Tetrahedron equation and its generalization from quantum groups

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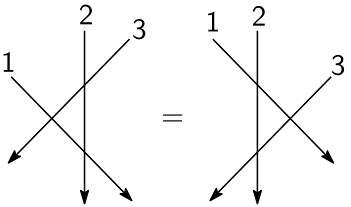
Yang-Baxter equation

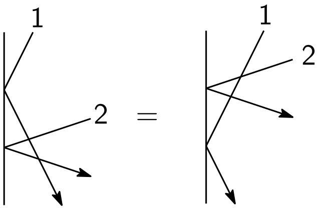
$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$



R : 2 particle scattering(2DR)

Reflection equation

$R_{21}K_2R_{12}K_1 = K_1R_{21}K_2R_{12}$



K : Reflection at boundary

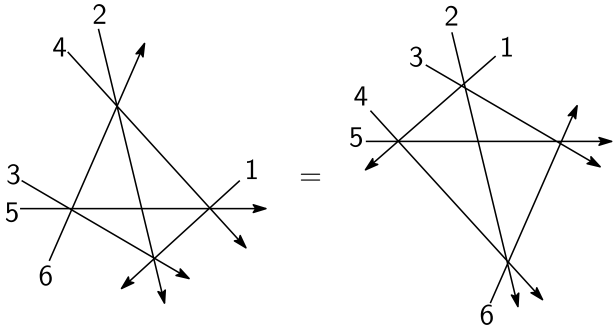
They are "factorization conditions" in 2D system.

Main issue: What about 3D?

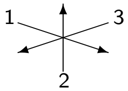
Tetrahedron equation (A.B. Zamolodchikov 1980)

$$R : F \otimes F \otimes F \to F \otimes F \otimes F \qquad \text{(3D } R\text{)}$$

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123} \in \mathrm{End}(F^{\otimes 6})$$



R_{123} =



- **Relevant quantum algebra**

  2D R : $U_q(\hat{g})$; quantized universal enveloping algebra of $g = \mathrm{Lie}\,\mathrm{G}$
  3D R : $A_q(G)$; quantized algebra of functions on G

- **What is** $A_q(G)$?

  It is a q-deformation of the dual of (i.e. function on) the enveloping algebra $U(g)$.

  Studied by Drinfeld (87), Vaksman-Soibelman (89), Reshetikhin-Takhatajan-Faddeev (90), Soibelman (91), Noumi-Yamada-Mimachi (92), Kashiwara (93), Geiss-Leclerc-Schröer (2011-) etc.

- **Simplest example:** $\mathbf{A_q(SL_2) = \langle t_{11}, t_{12}, t_{21}, t_{22}\rangle}$

$$t_{11}t_{21} = qt_{21}t_{11},\ t_{12}t_{22} = qt_{22}t_{12},\ t_{11}t_{12} = qt_{12}t_{11},\ t_{21}t_{22} = qt_{22}t_{21},$$
$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1.$$

Hopf algebra with coproduct $\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$.

- **Fock representation** $\pi_1 : \mathbf{A_q(SL_2)} \to \mathbf{End(F_q)}$

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} \\ -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix} \text{ acting on } F_q = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \oplus \cdots$$

$$\mathbf{k}|m\rangle = q^m|m\rangle,\ \mathbf{a}^+|m\rangle = |m+1\rangle,\ \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle.$$

Theorem (Classification of irreducible reps. Soibelman 1991)

1. *$A_q(G)$ has an irreducible representation π_i attached to each vertex i of the Dynkin diagram of G.*
2. *Irreducible representations are in 1:1 correspondence with elements of the Weyl group $W(G)$ (up to a "torus degree of freedom").*
3. *For any reduced expression $w = s_{i_1} \cdots s_{i_r} \in W(G)$, the irreducible representation for w is realized as $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$.*

Crucial Corollary

Unique (up to normalization) existence of the intertwiner for

$$\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}$$

for any other reduced expression $w = s_{j_1} \cdots s_{j_r}$.

