Combinatorial Bethe ansatz
and
ultradiscrete integrable systems

Atsuo Kuniba (Univ. Tokyo)

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“***Unsere Method liefert also alle Lösungen des Problems.***”

··· H. Bethe (1931)

“ ... composition of our bijection with the Robinson-Schensted-Knuth correspondence may be viewed as a combinatorial version of the Bethe ansatz.”

··· Kerov-Kirillov-Reshetikhin (1986)

Plan

Part 1: Bethe ansatz (1931-) and its combinatorial version (1986-)

Part 2: Box-ball system (1990-)

Part 3: Their interplay including periodic case (2002-)

1 dimensional Heisenberg chain

$$\boldsymbol{H} = \sum_{k=1}^{L} (\boldsymbol{\sigma}_k^x \boldsymbol{\sigma}_{k+1}^x + \boldsymbol{\sigma}_k^y \boldsymbol{\sigma}_{k+1}^y + \boldsymbol{\sigma}_k^z \boldsymbol{\sigma}_{k+1}^z - 1)$$

$$\boldsymbol{\sigma}_k^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_k, \quad \boldsymbol{\sigma}_k^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_k, \quad \boldsymbol{\sigma}_k^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_k, \quad (\boldsymbol{\sigma}_{L+1}^a = \boldsymbol{\sigma}_1^a)$$

$\boldsymbol{H}$ acts on

$$(\mathbb{C}^2)^{\otimes L} = \otimes(\text{spin up or down})$$

Symmetry: $[sl_2, \boldsymbol{H}] = 0.$

In particular, $\boldsymbol{H}$ preserves $\boldsymbol{N} := \sharp$ of down spins.

Diagonalization of H. (Bethe 1931)

$$H|u_1,\dots,u_N\rangle = E|u_1,\dots,u_N\rangle, \quad E = \sum_{j=1}^{N} \frac{-8}{u_j^2+1},$$

if $u_1,\dots,u_N$ satisfy Bethe equation

$$\left(\frac{u_j+i}{u_j-i}\right)^L = \prod_{k=1\ (k\neq j)}^{N} \frac{u_j-u_k+2i}{u_j-u_k-2i} \qquad (j=1,\dots,N).$$

$|u_1,\dots,u_N\rangle$: Bethe vector

- contains N down spins
- symmetric under permutations of u_j's.

Example. Length $L = 6$ chain with $N = 3$ down spins.

Unknowns: u_1, u_2, u_3

$$\left(\frac{u_1+i}{u_1-i}\right)^6 = \frac{(u_1-u_2+2i)(u_1-u_3+2i)}{(u_1-u_2-2i)(u_1-u_3-2i)},$$
$$\left(\frac{u_2+i}{u_2-i}\right)^6 = \frac{(u_2-u_1+2i)(u_2-u_3+2i)}{(u_2-u_1-2i)(u_2-u_3-2i)},$$
$$\left(\frac{u_3+i}{u_3-i}\right)^6 = \frac{(u_3-u_1+2i)(u_3-u_2+2i)}{(u_3-u_1-2i)(u_3-u_2-2i)}.$$

How may solutions up to permutations of u_1, u_2 and u_3?

We should have 5 vectors if Bethe ansatz is complete.

Reason:

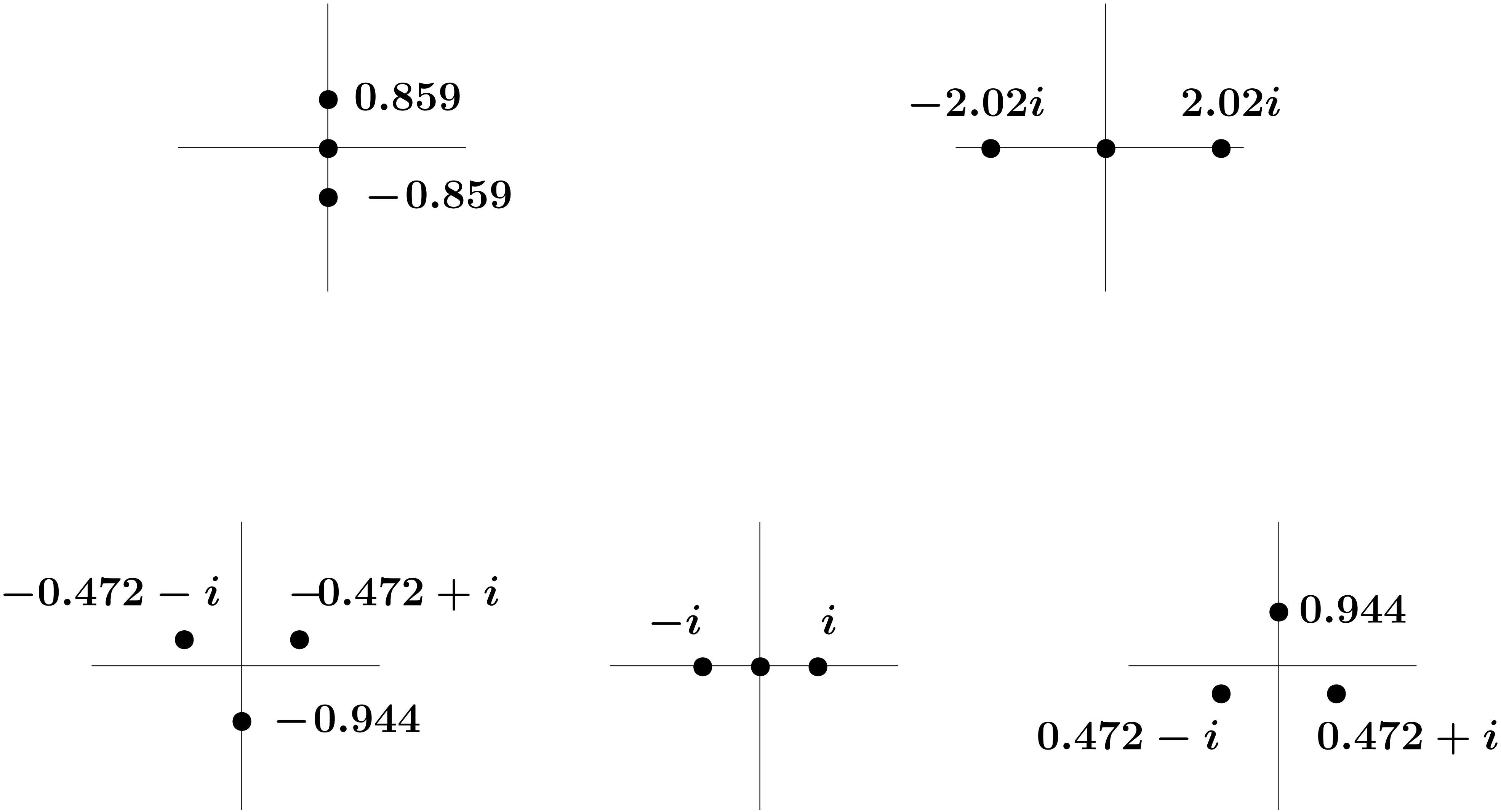
- **Bethe vector is highest weight vector with respect to sl_2.**
- **Irreducible decomposition:**

