Density and current profiles for $U_q(A^{(1)}_2)$ zero range process

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Based on [K & Mangazeev, arXiv:1705.10979, NPB in press]

Matrix Program: Integrability in low-dimensional quantum systems

University of Melbourne, Creswick 13 July 2017

Non-equilibrium statistical mechanics

Stochastic dynamics,

Markov process, …

Integrable systems

Quantum groups,

Yang-Baxter equation, …

Integrable Markov process

Spectral problem of the Markov matrix: solvable by Bethe ansatz

Exact asymptotic analysis: connection to random matrices, etc

Prototype examples

Asymmetric simple exclusion process

Asymmetric zero range process ← Today

Prominent features in multispecies (higher rank) models

Stochastic R matrix

Matrix product structure of stationary states

Zamolodchikov-Faddeev algebra

Hidden 3D structure related to the tetrahedron equation

Generalization of Ferrari-Martin type algorithm by crystal base

} skipped today

Outline

Stochastic R matrix for $U_q(A^{(1)}_n)$.

Integrable n species zero range process.

Stationary states and matrix product formula.

n=2 case. Grand canonical ensemble treatment of the 2nd class particles, where the 1st class particles are fixed as defects.

Exact profiles of the local density and current of the 2nd class particles.

References:

K, Mangazeev, Maruyama, Okado,
Stochastic R matrix for $U_q(A^{(1)}_n)$, NPB 2016

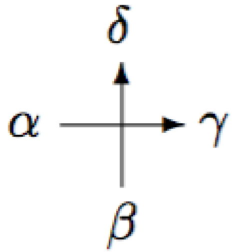
K, Okado,
A q-boson representation of Zamolodchikov-Faddeev algebra for stochastic R matrix of $U_q(A^{(1)}_n)$, LMP 2017

K, Mangazeev
Density and current profiles in $U_q(A^{(1)}_2)$-zero range process, arXiv:1705.10979 , NPB

Stochastic R matrix for $U_q(A^{(1)}_n)$

$$W = \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}} \mathbb{C}|\alpha\rangle, \qquad \mathcal{S}(\lambda,\mu) \in \mathrm{End}(W\otimes W)$$

$$\mathcal{S}(\lambda,\mu)(|\alpha\rangle\otimes|\beta\rangle) = \sum_{\gamma,\delta\in\mathbb{Z}^n_{\geq 0}} \mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta}\,|\gamma\rangle\otimes|\delta\rangle$$



$$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = \quad = q^{\sum_{i<j}(\beta_i-\gamma_i)\gamma_j}\left(\frac{\mu}{\lambda}\right)^{|\gamma|}\frac{(\lambda)_{|\gamma|}(\frac{\mu}{\lambda})_{|\beta|-|\gamma|}}{(\mu)_{|\beta|}}\prod_{i=1}^{n}\binom{\beta_i}{\gamma_i}_q$$

$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = 0$ unless $\alpha+\beta=\gamma+\delta$ and $\beta_i \geq \gamma_i$.

$$\alpha = (\alpha_1,\ldots,\alpha_n), \qquad |\alpha| = \alpha_1+\cdots+\alpha_n,$$

$$(z)_m = (z;q)_m = \prod_{j=0}^{m-1}(1-zq^j), \qquad \binom{m}{k}_q = \frac{(q)_m}{(q)_k(q)_{m-k}}.$$

Theorem [K, Mangazeev, Maruyama, Okado 2016]

$$\text{YBE}: \quad \mathcal{S}_{1,2}(\nu_1,\nu_2)\mathcal{S}_{1,3}(\nu_1,\nu_3)\mathcal{S}_{2,3}(\nu_2,\nu_3) = \mathcal{S}_{2,3}(\nu_2,\nu_3)\mathcal{S}_{1,3}(\nu_1,\nu_3)\mathcal{S}_{1,2}(\nu_1,\nu_2),$$

$$\text{Sum-to-1}: \quad \sum_{\gamma,\delta\in\mathbb{Z}^n_{\geq 0}} \mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = 1 \qquad (\forall \alpha,\beta\in\mathbb{Z}^n_{\geq 0}).$$

The stochastic R matrix originates in the factorized(!) special value of the $U_q(A_n^{(1)})$ quantum R matrix on (spin $l/2$) $\otimes$ (spin $m/2$) representations ($l \leq m$):

$$R(z=q^{l-m})^{\gamma,\delta}_{\alpha,\beta} = q^{\sum_{i<j}(\alpha_i(\beta_j-\gamma_j)+(\beta_i-\gamma_i)\gamma_j)} \binom{m}{l}^{-1}_{q^2} \prod_i \binom{\beta_i}{\gamma_i}_{q^2}$$

This factorization had escaped notice more than 30 years.
It will turn out to fit the stochastic setting perfectly.
The parameters in the stochastic R matrix originate in

$$\lambda = q^{-2l}, \qquad \mu = q^{-2m}$$

The case n=1 goes back to Povolotsky (2013) and Corwin-Petrov (arXiv:2015)

$$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = q^{\sum_{i<j}(\beta_i-\gamma_i)\gamma_j} \left(\frac{\mu}{\lambda}\right)^{|\gamma|} \frac{(\lambda)_{|\gamma|}(\frac{\mu}{\lambda})_{|\beta|-|\gamma|}}{(\mu)_{|\beta|}} \prod_{i=1}^{n} \binom{\beta_i}{\gamma_i}_q$$

Not difference type : $\mathcal{S}(\lambda,\mu) \neq \mathcal{S}(c\lambda, c\mu)$

Defines an infinite state-vertex model obeying the conservation law

$$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = 0 \quad \text{unless } \alpha+\beta=\gamma+\delta \in \mathbb{Z}^n_{\geq 0}$$

Sum-to-1 property $\sum_{\gamma,\delta} \mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = 1$ follows from

$U_q(A_n)$ –orbit of the unit normalization condition of R(z):

$$\Delta U_q(A_n)\Big(R(z)|\text{highest}\rangle \otimes |\text{highest}\rangle - |\text{highest}\rangle \otimes |\text{highest}\rangle\Big) = 0$$

It will eventually lead to the total probability conservation in the associated zero range process.