Example

$A_q(\mathrm{SL}_3) = \langle t_{ij} \rangle_{i,j=1}^3$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}}$$

$$W(\mathrm{SL}_3) = \langle s_1, s_2 \rangle. \qquad s_i^2 = 1, \quad s_2 s_1 s_2 = s_1 s_2 s_1.$$

$$\Longrightarrow \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_1 \otimes \pi_2 \otimes \pi_1.$$

$\exists 1$ Intertwiner $\Phi : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}$ such that

$$(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi = \Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1),$$

$$\Phi(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle.$$

Explicit form

$$R = \Phi P_{13}, \quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x,$$
$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R^{abc}_{ijk} |a\rangle \otimes |b\rangle \otimes |c\rangle.$$

$$R^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c} \sum_{\lambda,\mu\geq 0, \lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \times \begin{bmatrix} i, j, c+\mu \\ \mu, \lambda, i-\mu, j-\lambda, c \end{bmatrix}.$$

$$(q)_i = \prod_{j=1}^{i}(1-q^j), \quad \begin{bmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{bmatrix} = \begin{cases} \frac{\prod_{m=1}^{r}(q^2)_{i_m}}{\prod_{m=1}^{s}(q^2)_{j_m}} & \forall i_m, j_m \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise.} \end{cases}$$

$$R = \oplus(\text{finite dimensional matrix}).$$

Theorem (Kapranov-Voevodsky 1994)

R satisfies the tetrahedron equation
$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123} \in \mathrm{End}((F_q)^{\otimes 6})$.

Sketch of proof. $W(\mathrm{SL}_4) = \langle s_1, s_2, s_3 \rangle$.

$s_2s_1s_2 = s_1s_2s_1, \;\; s_3s_2s_3 = s_2s_3s_2, \;\; s_1s_2s_3s_1s_2s_1 = s_3s_2s_3s_1s_2s_3$ (longest el.)

$\exists 1$ intertwiners $\Phi^{(i)}$ for $A_q(\mathrm{SL}_4)$ modules:

$$(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi^{(1)} = \Phi^{(1)} \circ (\pi_1 \otimes \pi_2 \otimes \pi_1),$$
$$(\pi_3 \otimes \pi_2 \otimes \pi_3) \circ \Phi^{(2)} = \Phi^{(2)} \circ (\pi_2 \otimes \pi_3 \otimes \pi_2),$$
$$(\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1) \circ \Phi^{(3)} = \Phi^{(3)} \circ (\pi_3 \otimes \pi_2 \otimes \pi_3 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3)$$

- $\Phi^{(1)} = \Phi^{(2)} = RP_{13}$ as matrices.
- $\Phi^{(3)}$ can be realized in 2 ways as in the next page.

$123\underline{121}$	$\Phi^{(1)}_{456}$	$12\underline{31}21$	P_{34}
$1\underline{232}12$	$\Phi^{(2)}_{234}$	$\underline{121}321$	$\Phi^{(1)}_{123}$
$\underline{13}2\underline{31}2$	$P_{12}P_{45}$	$21\underline{232}1$	$\Phi^{(2)}_{345}$
$3\underline{121}32$	$\Phi^{(1)}_{234}$	$2\underline{13}2\underline{31}$	$P_{23}P_{56}$
$321\underline{232}$	$\Phi^{(2)}_{456}$	$23\underline{121}3$	$\Phi^{(1)}_{345}$
$32\underline{13}23$	P_{34}	$\underline{232}123$	$\Phi^{(2)}_{123}$
323123		323123	

Intertwiners on the right are applied on the underlined components.

$$P_{34}\Phi^{(2)}_{456}\Phi^{(1)}_{234}P_{12}P_{45}\Phi^{(2)}_{234}\Phi^{(1)}_{456} = \Phi^{(2)}_{123}\Phi^{(1)}_{345}P_{23}P_{56}\Phi^{(2)}_{345}\Phi^{(1)}_{123}P_{34}.$$

Substitute $\Phi^{(\bullet)}_{ijk} = R_{ijk}P_{ik}$ and cancel P_{jk}'s.