$$\square^{\otimes 6} = (6) + 5\,(5,1) + 9\,(4,2) + \mathbf{5}\,(3,3)$$

$$2^6 = 7 + 5 \times 5 + 9 \times 3 + 5 \times 1$$

$$(N = 0) \qquad (N = 1) \qquad (N = 2) \qquad (N = 3)$$

General case: Kostka number $K_{(L-N,N),(1^L)} = \binom{L}{N} - \binom{L}{N-1}$



0.859
−0.859
−2.02i
2.02i
−0.472 − i
−0.472 + i
−0.944
−i
i
0.944
0.472 − i
0.472 + i

String hypothesis: Bethe roots $\{u_1, \ldots, u_N\}$ = collection of **strings**.

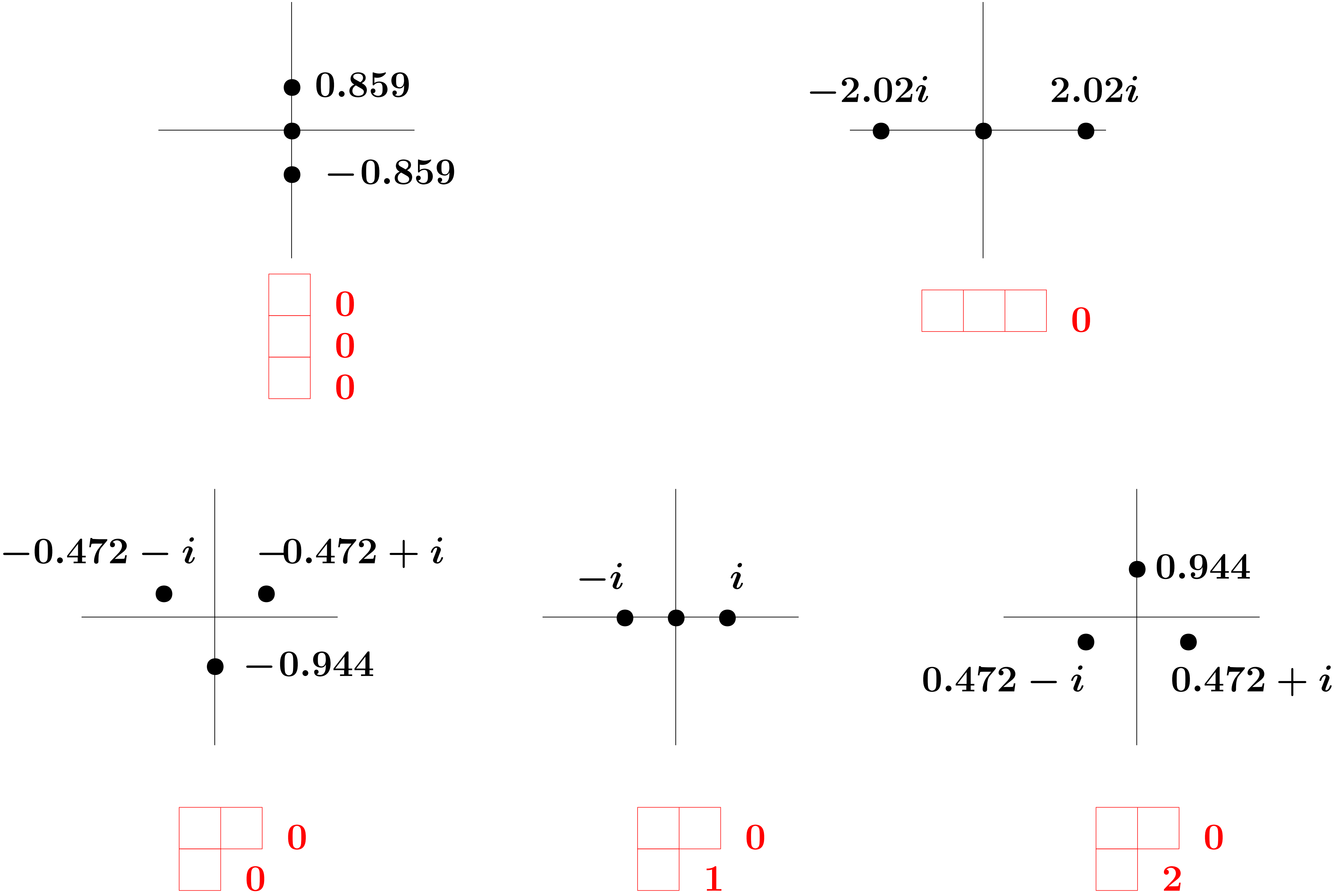
1-string	2-string	3-string	$\cdots$
α	$\beta - i$, $\beta + i$	$\gamma - 2i$, γ, $\gamma + 2i$	$\cdots$

with small distortions ($\alpha, \beta, \gamma, \ldots \in \mathbb{R}$: string center).