Commuting Markov transfer matrices

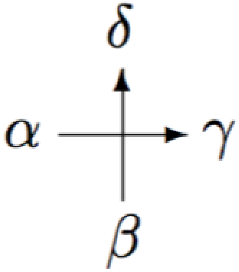
Consider the tensor product $W_0 \otimes W_1 \otimes \cdots \otimes W_L$ $(W_i = W)$ and define

$$T(\lambda|\mu_1,\ldots,\mu_L) = \mathrm{Tr}_{W_0}\left(\mathcal{S}_{W_0,W_L}(\lambda,\mu_L)\cdots\mathcal{S}_{W_0,W_1}(\lambda,\mu_1)\right) \in \mathrm{End}(W^{\otimes L}).$$

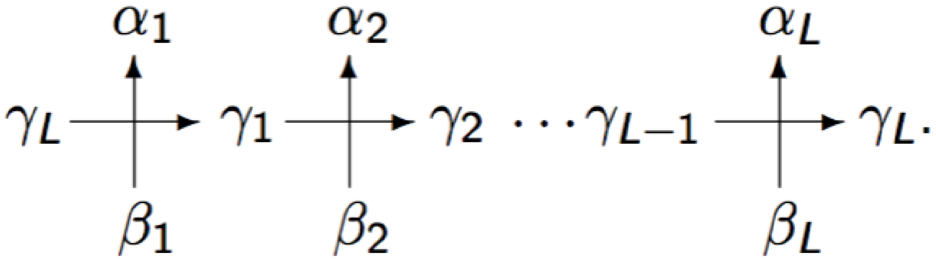
To illustrate

$$T|\beta_1,\ldots,\beta_L\rangle = \sum_{\alpha_1,\ldots,\alpha_L} T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L}|\alpha_1,\ldots,\alpha_L\rangle \in W^{\otimes L},$$

$$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} =$$



$$T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = \sum_{\gamma_1,\ldots,\gamma_L}$$



Discrete time Markov Process

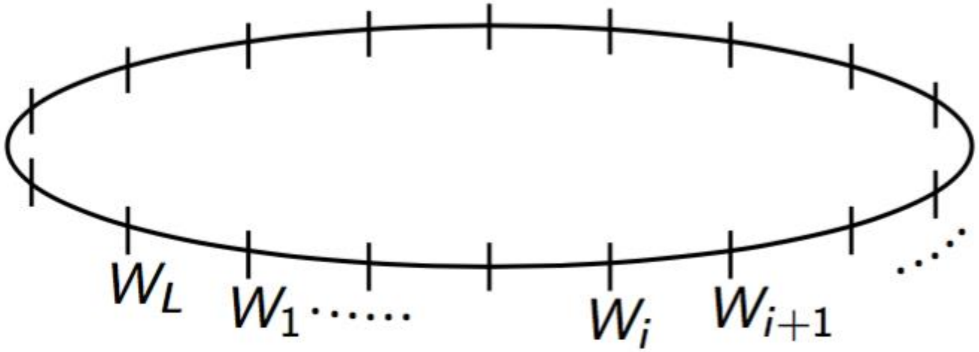
Proposition

1. *Sum-to-1:* $\sum_{\alpha_1,\ldots,\alpha_L} T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = 1$.
2. *Nonnegativity: Matrix elements of* $T(\lambda|\mu_1,\ldots,\mu_L) \in \mathbb{R}_{\geq 0}$ *when* $0 < \mu_i < \lambda < 1, 0 < q < 1$.
3. *YBE for* $\mathcal{S}(\lambda,\mu)$ *implies* $[T(\lambda|\mu_1,\ldots,\mu_L), T(\lambda'|\mu_1,\ldots,\mu_L)] = 0$.

Therefore

$$|P(t+1)\rangle = T(\lambda|\mu_1,\ldots,\mu_L)|P(t)\rangle \in W^{\otimes L}$$

defines a family of **discrete time Markov processes** that is simultaneously diagonalizable with respect to λ.



$(W_i = W)$

Continuous time Markov Process (1)

Set $\mu_1 = \cdots = \mu_L = \mu$, $T(\lambda|\mu) = T(\lambda|\mu, \ldots, \mu)$ and

$$H_+ = -\mu^{-1} \frac{\partial \log T(\lambda|\mu)}{\partial \lambda}\bigg|_{\lambda=1}, \qquad H_- = \mu \frac{\partial \log T(\lambda|\mu)}{\partial \lambda}\bigg|_{\lambda=\mu}.$$

Since $[T(\lambda|\mu), T(\lambda'|\mu)] = 0$, we have $[H_+, H_-] = 0$ and $T(\lambda|\mu), H_\pm$ all have common eigenvectors.

Baxter's formula works at **two** Hamiltonian points $\lambda = 1, \mu$.
$H_\pm$ are related by a daulity. Moreover, we have

1. Positivity; all the off-diagonal elements are nonnegative,
2. Sum-to-0; the sum of elements in any column is zero.

$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle \in W^{\otimes L}, \quad H = aH_+ + bH_- \ (a, b \in \mathbb{R}_{\geq 0})$$

defines a **continuous time Markov process**.

Continuous time Markov Process (2)

n=2 example

Particles 1 2

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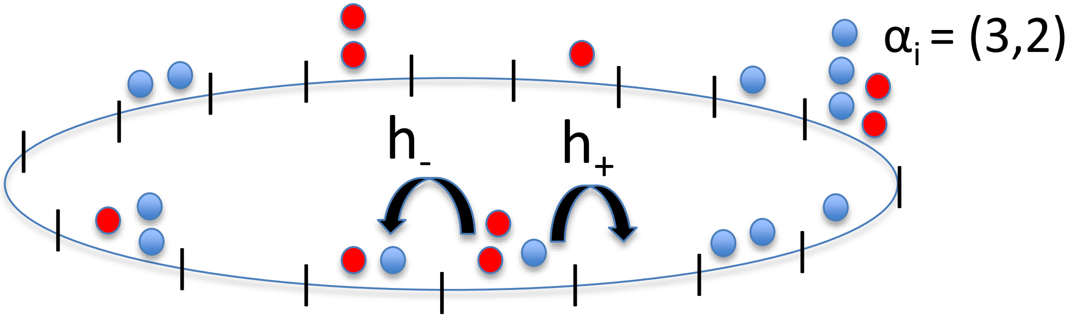