The result is $R_{356}R_{246}R_{145}R_{123} = R_{123}R_{145}R_{246}R_{356}$. $\square$

Summary so far (type A case)

Weyl group elements $\longleftrightarrow$ "Particle (string) states"
Cubic Coxeter relation $\longleftrightarrow$ 3D R matrix
Transformation of longest element $\longleftrightarrow$ Tetrahedron equation

Remark. 3D R here = Quantization of Miquel's theorem (1838) (Bazhanov-Sergeev-Mangazeev 2008).

Generalization to B, C, F_4 etc. (K-Okado, arXiv:1208.1586, 1210.6430)

1. Explicit intertwiner K for the quartic Coxeter relation.
2. B, C cases: 3D reflection equation involving R and K.
3. $q = 0$: Set theoretical solition to tetrahedron and 3D reflection eqs.

$A_q(\mathrm{Sp}_6) = \langle t_{ij} \rangle_{i,j=1}^{6}$: (Reshetikhin-Takhtadzhyan-Faddeev 1990)

Representations $\pi_1(t_{ij}), \pi_2(t_{ij}), \pi_3(t_{ij})$.

$$\pi_1 : \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} \\ 0 & 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 \\ 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} & 0 \\ 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \langle \mathbf{A}^\pm, \mathbf{K} \rangle = \langle \mathbf{a}^\pm, \mathbf{k} \rangle|_{q \to q^2}.$$

$W(\mathrm{Sp}_6) = \langle s_1, s_2, s_3 \rangle$

$$s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2.$$

Write $\pi_{i_1,\ldots,i_r}$ to mean $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$ to save space.

Equivalence	Intertwiner	
$\pi_{13} \simeq \pi_{31}$,	$P_{12}(x \otimes y) = y \otimes x$,	
$\pi_{121} \simeq \pi_{212}$,	$\Phi = R P_{13}$	(same as type A),
$\pi_{2323} \simeq \pi_{3232}$,	$\Psi = K P_{14} P_{23}$	(New object).

$$K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q), \quad R \in \mathrm{End}((F_q)^{\otimes 3}).$$

Matrix elements

$$K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K^{cmdn}_{a\,i\,b\,j} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle.$$

$K^{cmdn}_{a\,i\,b\,j} = 0$ unless $c+m+d = a+i+b, \;\; d+n-c = b+j-a.$

Theorem (Explicit form)

$$K^{c,m,0,n}_{a,i,0,j} = \sum_{\lambda \geq 0} (-1)^{m+\lambda} \frac{(q^4)_{c+\lambda}}{(q^4)_c} q^{\phi_2} \begin{bmatrix} i,j \\ \lambda, j-\lambda, m-\lambda, i-m+\lambda \end{bmatrix},$$

$$\phi_2 = (a+c+1)(m+j-2\lambda)+m-j.$$

$$K^{cmdn}_{a\,i\,b\,j} = \frac{(q^4)_a}{(q^4)_c} \sum_{\alpha,\beta,\gamma \geq 0} \frac{(-1)^{\alpha+\gamma}}{(q^4)_{d-\beta}} q^{\phi_1} K^{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma}_{c,m+d-\alpha-\beta-\gamma,0,n+d-\alpha-\beta-\gamma}$$

$$\times \begin{bmatrix} b, d-\beta, i+b-\alpha-\beta, j+b-\alpha-\beta \\ \alpha, \beta, \gamma, m-\alpha, n-\alpha, b-\alpha-\beta, d-\beta-\gamma \end{bmatrix},$$

$$\phi_1 = \alpha(\alpha+2d-2\beta-1)+(2\beta-d)(m+n+d)+\gamma(\gamma-1)-b(i+j+b).$$

Theorem

R and K yield the first nontrivial solution to the **3D reflection equation** *proposed by Isaev-Kulish in 1997:*

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

- The proof is parallel with type A.
- Uses the reduced expressions of the longest element $s_1s_2s_3s_2s_1s_2s_3s_2s_3 = s_3s_2s_3s_2s_1s_2s_3s_2s_1 \in W(\mathrm{Sp}_6)$.
- The two sides come from the 2 ways of constructing the intertwiners for $\pi_{123212323} \simeq \pi_{323212321}$ out of R and K.