Bethe equation + string hypothesis

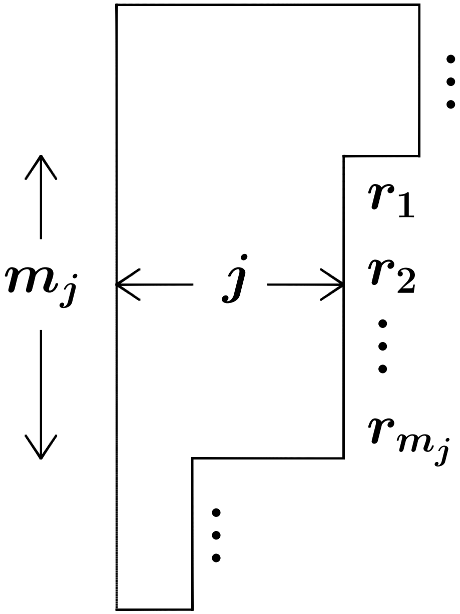
$\downarrow$ physicist's argument

$$\{\text{Bethe roots}\} \overset{1:1?}{\longleftrightarrow} \{\text{Rigged configuration}\}$$


0.859
−0.859
0
0
0
−2.02i
2.02i
0
−0.472 − i
−0.472 + i
−0.944
0
0
−i
i
0
1
0.944
0.472 − i
0.472 + i
0
2

Rigged configuration for sl_2 (spin $\frac{1}{2})^{\otimes L}$

Young diagram = configuration, $\{r_i\}$ = rigging



$$0 \le r_1 \le \cdots \le r_{m_j} \le p_j$$

$\cdots$ (fermionic) selection rule

$$p_j = L - 2\sum_{k\ge 1} \min(j,k) m_k$$

$\cdots$ vacancy number

$$\sharp \text{ of rigged configurations} = \sum_{\{m_i\}} \prod_{i \ge 1} \binom{p_i + m_i}{m_i}$$

{ Bethe roots } $\overset{1:1?}{\longleftrightarrow}$ { irreducible sl_2 components }

$\updownarrow$ 1 : 1 rigorously !

{ Bethe roots } $\underset{1:1?}{\longleftrightarrow}$ { rigged configuration }

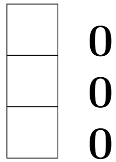
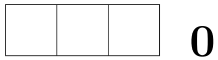
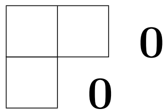
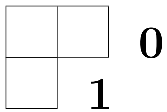
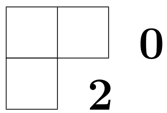
Theorem. ("alle Lösungen", Bethe's fermionic formula 1931)

$$\text{Kostka number } K_{(L-N,N),(1^L)} = \sum_{\{m_i\}} \prod_{i \geq 1} \binom{p_i + m_i}{m_i}$$

KKR theory. (Kerov-Kirillov-Reshetikhin 1986)

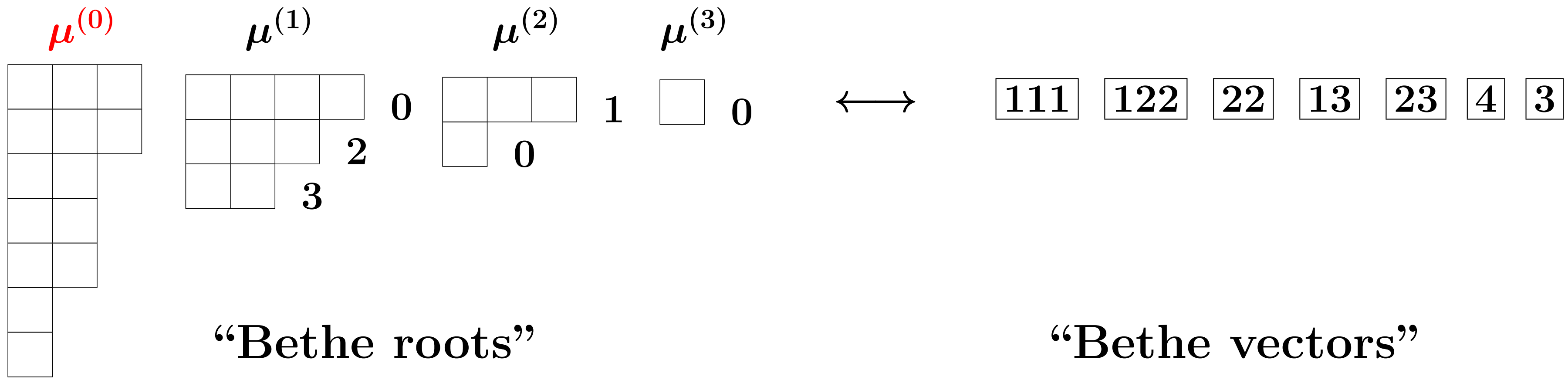
Canonical bijection and q-analogue of Bethe's formula from integrable spin chain with sl_n symmetry.

