Matrix products in integrable probability

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Non-equilibrium statistical mechanics

Stochastic dynamics,

Markov process, …

Integrable systems

Quantum groups,

Yang-Baxter equation, …

Integrable Markov process

Spectral problem of the Markov matrix: solvable by Bethe ansatz

Exact asymptotic analysis: connection to random matrices, etc

Prototype examples

Asymmetric simple exclusion process (ASEP)

Asymmetric zero range process (ZRP)

Key features

- Stochastic R matrix
- Stationary states: matrix product structure
- Zamolodchikov-Faddeev algebra

Hidden 3D structure related to the tetrahedron equation (no detail today)

This talk is mainly based on

K, Mangazeev, Okado, Stochastic R matrix for $U_q(A^{(1)}_n)$, Nucl. Phys. B913 (2016)

K and Okado, A q-boson representation of Zamolodchikov –Faddeev algebra for stochastic R matrix of $U_q(A^{(1)}_n)$, Lett. Math. Phys. 50 (2017)

K, Maruyama, Okado, Multispecies totally asymmetric zero range process: II. Hat relation and tetrahedron equation, J. Integrable Syst. 1 (2015)

Contents.

I. Quantum/stochastic R matrices

Can a quantum R matrix be made stochastic?

$U_q(A^{(1)}_n)$, symmetric tenor representation, quantum R matrix, stochastic gauge, specialization manifesting nonnegativity, stochastic R matrix

II. Integrable Markov process

III. Stationary states and matrix product formula

Preliminary on quantum groups

$U_q = U_q(A_n^{(1)})$: Drinfeld-Jimbo quantum affine algebra with Cartan matrix: $(a_{ij})_{i,j\in I}$ where $a_{ij} = 2\delta_{ij}^{(n+1)} - \delta_{i,j+1}^{(n+1)} - \delta_{i,j-1}^{(n+1)}$ generated by $e_i, f_i, k_i^{\pm 1} (i \in \{0, 1, \ldots, n\})$ satisfying

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}$$

$+$ Serre relations.

U_q is a Hopf algebra, so there exists an algebra homomorphism (**coproduct**) $\Delta : U_q \to U_q \otimes U_q$, such that

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i,$$
$$\Delta^{\mathrm{op}}(e_i) = e_i \otimes 1 + k_i \otimes e_i, \text{ etc.}$$

Symmetric tensor representation

For $l \in \mathbb{Z}_{>0}$ set

$$B_l = \{\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \mid |\alpha| := \sum_{i=1}^{n+1} \alpha_i = l\}$$

$$V_l = \bigoplus_{\alpha=(\alpha_1,\dots,\alpha_{n+1})\in B_l} \mathbb{Q}(q)|\alpha_1, \dots, \alpha_{n+1}\rangle.$$

There exists a representation of U_q with **spectral parameter** x

$$\pi_x^l : U_q \to \mathrm{End}(V_l),$$

$$\pi_x^l(k_i)|\alpha\rangle = q^{\alpha_{i+1}-\alpha_i}|\alpha\rangle, \quad \pi_x^l(e_i)|\alpha\rangle = x^{\delta_{i,0}}[\alpha_i]|\alpha - \varepsilon_i + \varepsilon_{i+1}\rangle,$$
$$\pi_x^l(f_i)|\alpha\rangle = x^{-\delta_{i,0}}[\alpha_{i+1}]|\alpha + \varepsilon_i - \varepsilon_{i+1}\rangle,$$

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and ε_i is the i-th standard basis vector of $\mathbb{Z}^{n+1}$.

Quantum R matrix

There exists a unique, up to overall normalization, intertwiner

$$R(x/y) = R^{l,m}(x/y) : V_l \otimes V_m \to V_l \otimes V_m,$$

satisfying

$$R(x/y)(\pi_x^l \otimes \pi_y^m) \circ \Delta(u) = (\pi_x^l \otimes \pi_y^m) \circ \Delta^{\mathrm{op}}(u)\, R(x/y), \qquad \forall u \in U_q.$$

Employ the unit normalization condition

$$R(z)(|0,\ldots,0,l\rangle \otimes |0,\ldots,0,m\rangle) = |0,\ldots,0,l\rangle \otimes |0,\ldots,0,m\rangle.$$

When $n = 1$, $l = m = 1$ case corresponds to the **6 vertex model** and arbitray l, m case **higher spin** generalizations.