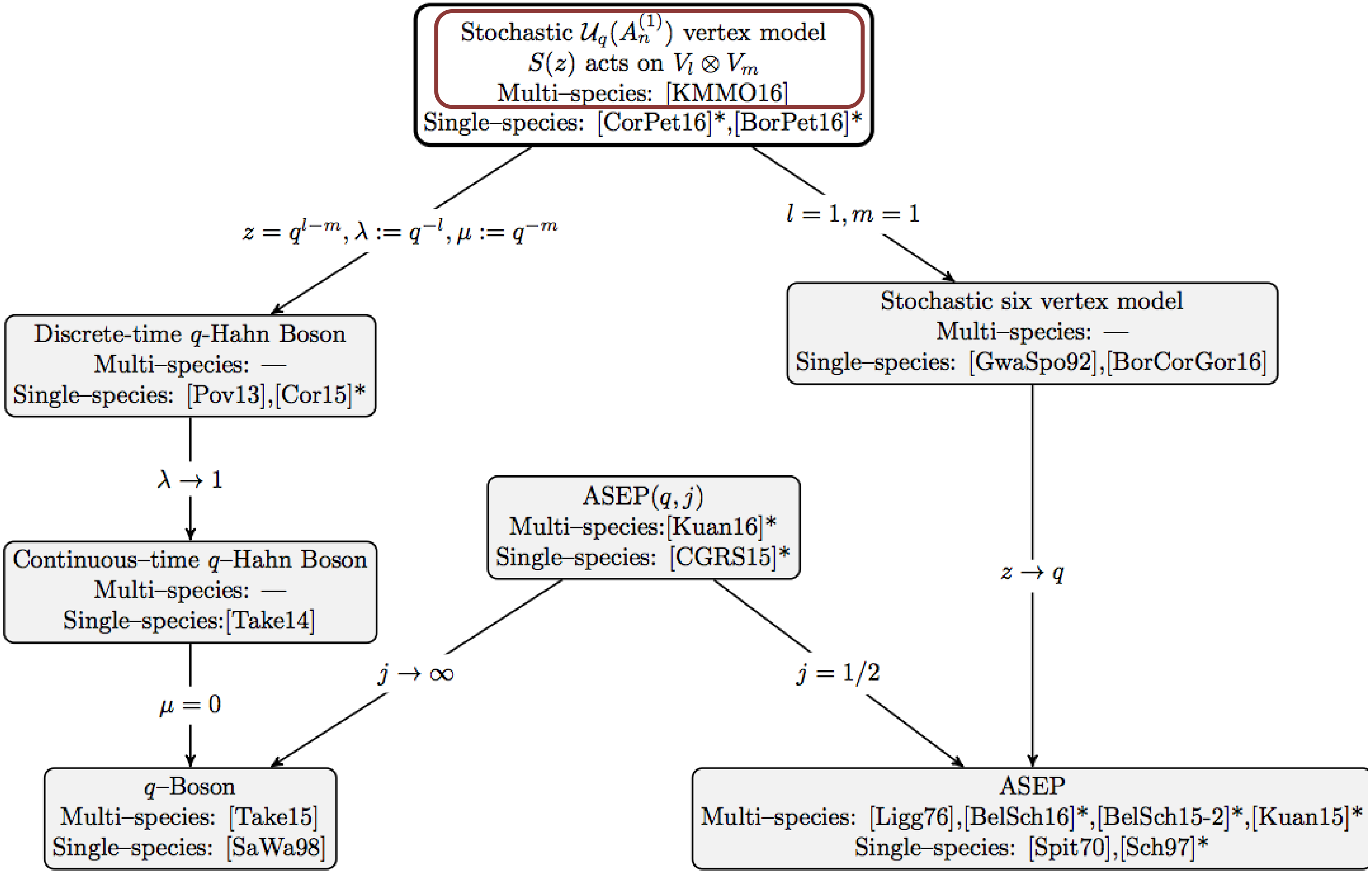
$H_\pm = \sum_{i\in\mathbb{Z}_L} h_{\pm,i,i+1}$ where $h_\pm$ is the **local** Markov matrix.

$$h_+|\alpha,\beta\rangle = \sum_{\gamma\in\mathbb{Z}^n_{\geq 0}\setminus\{0\}} \frac{q^{\sum_{1\leq i<j\leq n}(\alpha_i-\gamma_i)\gamma_j}\mu^{|\gamma|-1}(q)_{|\gamma|-1}}{(\mu q^{|\alpha|-|\gamma|};q)_{|\gamma|}} \prod_{i=1}^{n} \binom{\alpha_i}{\gamma_i}_q |\alpha-\gamma,\beta+\gamma\rangle,$$

$$h_-|\alpha,\beta\rangle = \sum_{\gamma\in\mathbb{Z}^n_{\geq 0}\setminus\{0\}} \frac{q^{\sum_{1\leq i<j\leq n}\gamma_i(\beta_j-\gamma_j)}(q)_{|\gamma|-1}}{(\mu q^{|\beta|-|\gamma|};q)_{|\gamma|}} \prod_{i=1}^{n} \binom{\beta_i}{\gamma_i}_q |\alpha+\gamma,\beta-\gamma\rangle$$

up to diagonal terms.

Defines a Zero Range Process of n-species of particles where the transition rate depends on the occupancy of the departure site only.



Stochastic $\mathcal{U}_q(A_n^{(1)})$ vertex model
$S(z)$ acts on $V_l \otimes V_m$
Multi–species: [KMMO16]
Single–species: [CorPet16]*,[BorPet16]*

$z = q^{l-m}, \lambda := q^{-l}, \mu := q^{-m}$

$l = 1, m = 1$

Discrete-time q-Hahn Boson
Multi–species: —
Single–species: [Pov13],[Cor15]*

Stochastic six vertex model
Multi–species: —
Single–species: [GwaSpo92],[BorCorGor16]

$\lambda \to 1$

ASEP(q, j)
Multi–species:[Kuan16]*
Single–species: [CGRS15]*

Continuous–time q–Hahn Boson
Multi–species: —
Single–species:[Take14]

$z \to q$

$\mu = 0$

$j \to \infty$

$j = 1/2$

q–Boson
Multi–species: [Take15]
Single–species: [SaWa98]