{rigged configurations} $\overset{\text{KKR}}{\longleftrightarrow}$ {highest paths}

	121212
	111222
	121122
	112212
	112122

KKR bijection for sl_n

$$\{\text{rigged configurations}\} \overset{1:1}{\longleftrightarrow} \{\text{highest paths}\}$$



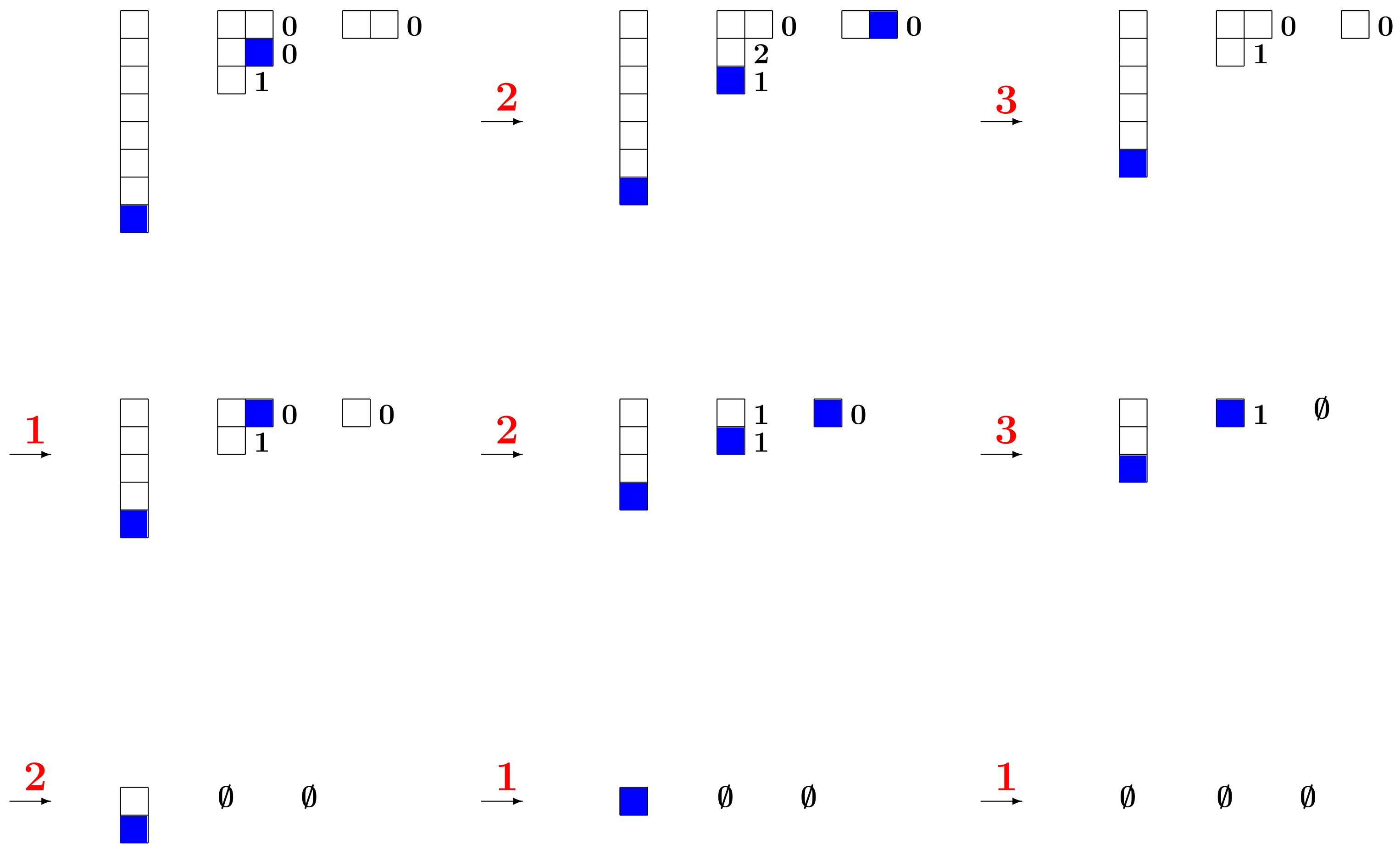
"Bethe roots" "Bethe vectors"

- highest path $= b_1 b_2 \dots b_L$, $\quad b_i =$ shape $(\mu_i^{(0)})$ semistandard tab.
  (+ highest condition)

- rigged configuration:

$$\left.\begin{array}{c}\mu^{(0)}, \mu^{(1)}, \dots, \mu^{(n-1)} : \text{configuration} \\ \text{attached with rigging}\end{array}\right\} \text{(+selection rule)}$$

Example of KKR algorithm



Top left rigged configuration $\xrightarrow{\text{KKR}}$ **11232132**

Summary of Part 1

KKR bijection

$$\{\text{rigged configurations}\} \overset{1:1}{\longleftrightarrow} \{\text{highest paths}\}$$

is a combinatorial analogue of the Bethe ansatz

$$\{\text{Bethe roots}\} \longrightarrow \{\text{Bethe vectors}\}$$

that establishes the Fermionic formula.

$\exists$ Conjectural fermionic formula for $\forall$affine Lie algebra (Hatayama et al. 1999, 2002).

Box-ball system on ∞ lattice (Takahashi-Satsuma 1990)

$$\cdots 11111111443221111111111111111111111111 \cdots$$
$$\cdots 11111111111114432211111111111111111111 \cdots$$
$$\cdots 11111111111111111144322111111111111111 \cdots$$

1 =empty box, $\quad 2, 3, 4 =$ color of balls

time evolution: $\quad T_\infty =$ (move 2)(move 3)(move 4)

(move i) · Pick the leftmost ball with color i and move it to the nearest right empty box.

· Do the same for the other color i balls.

- soliton=consecutive balls $i_1 \ldots i_a$ with color $i_1 \geq \cdots \geq i_a \geq 2$.
- velocity=amplitude.

