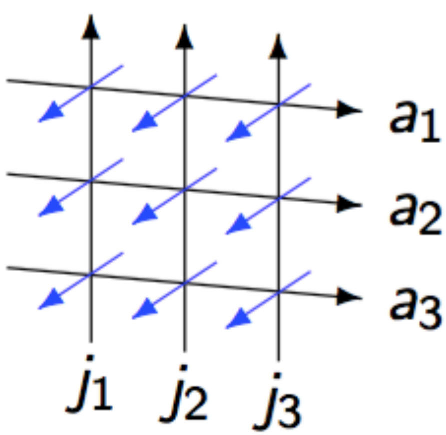
$1 \quad 1 \quad z^{-1}\mathbf{a}^+ \quad z\mathbf{a}^- \quad \mathbf{k} \quad q\mathbf{k}$

 ⋯ q-boson Fock space F

attach spectral parameter

back for q≠0

Define **Layer transfer matrix** $T(z)^{\mathbf{a}}_{\mathbf{j}} \in \mathrm{End}(F^{\otimes n^2})$ as

$T(z)^{\mathbf{a}}_{\mathbf{j}} =$ 

NW-free/SE-fixed boundary condition

$\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{j} = (j_1, \ldots, j_n)$

All vertices are $\mathcal{L}(z)$

Sum over all **black** edge spins except $\mathbf{a}, \mathbf{j}$

Bilinear relations of Layer transfer matrices $\mathbf{T(z)^a_j}$

Example. ($\cdots$ is a common sequence for $T(x)$ and $T(y)$)

(1) $T(x)^{\mathbf{a}}_{\mathbf{j}} T(y)^{\mathbf{a}}_{\mathbf{j}} = T(y)^{\mathbf{a}}_{\mathbf{j}} T(x)^{\mathbf{a}}_{\mathbf{j}}$ Commutativity

(2) $$y^2 T(x)^{0,\cdots}_{0,\cdots} T(y)^{1,\cdots}_{1,\cdots} + xyT(x)^{0,\cdots}_{1,\cdots} T(y)^{1,\cdots}_{0,\cdots} + xyT(x)^{1,\cdots}_{0,\cdots} T(y)^{0,\cdots}_{1,\cdots} + x^2 T(x)^{1,\cdots}_{1,\cdots} T(y)^{0,\cdots}_{0,\cdots} = (x \leftrightarrow y)$$

Fact: The ZF relations are included (!) in (1) and (2) at **q=0**.

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Fact: The ZF relations are included (!) in (1) and (2) at **q=0**.

(1) and (2) are r=s=0 and r=s=1 special cases of

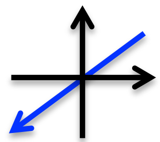
Prop. General bilinear relations of the layer transfer matrices

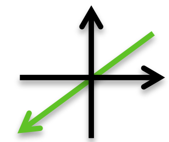
$$\sum x^{|\alpha|+|\beta|} y^{|\bar{\alpha}|+|\bar{\beta}|} T(x)^{..,\alpha_1,\ldots,\alpha_r,..}_{..,\beta_1,\ldots,\beta_s,..} T(y)^{..,\bar{\alpha}_1,\ldots,\bar{\alpha}_r,..}_{..,\bar{\beta}_1,\ldots,\bar{\beta}_s,..} = (x \leftrightarrow y).$$

- Sum over $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \{0,1\}^r$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s) \in \{0,1\}^s$.
- $0 \le r \le n$, $0 \le s \le n$ are arbitrary, $\bar{\alpha}_i = 1 - \alpha_i$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_r$, etc.
- Arrays "...." are arbitrary, but to be taken common for $T(x)$ and $T(y)$.

Proof of the bilinear relations of $\mathbf{T(z)^a_j}$

Introduce a variant of the 3D L-operator: $\mathcal{M}(z) := \mathcal{L}(z)|_{q\to -q}$.