ASEP
Multi–species: [Ligg76],[BelSch16]*,[BelSch15-2]*,[Kuan15]*
Single–species: [Spit70],[Sch97]*

Contains many integrable stochastic models known earlier (taken from Kuan ArXiv:1701.04468)

Stationary states

Stationary states are those satisfying

$$|\overline{P}\rangle = T(\lambda|\mu_1,\ldots,\mu_L)|\overline{P}\rangle \in W^{\otimes L}.$$

Because of the weight conservation

$$T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = 0 \text{ unless } \alpha_1+\cdots+\alpha_L = \beta_1+\cdots+\beta_L \in \mathbb{Z}^n_{\geq 0},$$

T is a direct sum of matrices acting on finite-dimensional subspaces (**sectors**) of $W^{\otimes L}$ parametrized by $m = (m_1,\ldots,m_n) \in \mathbb{Z}^n_{\geq 0}$.

$$S(m) = \{(\sigma_1,\ldots,\sigma_L) \in (\mathbb{Z}^n_{\geq 0})^L \mid \sigma_1+\cdots+\sigma_L = m\},$$

$$|\overline{P}(m)\rangle = \sum_{(\sigma_1,\ldots,\sigma_L)\in S(m)} \underbrace{\mathbb{P}(\sigma_1,\ldots,\sigma_L)}_{\text{Stationary probability}}|\sigma_1,\ldots,\sigma_L\rangle.$$

Example

$n = 2, m = (2,1), \mu_1 = \mu_2 = \mu_3 = \mu$. The stationary states for $L = 2, 3$ are:

$$\begin{aligned}|\overline{P}(2,1)\rangle = {} & (1-q^2\mu)(3+q-\mu-3q\mu)|\emptyset, 112\rangle \\ & +(1-\mu)(1+q+2q^2-2q\mu-q^2\mu-q^3\mu)|2,11\rangle \\ & +(1+q)(1-\mu)(2+q+q^2-\mu-q\mu-2q^2\mu)|1,12\rangle + \text{cyclic.}\end{aligned}$$

$$\begin{aligned}|\overline{P}(2,1)\rangle = {} & 3(1-q\mu)(1-q^2\mu)(2+q-(1+2q)\mu)|\emptyset,\emptyset,112\rangle \\ & +(1-\mu)(1-q\mu)(3+3q+3q^2-(1+5q+2q^2+q^3)\mu)|\emptyset,2,11\rangle \\ & +(1+q)(1-\mu)(1-q\mu)(3+3q+3q^2-(2+2q+5q^2)\mu)|\emptyset,1,12\rangle \\ & +(1+q)(1-\mu)(1-q\mu)(5+2q+2q^2-(3+3q+3q^2)\mu)|\emptyset,12,1\rangle \\ & +(1-\mu)(1-q\mu)(1+2q+5q^2+q^3-(3q+3q^2+3q^3)\mu)|\emptyset,11,2\rangle \\ & +(1+q)(1+q+q^2)(1-\mu)^2(2+q-(1+2q)\mu)|1,1,2\rangle + \text{cyclic.}\end{aligned}$$

Conjecturally $\mathbb{P}(\sigma_1, \ldots, \sigma_L) \in \mathbb{Z}_{\geq 0}[q, -\mu_1, \ldots, -\mu_L]$ in a certain normalization.

Matrix product formula

Stationary state is

Perron-Frobenius Bethe ansatz

algebraic $\cap$ transcendental $\simeq$ Matrix product structure

$$\mathbb{P}(\sigma_1,\ldots,\sigma_L) = \mathrm{Tr}(\underbrace{X_{\sigma_1}(\mu_1)\cdots X_{\sigma_L}(\mu_L)}_{\text{Operators acting on some } \textbf{\textit{auxiliary space}}}).$$

Theorem

If the operators $X_\alpha(\mu)$ $(\alpha \in \mathbb{Z}^n_{\geq 0})$ satisfy the Zamolodchikov-Faddeev (ZF) algebra

$$X_\alpha(\mu)X_\beta(\lambda) = \sum_{\gamma,\delta} \mathcal{S}(\lambda,\mu)^{\beta,\alpha}_{\gamma,\delta} X_\gamma(\lambda)X_\delta(\mu)$$

and the trace is nonzero, the matrix product formula holds.

q-Boson realization

Consider the Fock space $F = \oplus_{m\geq 0}\mathbb{Q}(q)|m\rangle$ and the operators $\mathbf{b}, \mathbf{c}$ and $\mathbf{k}$ acting on it as

$$\mathbf{b}|m\rangle = |m+1\rangle, \qquad \mathbf{c}|m\rangle = (1-q^m)|m-1\rangle, \qquad \mathbf{k}|m\rangle = q^m|m\rangle$$

A q-boson representation of the ZF algebra on $F^{\otimes n(n-1)/2}$ has been obtained in [K-Okado, 2017]. For $n = 2$, it is effectively given by

$\alpha = (\alpha_1, \alpha_2)$

$$X_\alpha = \frac{(\mu)_{\alpha_1+\alpha_2}}{(q)_{\alpha_1}(q)_{\alpha_2}} \frac{(\mu\mathbf{b})_\infty}{(\mathbf{b})_\infty}\mathbf{k}^{\alpha_2}\mathbf{c}^{\alpha_1}, \qquad \frac{(\mu\mathbf{b})_\infty}{(\mathbf{b})_\infty} = \sum_{\boxed{j\geq 0}} \frac{(\mu;q)_j}{(q;q)_j}\mathbf{b}^j$$

Therefore in order to have non-vanishing stationary probability

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(X_{\sigma_1}(\mu_1)\cdots X_{\sigma_L}(\mu_L))$$

sum of the expansion indices **j** = total number of the first class particles.