- Collisions of 2 solitons

$$
\begin{array}{l}
\cdots 111\color{red}{44322}\color{black}{1111}\color{red}{433}\color{black}{11111111111111111111111} \cdots \\
\cdots 11111111443221143311111111111111111111 \cdots \\
\cdots 11111111111114432243311111111111111111 \cdots \\
\cdots 11111111111111111132244433111111111111 \cdots \\
\cdots 11111111111111111111132211444331111111 \cdots \\
\cdots 111111111111111111111111\color{red}{322}\color{black}{1111}\color{red}{44433}\color{black}{11} \cdots
\end{array}
$$

- Amplitudes are individually conserved.

- Two body scattering:

  Exchange of internal labels (colors) like quarks in hadrons

  Phase shift

Collision of 3 solitons

$\cdots$ 11432114211113111111111111111 $\cdots$
$\cdots$ 11111432142111311111111111111 $\cdots$
$\cdots$ 11111111431422131111111111111 $\cdots$
$\cdots$ 11111111114311423211111111111 $\cdots$
$\cdots$ 11111111111143112143211111111 $\cdots$
$\cdots$ 11111111111111431211143211111 $\cdots$
$\cdots$ 11111111111111114132111143211 $\cdots$

$\cdots$ 11432111142113111111111111111 $\cdots$
$\cdots$ 11111432111421311111111111111 $\cdots$
$\cdots$ 11111111432114231111111111111 $\cdots$
$\cdots$ 11111111111432124311111111111 $\cdots$
$\cdots$ 11111111111111413243211111111 $\cdots$
$\cdots$ 11111111111111141132143211111 $\cdots$
$\cdots$ 11111111111111114111321143211 $\cdots$

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

Double origin of integrability

(1) UltraDiscretization (UD) of soliton equations

- **Key formula**

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) + \exp(\frac{b}{\varepsilon}) \right) = \max(a, b)$$

$$(+, \times) \longrightarrow (\max, +)$$

keeps distributive law:

$$AB + AC = A(B + C) \to \max(a + b, a + c) = a + \max(b, c)$$

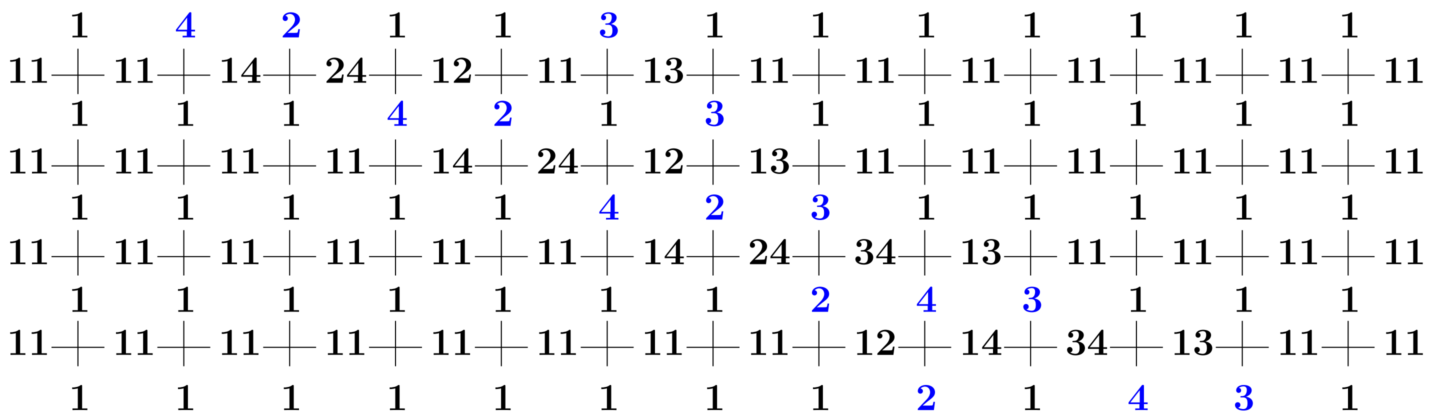
- UD of a discrete KdV equation turns out to be an evolution eq. of the 2-state box-ball system. (Tokihiro et al. 1996)

(2) Solvable lattice model at "temperature 0"

Time evolution pattern

$$\cdots 1421131111111 \cdots$$
$$\cdots 1114213111111 \cdots$$
$$\cdots 1111142311111 \cdots$$
$$\cdots 1111111243111 \cdots$$
$$\cdots 1111111121431 \cdots$$

is a configuration of a 2D lattice model in statistical mechanics



- **n-state box-ball system**

  = 2D solvable vertex model associated with quantum group

  $U_q(\widehat{sl}_n)$ at $q = 0$. ($q \sim$ temperature)

- **Row transfer matrix at $q = 0$**

  = deterministic map

  = time evolution of box-ball system
  (forms a commuting family $T_1, T_2, \ldots T_\infty$.)

- **Proper formulation by Kashiwara's crystal base theory**
  (Hatayama et al. 2000, Fukuda et al. 2000)

Some outcome.

- **integrable cellular automata associated with affine Lie algebras**

example: $D_5^{(1)} = \widehat{so}_{10}$ **-automaton**

$$
\begin{array}{l}
\cdots 111\bar{3}\bar{5}3221111\bar{2}\bar{2}\bar{5}11111111111111111111111111111111 \cdots \\
\cdots 11111111\bar{3}\bar{5}32211\bar{2}\bar{2}\bar{5}11111111111111111111111111111 \cdots \\
\cdots 1111111111111\bar{3}\bar{5}322\bar{2}\bar{2}\bar{5}11111111111111111111111111 \cdots \\
\cdots 111111111111111111\bar{3}\bar{5}3\bar{1}\bar{1}\bar{5}11111111111111111111111 \cdots \\
\cdots 111111111111111111111114\bar{1}\bar{1}\bar{4}5\bar{5}111111111111111111 \cdots \\
\cdots 1111111111111111111111111142 2\bar{2}\bar{2}\bar{4}5\bar{5}1111111111111 \cdots \\
\cdots 11111111111111111111111111111422 11\bar{2}\bar{2}\bar{4}5\bar{5}11111111 \cdots \\
\cdots 11111111111111111111111111111111422 1111\bar{2}\bar{2}\bar{4}5\bar{5}111 \cdots
\end{array}
$$