$\mathcal{L}(z) =$ 

$\mathcal{M}(z) =$ 

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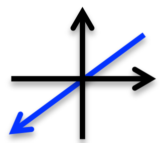
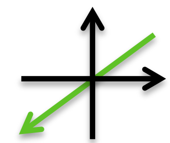
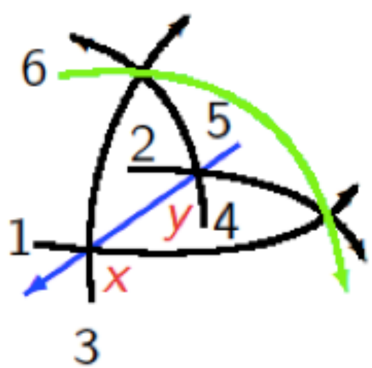
$\mathcal{L}(z) =$  $\mathcal{M}(z) =$ 

Theorem ($\mathcal{L}(z)$ and $\mathcal{M}(z)$ satisfy the tetrahedron equation)

$$\mathcal{M}(\tfrac{ux}{y})_{126}\mathcal{M}(u)_{346}\mathcal{L}(x)_{135}\mathcal{L}(y)_{245} = \mathcal{L}(y)_{245}\mathcal{L}(x)_{135}\mathcal{M}(u)_{346}\mathcal{M}(\tfrac{ux}{y})_{126}$$



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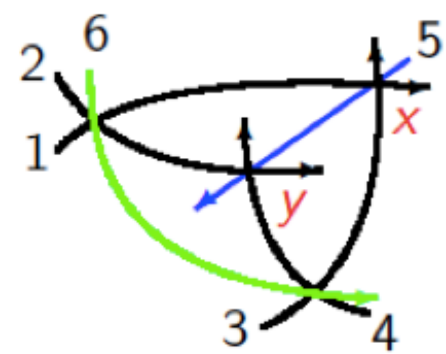
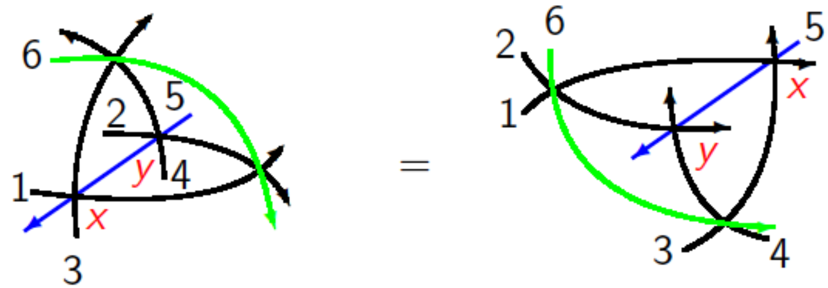


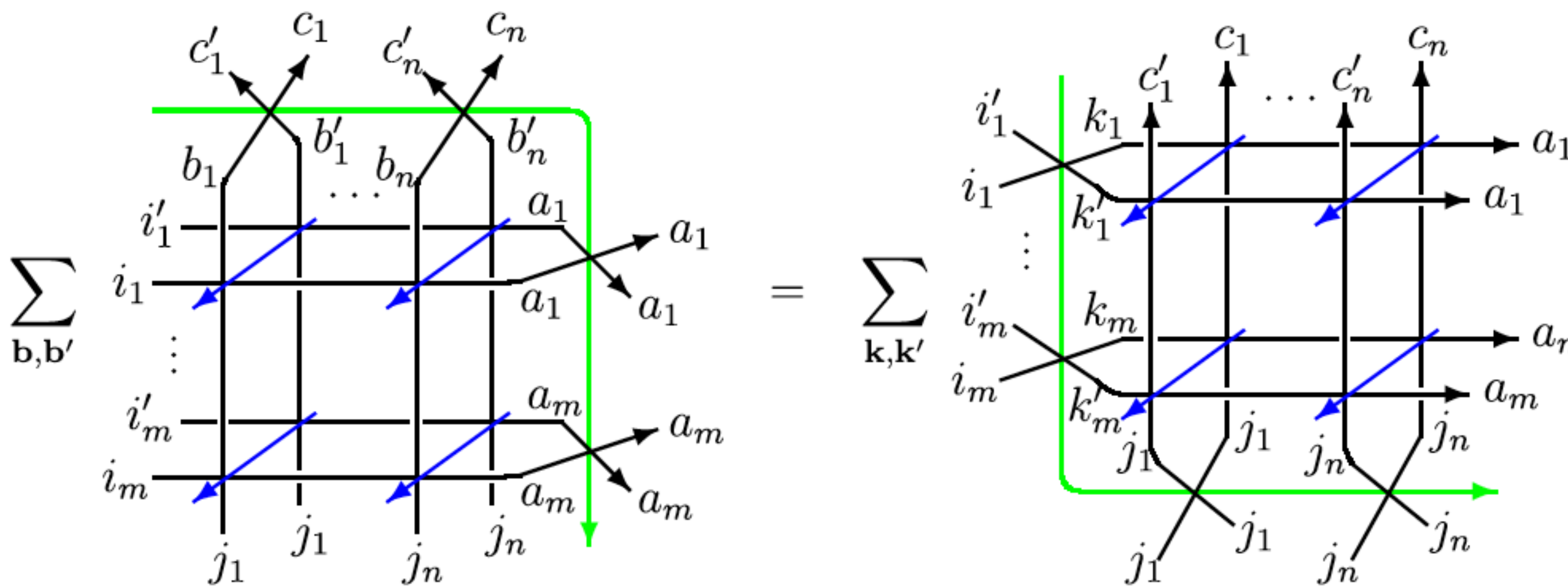
Equality of linear operators on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F \otimes F$