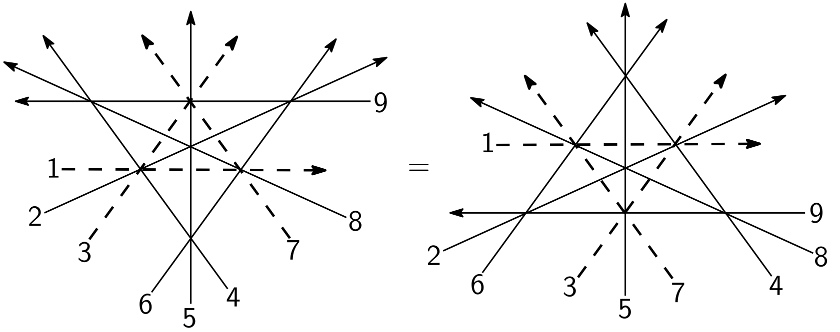
Physical and geometric interpretation of the 3D reflection eq.

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

is a "factorization" of 3 string scattering with boundary reflections.

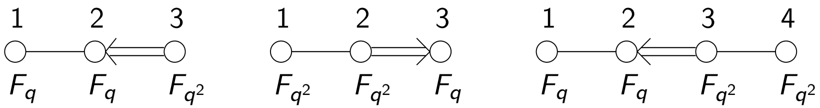
R : Scattering amplitude of 3 strings.

K: Reflection amplitude with boundary freedom signified by spaces $1, 3, 7$.



B, F_4 cases

C_3 $\qquad$ B_3 $\qquad$ F_4



C_3:
$R : 121 = 212$
$K : 2323 = 3232$

B_3:
$S : 121 = 212$
$J : 2323 = 3232$

F_4:
$R : 121 = 212$
$K : 2323 = 3232$
$S : 434 = 343$

$$R \in \mathrm{End}(F_q \otimes F_q \otimes F_q), \qquad K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$$

$$S = R|_{q \to q^2} \in \mathrm{End}(F_{q^2} \otimes F_{q^2} \otimes F_{q^2})$$

$$J = P_{14}P_{23}KP_{23}P_{14} \in \mathrm{End}(F_q \otimes F_{q^2} \otimes F_q \otimes F_{q^2}).$$

Both (R, K) and (S, J) satisfy the 3D reflection equation.

A reduced expression of the longest element of $W(F_4)$ is

$$s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1 \quad \text{(length 24).}$$

The intertwiner for $\pi_{434234232123423123412321} \simeq \pi_{\text{reverse order}}$ can be constructed by composition of R, K, S in two ways, which must coincide. This leads to the F_4-analogue of the tetrahedron equation:

$$\begin{aligned}
& S_{14,15,16} S_{9,11,16} K_{16,10,8,7} K_{9,13,15,17} S_{4,5,16} R_{7,12,17} S_{1,2,16} R_{6,10,17} S_{9,14,18} K_{1,3,5,17} \\
\times\ & S_{11,15,18} K_{18,12,8,6} S_{1,4,18} S_{1,8,15} R_{7,13,19} K_{1,6,11,19} K_{4,12,15,19} R_{3,10,19} S_{4,8,11} K_{1,7,14,20} \\
\times\ & S_{2,5,18} R_{6,13,20} R_{3,12,20} S_{1,9,21} K_{2,10,15,20} S_{4,14,21} K_{21,13,8,3} S_{2,11,21} S_{2,8,14} R_{6,7,22} \\
\times\ & K_{2,3,4,22} S_{5,15,21} K_{11,13,14,22} R_{10,12,22} K_{2,6,9,23} R_{3,7,23} R_{19,20,22} K_{16,17,18,22} R_{10,13,23} \\
\times\ & K_{5,12,14,23} R_{3,6,24} K_{16,19,21,23} K_{4,7,9,24} R_{17,20,23} K_{5,10,11,24} R_{12,13,24} R_{17,19,24} \\
\times\ & K_{18,20,21,24} S_{5,8,9} R_{22,23,24} \ = \ \text{product in reverse order.}
\end{aligned}$$