- **particles and anti-particles undergo pair-creations/annihilations**

- **solitons and their scattering rules characterized by crystal theory. (Hatayama-K-Okado-Takagi-Yamada 2002)**

Summary of Part 1 and 2

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	$\overset{\text{UD}}{\longrightarrow}$	Cellular automata Box-ball systems	$\overset{0 \leftarrow q}{\longleftarrow}$	Lattice statistical models Solvable vertex models
Inverse scattering method		Combinatorial Bethe ansatz		Bethe ansatz

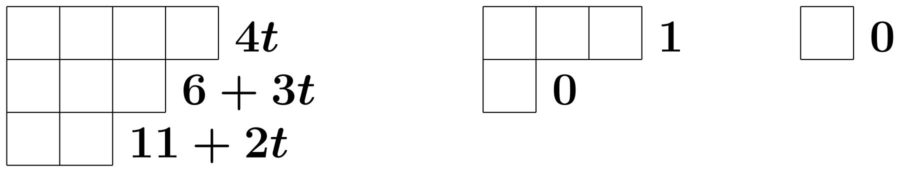
Natural to apply KKR theory to box-ball systems.

Dynamics of box-ball system in terms of rigged configuration

$t = 0$: 111122221111332114311111111111111111111111111111
$t = 1$: 111111122221113321431111111111111111111111111111
$t = 2$: 111111111111222211133243111111111111111111111111
$t = 3$: 111111111111111122221132433111111111111111111111
$t = 4$: 111111111111111111112221322433111111111111111111
$t = 5$: 111111111111111111111112211322433211111111111111
$t = 6$: 111111111111111111111111122111322143321111111111
$t = 7$: 111111111111111111111111111221111322114332111111

$\mu^{(0)}$ $\mu^{(1)}$ $\mu^{(2)}$ $\mu^{(3)}$

(1^{48})



configuration $\cdots$ conserved quantity (action variable !)

rigging $\cdots$ linear flow (angle variable !)

KKR bijection $\cdots$ direct/inverse scattering map (separation of variables)

Theorem. (K-Sakamoto-TY 2006)

- **KKR theory (a version of combinatorial Bethe ansatz) is the inverse scattering scheme of the box-ball system on ∞ lattice.**
- **Solution of the initial value probelm.**

KKR theory	box-ball system
rigged configuration	scattering data
KKR bijection	direct/inverse scattering

One can define

$$\tau_{k,i} = \text{piecewise linear function ("Energy") on rigged configuration}$$

such that

(1) τ **= UD of tau functions for KP hierarchy**

(2) τ **= "Baxter's corner transfer matrix" for box-ball system**

(3) Eq. of motion of box-ball system becomes

$$\tau + \tau = \max(\tau + \tau, \tau + \tau) \quad \cdots \text{ UD Hirota bilinear}$$

$$(\text{cf. } \tau\tau = \tau\tau + \tau\tau \quad \cdots \text{ usual Hirota bilinear})$$

(4) Image of KKR map $= \tau - \tau + \tau - \tau.$

	Bethe ansatz	Corner transfer matrix
main combinatorial object	rigged configuration	energy
role in box-ball system	action-angle variable	tau function
dynamics	linear	bilinear

Periodic box-ball system (sl_2 spin $\frac{1}{2}$, size L)

evolution under T_2

1 1 2 1 1 1 2 2 2 1 1 1 2 2
2 2 1 2 1 1 1 1 2 2 2 1 1 1
1 1 2 1 2 2 1 1 1 1 2 2 2 1
2 1 1 2 1 1 2 2 1 1 1 1 2 2
2 2 2 1 2 1 1 1 2 2 1 1 1 1
1 1 2 2 1 2 2 1 1 1 2 2 1 1
1 1 1 1 2 1 2 2 2 1 1 1 2 2
2 2 1 1 1 2 1 1 2 2 2 1 1 1

evolution under T_3

1 1 2 1 1 1 2 2 2 1 1 1 2 2
2 2 1 2 1 1 1 1 1 2 2 2 1 1
1 1 2 1 2 2 2 1 1 1 1 1 2 2
2 2 1 2 1 1 1 2 2 2 1 1 1 1
1 1 2 1 2 2 1 1 1 1 2 2 2 1
2 2 1 2 1 1 2 2 1 1 1 1 1 2
1 1 2 1 2 2 1 1 2 2 2 1 1 1
1 1 1 2 1 1 2 2 1 1 1 2 2 2

commuting family of time evolutions $T_1, T_2, \ldots$.

T_∞ = "ball moving" procedure (Yoshihara et al. 2003).

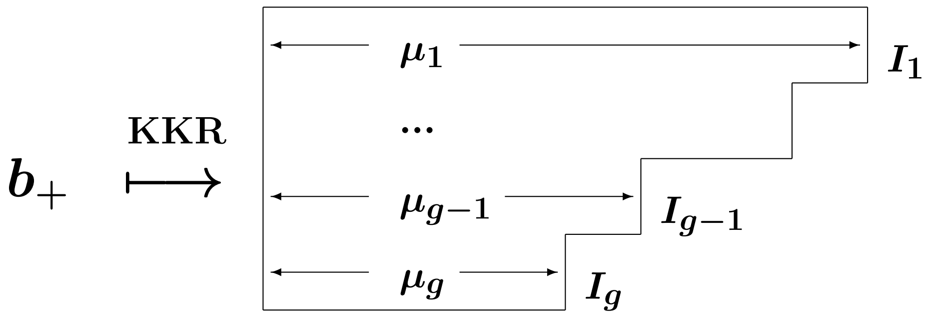
- **Action-angle variables**

Any path b is expressed as

$$b = T_1^{d}(b_+) \quad (T_1 : \text{cyclic shift}, \;\; b_+ : \text{highest path}, \; d \in \mathbb{Z}),$$

where (d, b_+) is not unique.