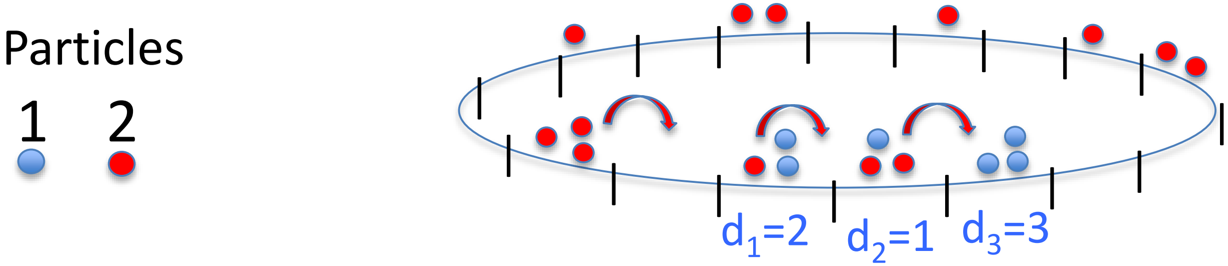
From now on $n=2$, and consider H_+ with homogeneous choice $\mu_1 = \ldots = \mu_L = \mu$.

Grand canonical ensemble treatment for the second class particles

Introduce the **fugacity** y and the generating series of matrix product operators

$$A_{\alpha_1} = \sum_{\alpha_2 \geq 0} y^{\alpha_2} X_{\alpha_1,\alpha_2} = \frac{(\mu;q)_{\alpha_1}}{(q;q)_{\alpha_1}} \frac{(\mu\mathbf{b};q)_\infty}{(\mathbf{b};q)_\infty} (y\mathbf{k};q)_\infty^{-1} \mathbf{c}^{\alpha_1} (\mu y \mathbf{k};q)_\infty$$

We fix $d_1, \ldots, d_s$ first class particles at site 1,…, s as *defects* and invetigate their influence on the grand canonical ensemble (GCE) of the second class particles.



Stationary average is replaced by the conditional one as

$$\frac{\cdots\cdots}{\sum_{\sigma_1,\ldots,\sigma_L} \mathrm{Tr}(X_{\sigma_1} \cdots X_{\sigma_L})} \Longrightarrow \frac{\cdots\cdots}{\mathrm{Tr}(A_{d_1} \cdots A_{d_s} A_0^{L-s})}$$

Defect-free case $d_1 = \dots = d_s = 0$

The system has translational invariance.

Partition function $\lim_{L\to\infty} \frac{1}{L} \log \mathrm{Tr}(A_0^L) = \frac{(\mu y)_\infty}{(y)_\infty}$

Density $\rho = \lim_{L\to\infty} \frac{1}{L} y \frac{\partial}{\partial y} \log \mathrm{Tr}(A_0^L) = \sum_{i\geq 0} \frac{(1-\mu)yq^i}{(1-yq^i)(1-\mu yq^i)}$

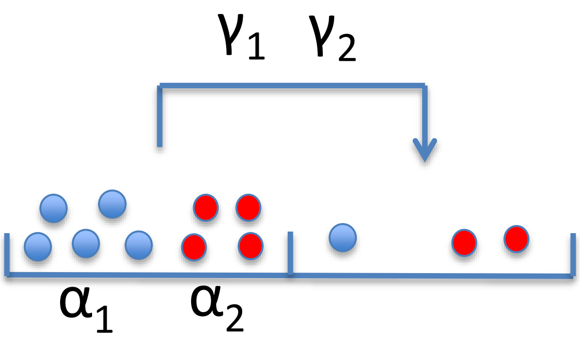
which is a difference f(y) - f(μy) of the q-digamma function $f(\zeta) = -\zeta \frac{\partial}{\partial \zeta} \log(\zeta)_\infty$

One to one correspondence between the fugacity $0 < y < 1$ and density $\rho > 0$ (no phase transition like ***condensation***) .

Probability of finding n second class particles at any site is

$$P(n) = \lim_{L\to\infty} \frac{y^n \mathrm{Tr}\big(X_{0,n} A_0^{L-1}\big)}{\mathrm{Tr}\big(A_0^L\big)} = y^n \frac{(\mu)_n}{(q)_n} \frac{(y)_\infty}{(\mu y)_\infty}$$

To define the local **current** J, denote the rate for the local transition by

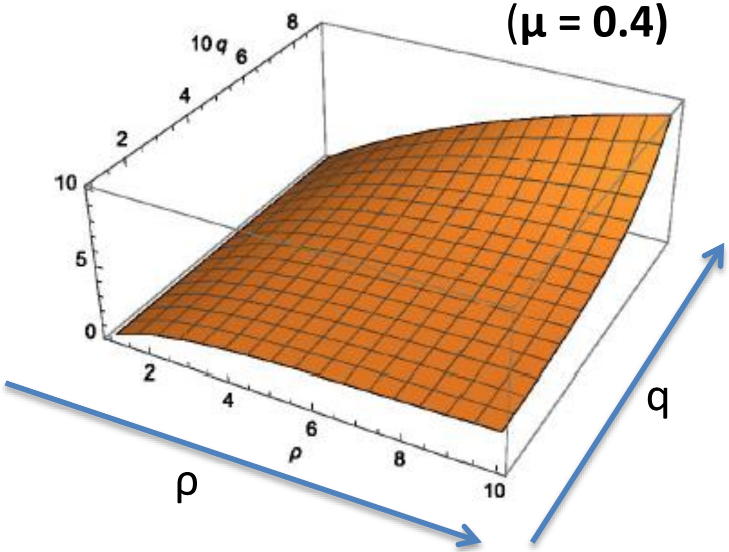


$$w_+(\gamma|\alpha) = \frac{q^{(\alpha_1-\gamma_1)\gamma_2}\mu^{\gamma_1+\gamma_2-1}(q)_{\gamma_1+\gamma_2-1}}{(\mu q^{\alpha_1+\alpha_2-\gamma_1-\gamma_2})_{\gamma_1+\gamma_2}} \frac{(q)_{\alpha_1}}{(q)_{\gamma_1}(q)_{\alpha_1-\gamma_1}} \frac{(q)_{\alpha_2}}{(q)_{\gamma_2}(q)_{\alpha_2-\gamma_2}}$$

The **current** (uniform) is defined by $J = \sum_{n \geq l \geq 1} l w_+((0,l)|(0,n))P(n)$