$16R$'s, $16S$'s and $18K$'s acting on $F_{q_{i_1}} \otimes \cdots \otimes F_{q_{i_{24}}}$.

Another aspect: Connection with PBW basis

$$U_q^+(sl_3) = \langle e_1, e_2 \rangle \text{ with Serre relation } [[e_1, e_2]_q, e_1]_q = [[e_2, e_1]_q, e_2]_q = 0.$$

$$\left([x, y]_q := xy - qyx, \;\; [a]! = \prod_{m=1}^{a} \frac{q^m - q^{-m}}{q - q^{-1}}\right)$$

Two PBW bases: $\{E^{a,b,c}\}_{(a,b,c)\in(\mathbb{Z}_{\geq 0})^3}$, $\{E'^{a,b,c}\}_{(a,b,c)\in(\mathbb{Z}_{\geq 0})^3}$

$$E^{a,b,c} = \frac{e_1^a([e_2, e_1]_q)^b e_2^c}{[a]![b]![c]!}, \quad E'^{a,b,c} = E^{a,b,c}|_{e_1 \leftrightarrow e_2}$$

Then $\displaystyle E^{a,b,c} = \sum_{ijk} R_{ijk}^{abc} E'^{kji}$ (Sergeev 2008)

3D R = transition matrix of the PBW bases of $U_q^+(sl_3)$

Theorem (K-Okado-Yamada 2013, Tanisaki preprint)

For any simple Lie group G and $\mathfrak{g} = \mathrm{Lie}(G)$, set
$\Phi :=$ Intertwiner of Soibelman irreducible representations of $A_q(G)$,
$\Gamma :=$ Transition matrix of the PBW bases of $U_q^+(\mathfrak{g})$.

Then $\Phi = \Gamma$.

Now we proceed to the next topic

2D Reduction : Tetrahedron equation $\to$ Yang-Baxter equation.

2D reduction

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$

$\Downarrow$ 2d reduction (eliminate spaces 4,5,6)

$$S_{12}(x)S_{13}(xy)S_{23}(y) = S_{23}(y)S_{13}(xy)S_{12}(x) \quad \cdots \text{Yang-Baxter equation}$$

Prescription $\langle \chi_s(x,y)|R_{124}R_{135}R_{236}R_{456}|\chi_t(1,1)\rangle$
$= \langle \chi_s(x,y)|R_{456}R_{236}R_{135}R_{124}|\chi_t(1,1)\rangle$

by the boundary vectors

$$\langle \chi_s(x,y)| = \langle \chi_s(x)| \otimes \langle \chi_s(xy)| \otimes \langle \chi_s(y)| \in \overset{4}{F^*} \otimes \overset{5}{F^*} \otimes \overset{6}{F^*},$$

$$|\chi_t(x,y)\rangle = |\chi_t(x)\rangle \otimes |\chi_t(xy)\rangle \otimes |\chi_t(y)\rangle \in \overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F}$$

satisfying $\langle \chi_s(x,y)|R_{456} = \langle \chi_s(x,y)|, \quad R_{456}|\chi_t(x,y)\rangle = |\chi_t(x,y)\rangle.$

Then $S_{12}(x) = \langle \chi_s(x)|R_{124}|\chi_t(1)\rangle \in \mathrm{End}(F \otimes F)$. $(F = F_q)$