But anyway let (μ, I) be the rigged configuration for b_+ :



$$\mu = (\mu_1, \ldots, \mu_g), \quad \mathrm{I} = (I_1, \ldots, I_g).$$

$$\begin{pmatrix} \text{For simplicity we assume } \mu_1 > \mu_2 > \cdots > \mu_g \\ p_i = L - 2\sum_{j \in \mu} \min(i, j) : \text{vacancy number} \end{pmatrix}$$

Lemma.

- μ and $(\mathrm{I} + d\mathrm{h}_1)/A\mathbb{Z}^g$ are unique, where

$$\mathrm{h}_l = (\min(l, i))_{i \in \mu} \in \mathbb{Z}^g, \quad A = \big(\delta_{ij} p_i + 2\min(i, j)\big)_{i,j \in \mu}.$$

- μ is invariant under $\{T_l\}$ (action variable).

  (= list of soliton's amplitudes)

Define

$\mathcal{P}(\mu) := \{\text{paths whose action variable} = \mu\}$: iso-level set

$\mathcal{J}(\mu) := \mathbb{Z}^g / A\mathbb{Z}^g$: set of angle variables

$\Phi(b) := (\mathrm{I} + d\mathrm{h}_1)/A\mathbb{Z}^g \ : \ \mathcal{P}(\mu) \longrightarrow \mathcal{J}(\mu)$

Theorem. **(KT-Takenouchi 2006)**

$\Phi : \mathcal{P}(\mu) \to \mathcal{J}(\mu)$ is a bijection.

$$\begin{array}{ccc} \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \\ {\scriptstyle T_l}\downarrow & & \downarrow{\scriptstyle T_l} \\ \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \end{array}$$

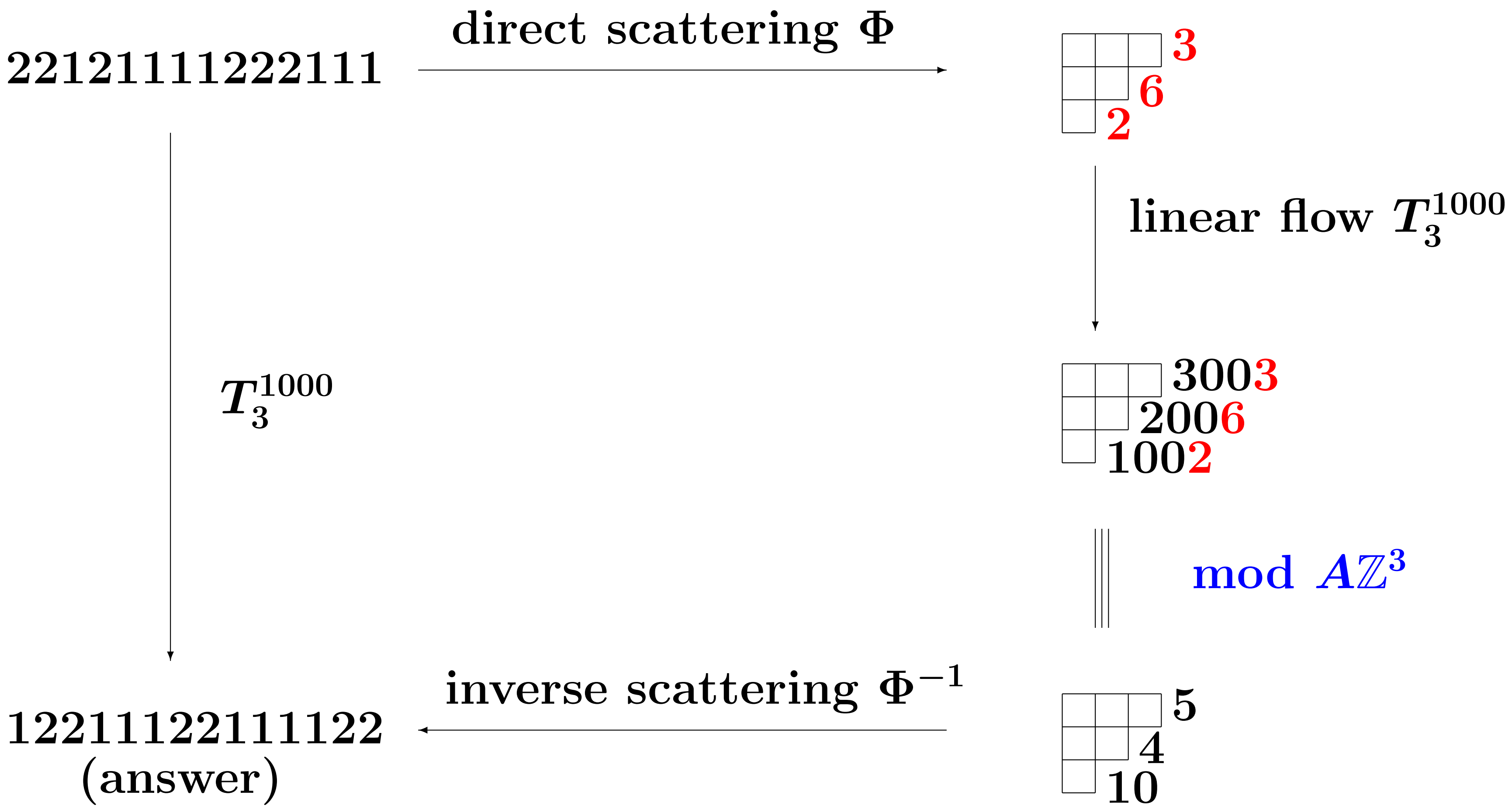
is commutative, where $T_l(\mathrm{J}) = \mathrm{J} + \mathrm{h}_l$ on $\mathcal{J}(\mu)$ ∎

Nonlinear dynamics becomes straight motions in

$$\mathcal{J}(\mu) = \mathbb{Z}^g / A\mathbb{Z}^g,$$

which is an ultradiscrete analogue of Jacobi variety.