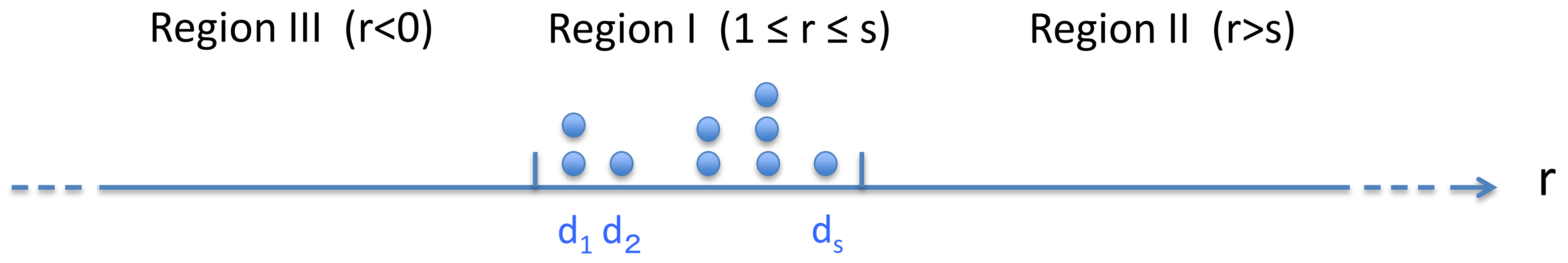
and the result is

$$J = \mu^{-1} \sum_{i \geq 1} \frac{i(\mu y)^i}{1-q^i}$$



(μ = 0.4)

Defect present case



Conditional Probability of finding n 2nd class particles ● at site r

$$\left.\begin{aligned}
P_{\mathrm{I}}(r,n) &= \lim_{L\to\infty} \frac{y^n \mathrm{Tr}\big(A_{d_1}\cdots A_{d_{r-1}} X_{d_r,n} A_{d_{r+1}} \cdots A_{d_s} A_0^{L-s}\big)}{\mathrm{Tr}\big(A_{d_1}\cdots A_{d_s} A_0^{L-s}\big)}\\
P_{\mathrm{II}}(r,n) &= \lim_{L\to\infty} \frac{y^n \mathrm{Tr}\big(A_{d_1}\cdots A_{d_s} A_0^{r-s-1} X_{0,n} A_0^{L-r}\big)}{\mathrm{Tr}\big(A_{d_1}\cdots A_{d_s} A_0^{L-s}\big)}\\
P_{\mathrm{III}}(r,n) &= \lim_{L\to\infty} \frac{y^n \mathrm{Tr}\big(X_{0,n} A_0^{|r|} A_{d_1}\cdots A_{d_s} A_0^{L-|r|-s-1}\big)}{\mathrm{Tr}\big(A_{d_1}\cdots A_{d_s} A_0^{L-s}\big)}
\end{aligned}\right\} =: \mathrm{P(r,n)}$$

Local density at site r

$$\rho(r) = \sum_{n\geq 0} nP(r,n)$$

Local current from site r to r+1

$$J(r) = \begin{cases} \sum_{n\geq l\geq 1} lw_{\pm}((0,l)|(d_r,n))P(r,n) & \text{in Region I} \\ \sum_{n\geq l\geq 1} lw_{\pm}((0,l)|(0,n))P(r,n) & \text{in Region II, III} \end{cases}$$

Average density **ρ** and the **fugacity** **y** of the 2nd class particles are related by the same relation as the defect-free case:

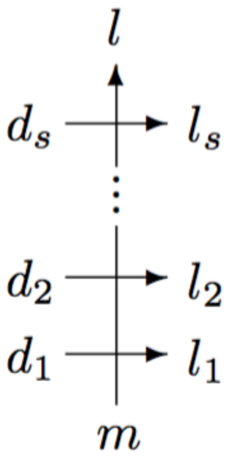
$$\rho = \sum_{i \geq 0} \frac{(1-\mu) y q^i}{(1 - yq^i)(1 - \mu y q^i)}$$

$$\eta_m = \frac{(y)_m}{(\mu y)_m} \qquad 1 = \eta_0 > \eta_1 > \eta_2 > \cdots \geq 0$$

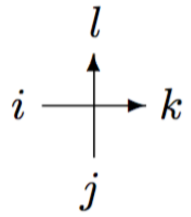
(quantity controlling the decay of defect influence)

The defect content will be incorporated in a column monodromy matrix type function **$G_{m,l}(d_1, . . . ,d_s)$** defined by

$$G_{m,l}(d_1, \ldots, d_s) = \sum_{l_1, \ldots, l_s}$$



(Sum over internal edges)



$$= \delta^{l+k}_{i+j} y^k \frac{(\mu)_k (y)_{j-k}}{(\mu y)_j} \binom{j}{k}_q$$

(n=1 stochastic vertex weight with fugacity y entering as a spectral parameter)

Result

(all the sums are finite ones)

Region I
(1 ≤ r ≤ s)

$$\left[\begin{aligned}
P(r,n) &= y^n \frac{(q^{d_r}\mu)_n (y)_\infty}{(q)_n(\mu y)_\infty} \sum_m q^{n(m-d_r)} \frac{(\mu y)_m}{(y)_{m-d_r}} G_{0,m}(d_1,\dots,d_r) \\
\rho(r) - \rho &= \sum_m G_{0,m}(d_1,\dots,d_r) \Big(\sum_{k=0}^{m-1} \frac{\mu y q^k}{1-\mu y q^k} - \sum_{k=0}^{m-d_r-1} \frac{y q^k}{1-yq^k} \Big) \\
J(r) - J &= -\sum_m G_{0,m}(d_1,\dots,d_r) \sum_{k=0}^{m-1} \frac{y q^k}{(1-\mu y q^k)^2}
\end{aligned}\right.$$

Region II
(r>s)