Boundary vectors

There are 2 such boundary vectors (K-Sergeev 2013):

$$\langle \chi_1(z)| = \sum_{m\geq 0} \frac{z^m}{(q)_m}\langle m| \quad \langle \chi_2(z)| = \sum_{m\geq 0} \frac{z^m}{(q^4)_m}\langle 2m|,$$

$$|\chi_1(z)\rangle = \sum_{m\geq 0} \frac{z^m}{(q)_m}|m\rangle, \quad |\chi_2(z)\rangle = \sum_{m\geq 0} \frac{z^m}{(q^4)_m}|2m\rangle.$$

So far: 1-layer version of reduction

Possible to extend it to n-layer version

n-layer version of the tetrahedron equation

$$\prod_{1\le i\le n}^{\to} (R_{1_i 2_i 4} R_{1_i 3_i 5} R_{2_i 3_i 6}) R_{456} = R_{456} \prod_{1\le i\le n}^{\to} (R_{2_i 3_i 6} R_{1_i 3_i 5} R_{1_i 2_i 4})$$

$1_1, \ldots, 1_n, 2_1, \ldots, 2_n, 3_1, \ldots, 3_n, 4, 5, 6$: copies of the Fock space F

The same reduction $\langle \chi_s(x,y)|(\cdots)|\chi_t(1,1)\rangle$ works.

$\Longrightarrow$ Solution of the Yang-Baxter equation constructed as

$$S^{s,t}(z) = \langle \chi_s(z)| R_{1_1 2_1 4} R_{1_2 2_2 4} \cdots R_{1_n 2_n 4} |\chi_t(1)\rangle \in \mathrm{End}(F^{\otimes n} \otimes F^{\otimes n}).$$

(The evaluation is done in the space 4.)

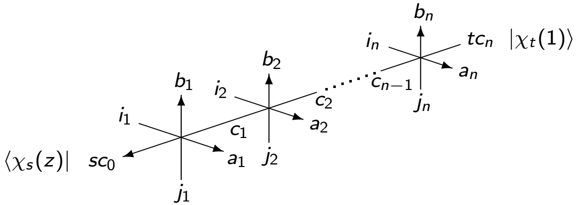
$S^{s,t}(z)$ have matrix product construction from 3D R

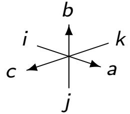
Notations:

$$|\mathbf{a}\rangle = |a_1\rangle \otimes \cdots \otimes |a_n\rangle \in F^{\otimes n} \quad \text{for} \quad \mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$$

$$S^{s,t}(z)\big(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle\big) = \sum_{\mathbf{a},\mathbf{b}} S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle,$$

$$S^{s,t}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \sum_{c_0,\ldots,c_n \geq 0} \frac{z^{c_0}(q^2)_{sc_0}}{(q^{s^2})_{c_0}(q^{t^2})_{c_n}} R^{a_1,b_1,sc_0}_{i_1,j_1,c_1} R^{a_2,b_2,c_1}_{i_2,j_2,c_2} \cdots R^{a_n,b_n,c_{n-1}}_{i_n,j_n,tc_n}$$



$R^{a,b,c}_{i,j,k} =$ 

Examples

Substitute the matrix elements of 3D R

$$R^{a,b,c}_{i,0,k} = q^{ac}\frac{(q^2)_i(q^2)_k}{(q^2)_a(q^2)_b(q^2)_c}\delta^{a+b}_i\delta^{b+c}_k, R^{0,b,c}_{i,j,k} = (-1)^j q^{j(c+1)}\frac{(q^2)_k}{(q^2)_c}\delta^b_{i+j}\delta^{b+c}_{j+k}.$$