Originally introduced as a 3D analogue of YBE [Zamolodchikov 1980].

Repeated application of the tetrahedron eq.



leads to

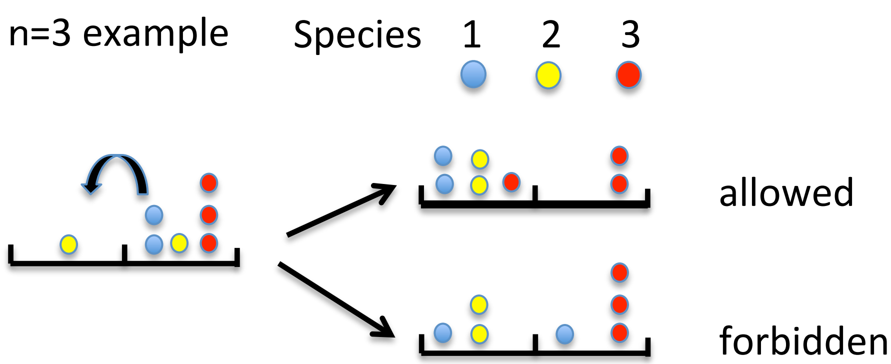


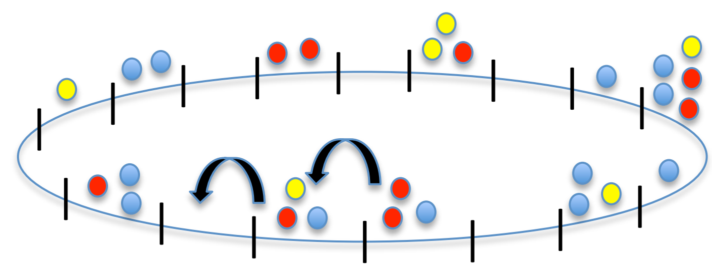
The bilinear relations are proved by evaluating this identity between various left/right eigenvectors of M(･) running along the green arrow.

QED

Remarks

Quite a parallel story in the *cross channel* holds also for a class of **n-species Totally Asymmetric Zero Range Process (n-TAZRP)** , where smaller species particles have the *priority* to hop to the left neighbor site.





Yang-Baxter:	R matrix for anti-symm. tensor rep.	R matrix for symm. tensor rep.
Tetrahedron:	3D **L**-operator	3D **R**-operator
Markov process:	n-TASEP (today's talk)	n-TAZRP [J. Integrable Sys. 2016]

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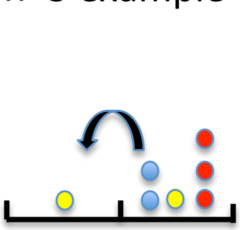
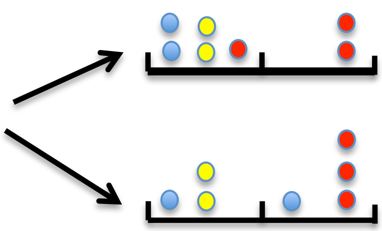
n=3 example

Species 1 2 3





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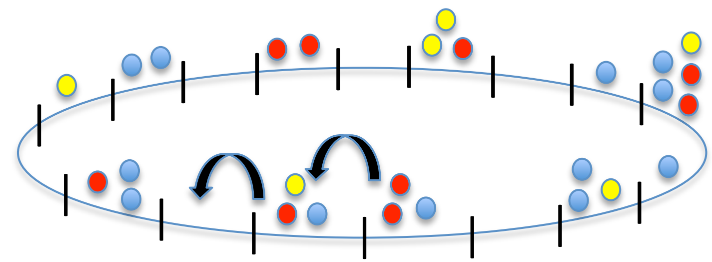

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A natural q-version of these models in the *direct channel* are formulated in terms of **Stochastic R matrices for $U_q(A^{(1)}_n)$.** [K-Mangazeev-Maruyama-Okado 2016].