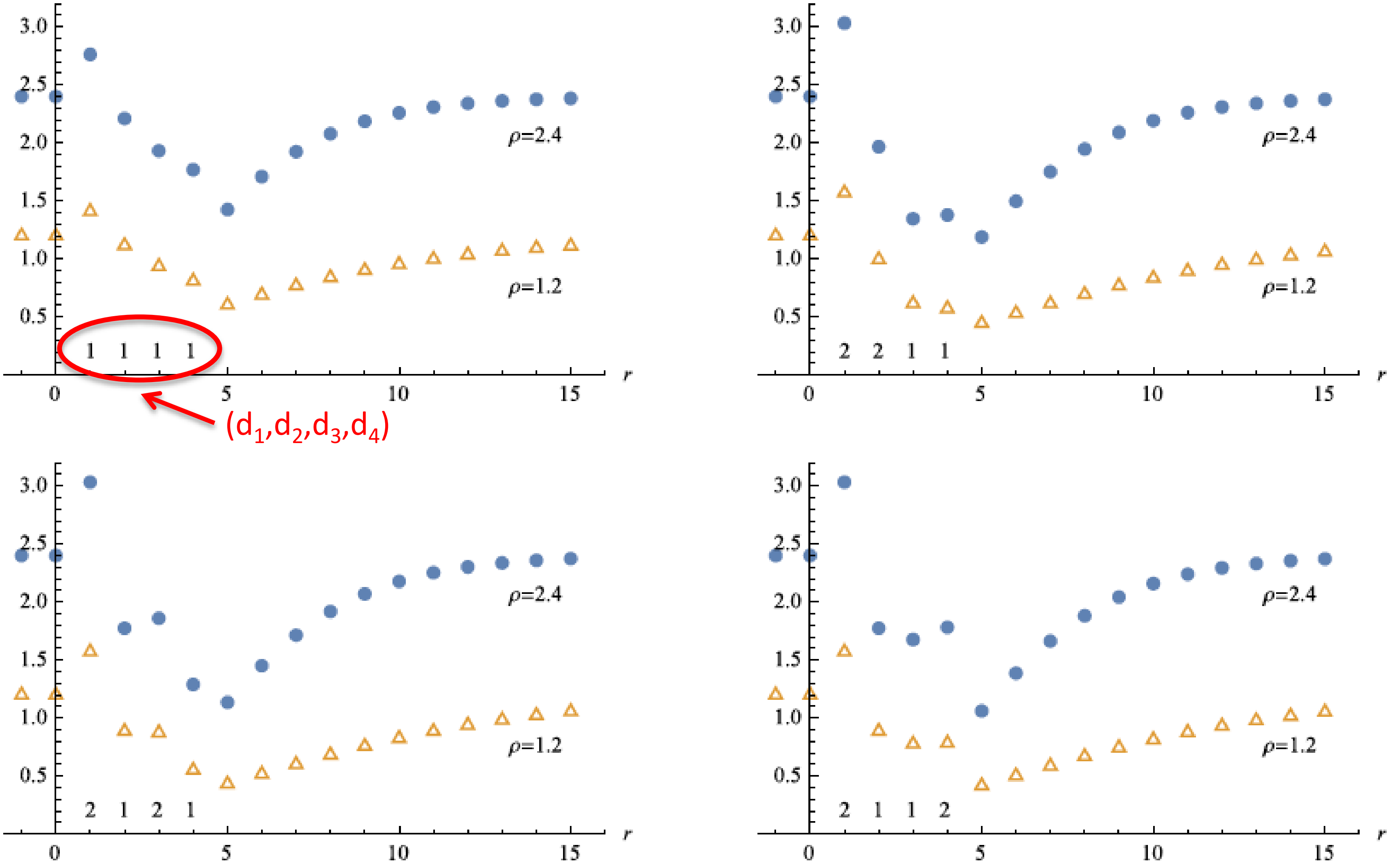
$$\left[\begin{aligned}
P(r,n) &= P(n) \sum_{i,j,m} G_{0,m}(d_1,\dots,d_s)(-1)^i q^{\frac{1}{2}i(i-1+2n)+j} \frac{(q^{-m})_j}{(q)_i(q)_{j-i}} \eta_i^{-1} \eta_j^{r-s} \\
\rho(r) - \rho &= \sum_{j,m} G_{0,m}(d_1,\dots,d_s) \frac{q^j (q^{-m})_j \eta_j^{r-s}}{1-q^j} \Big(\frac{1}{(y)_j} - \frac{1}{(\mu y)_j} \Big) \\
J(r) - J &= \sum_{j,m} G_{0,m}(d_1,\dots,d_s) \frac{q^j (q^{-m})_j \eta_j^{r-s}}{(1-q^j)(\mu y)_j} \sum_{k=0}^{j-1} \frac{y q^k}{1-\mu y q^k}
\end{aligned}\right.$$

Decay mode

Region III (r≤0)
=Region II (r = +∞)

$$\left[\ \underbrace{P(r,n) = P(n) \qquad \rho(r) = \rho \qquad J(r) = J}_{\text{Same as defect-free case}}\right.$$

Local density profile (continued)

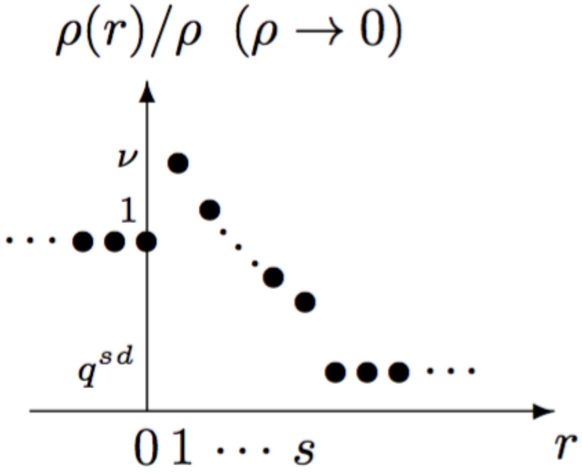


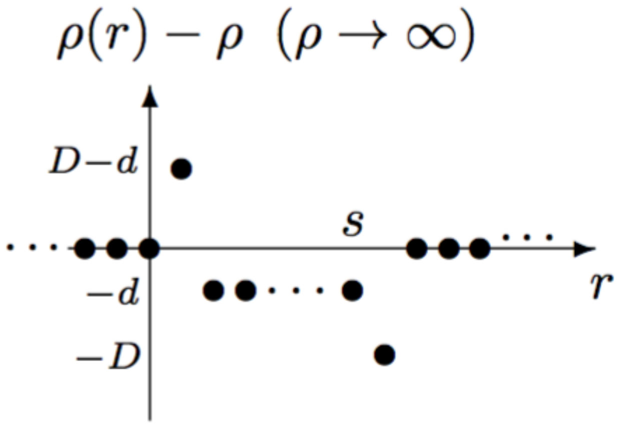
(d_1,d_2,d_3,d_4)

FIGURE 6. Plots of $\rho(r)$ for the densities $\rho = 1.2$ and $\rho = 2.4$ with the presence of the defects $(d_1, \dots, d_4)$ shown at sites $1, \dots, 4$. $(q, \mu) = (0.8, 0.5)$.