Solution of initial value problem (inverse method)



Riemann theta (with pure imaginary period matrix) :

$$\vartheta(\mathrm{z}) := \sum_{\mathrm{n}\in\mathbb{Z}^g} \exp\Big(-\frac{{}^t\mathrm{n}A\mathrm{n}/2 + {}^t\mathrm{n}\mathrm{z}}{\epsilon}\Big)$$

UD Riemann theta ($\mathrm{z} \in \mathbb{R}^g$):

$$\Theta(\mathrm{z}) := \lim_{\epsilon\to+0} \epsilon \log \vartheta(\mathrm{z}) = -\min_{\mathrm{n}\in\mathbb{Z}^g}\{{}^t\mathrm{n}A\mathrm{n}/2 + {}^t\mathrm{n}\mathrm{z}\}$$

<u>Theorem</u>. (KS 2006)

Action-angle variable $(\mu, \mathrm{I}) \mapsto$ path $b_1 b_2 \dots b_L \in \{1,2\}^L$ is given by

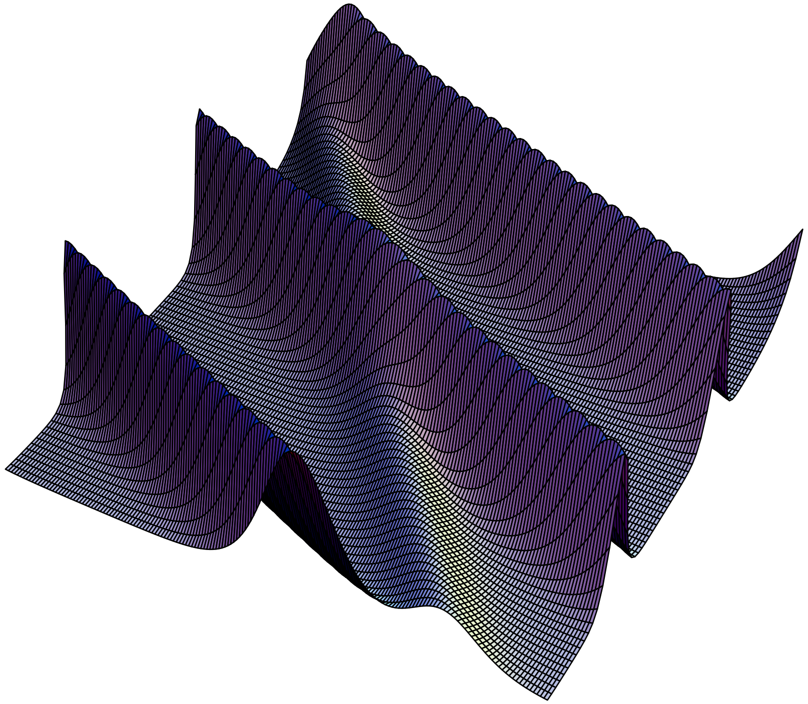
$$\begin{aligned} b_k = 1 &+ \Theta\big(\mathrm{J} - k\mathrm{h}_1\big) - \Theta\big(\mathrm{J} - (k-1)\mathrm{h}_1\big) \\ &- \Theta\big(\mathrm{J} - k\mathrm{h}_1 + \mathrm{h}_\infty\big) + \Theta\big(\mathrm{J} - (k-1)\mathrm{h}_1 + \mathrm{h}_\infty\big), \end{aligned}$$

with $\mathrm{J} = \mathrm{I} - {}^t(p_{\mu_1}, \dots, p_{\mu_g})/2$.

Inverse UD: double difference of $\Theta \longrightarrow$ double ratio of ϑ

$$b(k,t) = \frac{\vartheta\big(\mathrm{J} + t\mathrm{h}_\infty - k\mathrm{h}_1\big)\vartheta\big(\mathrm{J} + (t+1)\mathrm{h}_\infty - (k-1)\mathrm{h}_1\big)}{\vartheta\big(\mathrm{J} + t\mathrm{h}_\infty - (k-1)\mathrm{h}_1\big)\vartheta\big(\mathrm{J} + (t+1)\mathrm{h}_\infty - k\mathrm{h}_1\big)}.$$

Same structure as the quasi-periodic solution of the KdV/Toda eq. by Date-Tanaka and Kac-van Moerbeke (1976).



Two soliton state with amplitudes 6 and 2.
System size $L = 170$, duration $0 \leq t \leq 70$.

- **General case: UD Riemann theta with rational characteristics.**
- $\mathcal{J}(\mu)$ **from Jacobian of tropical hyperelliptic curve.**

Summary

- **Bethe ansatz persists in a combinatorial setting.**

- **Links the limits of soliton equations and solvable lattice models as an ultradiscrete integrable system.**

- **Provides combinatorial analogue of inverse scattering method, corner transfer matrix, tau functions, Jacobi-variety and Riemann theta function, etc.**

Reviews

- **Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry,** **R. Inoue, A. Kuniba, T. Takagi, J. Phys. A 45 (2012) 073001**

- ***Bethe ansatz and combinatorics*, A. Kuniba (2011), Asakura Publ.**