Up to an overall factor, the following formulas are valid ($t = 1, 2$):

$$S^{1,t}(z)^{\mathbf{a},\mathbf{0}}_{\mathbf{a},\mathbf{0}} = (-q)^{-|\mathbf{a}|}S^{1,t}(z)^{\mathbf{0},\mathbf{a}}_{\mathbf{0},\mathbf{a}} = \frac{(z^t;q^t)_{|\mathbf{a}|}}{(-z^tq;q^t)_{|\mathbf{a}|}} \quad (|\mathbf{a}| = a_1 + \cdots + a_n),$$

$$S^{1,1}(z)^{2\mathbf{e}_1,\mathbf{0}}_{\mathbf{e}_1,\mathbf{e}_1} = (-q)^{-1}S^{1,1}(z)^{\mathbf{0},2\mathbf{e}_1}_{\mathbf{e}_1,\mathbf{e}_1} = \frac{(1+q)(1-z)}{(1+zq)(1+zq^2)},$$

where $(z;q)_m = \prod_{j=1}^m(1-zq^{j-1})$,

$$\mathbf{e}_i = (0,\ldots,0,\overset{i}{1},0,\ldots,0), \quad \mathbf{0} = (0,0,\ldots,0)$$

Proposition (summary so far)

$S(z) = S^{s,t}(z) \in \mathrm{End}(F^{\otimes n} \otimes F^{\otimes n})$ satisfies the Yang-Baxter equation

$$S_{12}(x)S_{13}(xy)S_{23}(y) = S_{23}(y)S_{13}(xy)S_{12}(x)$$

Problem:

Find a characterization of $S^{1,1}(z), S^{1,2}(z), S^{2,2}(z)$ in the framework of the quantum group theory. ($S^{2,1}(z)$ is simply related to $S^{1,2}(z)$.)

Result

They are quantum R-matrices intertwining the q-oscillator representations of $U_q(D^{(2)}_{n+1}), U_q(A^{(2)}_{2n}), U_q(C^{(1)}_n)$.

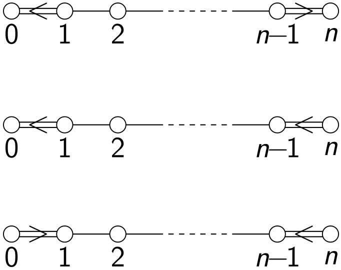
Affine Lie algebras $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, $C_n^{(1)}$

Dynkin diagrams

$D_{n+1}^{(2)}$

$A_{2n}^{(2)}$

$C_n^{(1)}$



q-oscillator representations

$V_x := F^{\otimes n}[x, x^{-1}]$ (x : spectral parameter).

Let $\langle e_j, f_j, k_j^{\pm 1}\rangle_{0\le j\le n}$ act on V_x by $([m] = (q^m - q^{-m})/(q - q^{-1}))$

$$
\begin{aligned}
e_0|\mathbf{m}\rangle &= x|\mathbf{m}+\mathbf{e}_1\rangle \\
f_0|\mathbf{m}\rangle &= \sqrt{-1}\kappa[m_1]x^{-1}|\mathbf{m}-\mathbf{e}_1\rangle \qquad \kappa = (q+1)/(q-1) \\
k_0|\mathbf{m}\rangle &= -\sqrt{-1}q^{m_1+\frac{1}{2}}|\mathbf{m}\rangle \\
e_j|\mathbf{m}\rangle &= [m_j]|\mathbf{m}-\mathbf{e}_j+\mathbf{e}_{j+1}\rangle \qquad (0<j<n) \\
f_j|\mathbf{m}\rangle &= [m_{j+1}]|\mathbf{m}+\mathbf{e}_j-\mathbf{e}_{j+1}\rangle \qquad (0<j<n) \\
k_j|\mathbf{m}\rangle &= q^{-m_j+m_{j+1}}|\mathbf{m}\rangle \qquad (0<j<n) \\
e_n|\mathbf{m}\rangle &= \sqrt{-1}\kappa[m_n]|\mathbf{m}-\mathbf{e}_n\rangle \\
f_n|\mathbf{m}\rangle &= |\mathbf{m}+\mathbf{e}_n\rangle \\
k_n|\mathbf{m}\rangle &= \sqrt{-1}q^{-m_n-\frac{1}{2}}|\mathbf{m}\rangle.
\end{aligned}
$$