High and low density asymptotic forms

Density profile for *homogeneous* case $d_1 = \ldots = d_s = d$



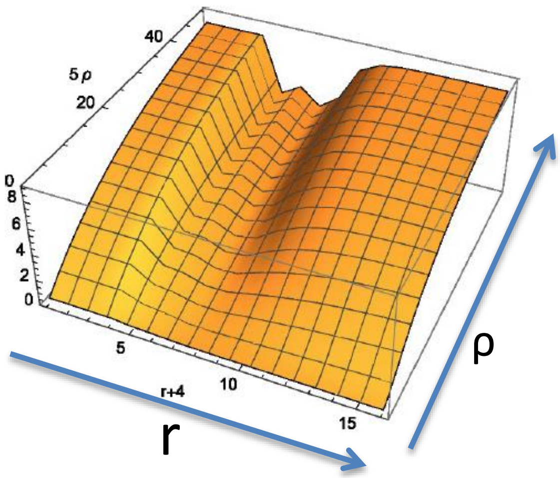


$$D = \sum_{k=0}^{d-1} \frac{1}{1 - \mu q^k}, \qquad \nu = \frac{1 - \mu q^d}{1 - \mu}$$

Peak at the left boundary and **Valley** at the right boundary of the defect cluster. Decreases toward right. In general, inhomogeneous depending on $d_1, \ldots, d_s$.

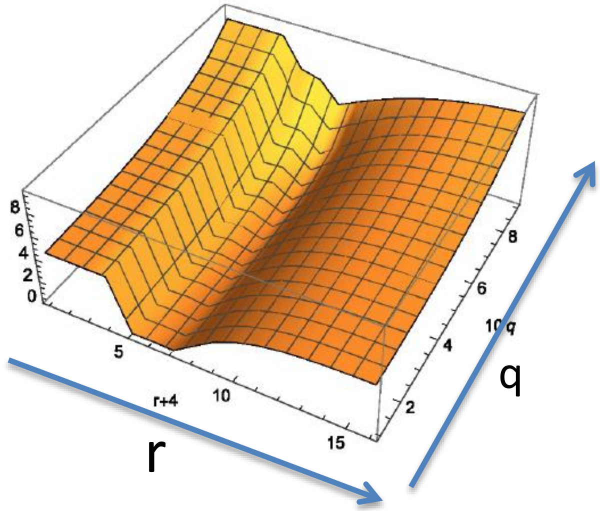
Profile of local current J(r)

$0<\rho<10$, $(q,\mu)=(0.8,0.5)$



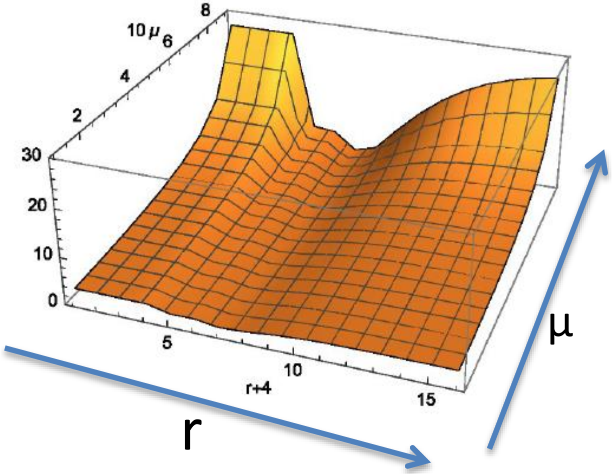
$(d_1,d_2,d_3,d_4)=(2,1,2,1)$

$0.1<q<0.9$, $(\rho,\mu)=(3,0.7)$



$(d_1,d_2,d_3)=(2,1,3)$

$0.1<\mu<0.9$, $(\rho,q)=(4,0.8)$



$(d_1,d_2,d_3,d_4)=(2,1,3,1)$

Generally suppressed by the defects. No peak at the left boundary.
Other aspects are similar to the density profiles.

Remarks

1. ***Total excess*** of the 2nd class particles is given by

$$\boxed{\Delta\rho_{\text{tot}} := \sum_{r=-\infty}^{\infty} (\rho(r) - \rho) = -(d_1 + \cdots + d_s)}$$

. . . ***back-reaction*** of defects

2. $U_q(A^{(1)}{}_2)$ ZRP / (GCE for the 2nd class particles) $\simeq$ $U_q(A^{(1)}{}_1)$ ZRP

$$\boxed{\lim_{L\to\infty} \frac{\mathrm{Tr}\left(A_{d_1}\cdots A_{d_s} A_0^{L-s}\right)}{\mathrm{Tr}\left(A_0^L\right)} = y^{-d_1-\cdots-d_s} \underbrace{g_{d_1}\cdots g_{d_s}}}$$

$$g_d = \frac{(\mu)_d}{(q)_d}$$

[Matrix product factor for *single* species model]

GCE of the 2nd class particles lets the 1st class particles acquire an effective fugacity **y^{-1}** as if Higgs mechanism.

. . . recursive structure in rank (reminding us of ***Nested Bethe Ansatz***)

3. GCE for the both classes of particles requires the large L asymptotics of the grand partition function in ***two*** fugacity variables x and y.

$$Z_L(x,y) = \mathrm{Tr}(V(x,y)^L) \qquad V(x,y) = \Xi(\mu x, \mu y, \mu)\Xi(x,y,1)^{-1}$$

$$\Xi(x,y,\mu) = (x\mathbf{c})_\infty (y\mathbf{k})_\infty (\mu\mathbf{b})_\infty$$

(← Vertex operator like)

Summary

From the quantum R matrix of $U_q(A^{(1)}_n)$ for higher spin representations,
one can extract the **stochastic R matrix** satisfying the YBE, positivity and **sum-to-1.**

It leads to discrete and continuous time integrable Markov processes described as **zero range process (ZRP)** of n classes of particles.

Stationary states of the $U_q(A^{(1)}_n)$ –ZRP have the **matrix product** structure related to a q-boson realization of the Zamolodchikov-Faddeev algebra.

For n=2, using the matrix product formula, local density and current of the 2nd class particles are exactly computed in the GCE with the 1st class particles fixed as defects.