$\mathbf{e}_j = (0,\ldots,\overset{j}{1},\ldots,0)$, $\mathbf{m} = \sum_{j=1}^n m_j\mathbf{e}_j \in \mathbb{Z}^n$, $|\mathbf{m}\rangle = |m_1\rangle \otimes\cdots\otimes |m_n\rangle \in F^{\otimes n}$

Proposition

V_x is an irreducible representation (q-oscillator representation) of the Drinfeld-Jimbo quantum affine algebra $U_q(D^{(2)}_{n+1}) = \langle e_j, f_j, k_j^{\pm 1} \rangle_{0 \le j \le n}$.

- $U_q(A^{(2)}_{2n})$ and $U_q(C^{(1)}_n)$ also have similar q-oscillator representations.
- The q-oscillator representations for $U_q(A^{(1)}_n), U_q(C_n)$ were known by Hayashi (1990).

Quantum R matrix for q-oscillator representation

For simplicity, consider $U_q = U_q(D^{(2)}_{n+1})$ for the time being.

$R(z) \in \mathrm{End}(V_x \otimes V_y)$ $(z = x/y)$ is characterized by

(i) Commutativity: $[PR(z), \Delta(g)] = 0 \quad \forall g \in U_q$

(Δ: coproduct of U_q, $P(u \otimes v) = v \otimes u$)

(ii) Normalization: $R(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = \frac{(-zq;q)_\infty}{(z;q)_\infty} |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$

Introduce a gauge transformed $R(z)$

$$\tilde{R}(z) := (K^{-1} \otimes 1) R(z) (1 \otimes K)$$

$$K|\mathbf{m}\rangle = (-\sqrt{-1}q^{\frac{1}{2}})^{m_1 + \cdots + m_n} |\mathbf{m}\rangle$$

Both $R(z)$ and $\tilde{R}(z)$ satisfy the Yang-Baxter equation.

$\tilde{R}_{\mathfrak{g}}(z) := \tilde{R}(z)$ of the q-oscillator representation of $U_q(\mathfrak{g})$

Theorem

$$S^{1,1}(z) = \tilde{R}_{D^{(2)}_{n+1}}(z), \quad S^{1,2}(z) = \tilde{R}_{A^{(2)}_{2n}}(z), \quad S^{2,2}(z) = \tilde{R}_{C^{(1)}_{n}}(z).$$

Proof: Can check the commutativity of $S^{s,t}(z)$ with U_q. $\square$

Remark: Boundary vector $\Longleftrightarrow$ End shape of the Dynkin diagram of $\mathfrak{g}$

$\langle \chi_1(z)|$ $\quad |\chi_1(1)\rangle$ $\quad \langle \chi_1(z)|$ $\quad |\chi_2(1)\rangle$ $\quad \langle \chi_2(z)|$ $\quad |\chi_2(1)\rangle$

Related results

- Bazhanov-Sergeev (2006) ($L = $ 3D L-operator satisfying $RLLL = LLLR$)

  $\mathrm{Tr}(R \cdots R)$, $\mathrm{Tr}(L \cdots L) = \oplus$ (R for type A sym or anti-sym tensor rep.)

- K-Sergeev (2013)

  $\langle \chi_s(z) | L \cdots L | \chi_t(1) \rangle = R$-matrix for spin rep. of $U_q(B_n^{(1)}), U_q(D_n^{(1)})$ etc.

- K-Okado (to appear in CMP)

  $\langle \chi_s(z) | R \cdots R | \chi_t(1) \rangle = R$-matrix for q-oscillator rep. of $U_q(D_{n+1}^{(2)})$ etc.

- K-Okado-Sergeev (arXiv:1409.1986)

  R-matrix for modular double of these U_q's.