$R^{l,m}(z)$ satisfies the Yang-Baxter equation (YBE)

$$R^{k,l}_{1,2}(x) R^{k,m}_{1,3}(xy) R^{l,m}_{2,3}(y) = R^{l,m}_{2,3}(y) R^{k,m}_{1,3}(xy) R^{k,l}_{1,2}(x) \quad \text{on } V_k \otimes V_l \otimes V_m.$$

Stochastic gauge: S(z)

$$R(z)(|\alpha\rangle \otimes |\beta\rangle) = \sum_{\gamma,\delta} R(z)^{\gamma,\delta}_{\alpha,\beta} |\gamma\rangle \otimes |\delta\rangle$$

Want to modify it so as to satisfy (i) Sum-to-1 and (ii) Nonnegativity

$$\text{(i)} \qquad S(z)^{\gamma,\delta}_{\alpha,\beta} = q^{\eta} R(z)^{\gamma,\delta}_{\alpha,\beta}, \qquad \sum_{\gamma,\delta} S(z)^{\gamma,\delta}_{\alpha,\beta} = 1 \qquad \text{(Sum-to-1)}$$

It is fulfilled with **stochastic gauge** $\eta = \sum_{1 \le i < j \le n+1} (\delta_i \gamma_j - \alpha_i \beta_j)$.

(Sum-to-1) $= U_q(A_n)$ – orbit of the unit normalization condition.

(Sum-to-1) eventually leads to the **total probability conservation** of the transition matrix of our discrete time Markov process.

$S(z)$ also satisfies YBE. $\quad n = 1$ case is studied by Corwin-Petrov.

Specialization manifesting (ii) Nonnegativity

$^{\exists}$Special value of z at which the matrix elements of $S(z)$ are nonnegative.

$$S(z = q^{l-m})^{\gamma,\delta}_{\alpha,\beta} = \delta_{\alpha+\beta,\gamma+\delta}\,\Phi_{q^2}(\bar{\gamma}|\bar{\beta}; q^{-2l}, q^{-2m}),$$

where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_n)$ for $\gamma = (\gamma_1, \ldots, \gamma_{n+1})$ and

$$\Phi_q(\bar{\gamma}|\bar{\beta}; \lambda, \mu) = q^{\xi}\left(\frac{\mu}{\lambda}\right)^{|\bar{\gamma}|}\frac{(\lambda; q)_{|\bar{\gamma}|}(\frac{\mu}{\lambda}; q)_{|\bar{\beta}|-|\bar{\gamma}|}}{(\mu; q)_{|\bar{\beta}|}}\prod_{i=1}^{n}\binom{\beta_i}{\gamma_i}_q,$$

$$\xi = \sum_{1\le i<j\le n}(\beta_i - \gamma_i)\gamma_j,\ (\lambda; q)_m = \prod_{i=0}^{m-1}(1-\lambda q^i),\ \binom{m}{k}_q = \frac{(q)_m}{(q)_k(q)_{m-k}}.$$

$n = 1$ case is introduced by Povolotsky.

Stochastic R matrix

In view of this formula, define an operator $\mathcal{S}(\lambda, \mu)$ acting on $W \otimes W$ by

$$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = \delta_{\alpha+\beta,\gamma+\delta}\Phi_q(\gamma|\beta;\lambda,\mu),$$

$$\mathcal{S}(\lambda,\mu)(|\alpha\rangle \otimes |\beta\rangle) = \sum_{\gamma,\delta} \mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta}|\gamma\rangle \otimes |\delta\rangle,$$

$$W = \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}} \mathbb{Q}(q)|\alpha_1,\ldots,\alpha_n\rangle.$$

Proposition (KMMO)

$\mathcal{S}(\lambda,\mu)$ *satisfies nonnegativity in* $0<\mu<\lambda<1, 0<q<1$, *Sum-to-1, YBE.*

$$\sum_{\gamma,\delta} \mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = 1,$$

$$\mathcal{S}_{1,2}(\lambda,\mu)\mathcal{S}_{1,3}(\lambda,\nu)\mathcal{S}_{2,3}(\mu,\nu) = \mathcal{S}_{2,3}(\mu,\nu)\mathcal{S}_{1,3}(\lambda,\nu)\mathcal{S}_{1,2}(\lambda,\mu) \quad \text{on } W^{\otimes 3}.$$

Note that $\mathcal{S}(\lambda,\mu)$ does **not** take the form $\mathcal{S}(\lambda/\mu)$.

Contents.

Commuting Markov transfer matrices

Consider the tensor product $W_0 \otimes W_1 \otimes \cdots \otimes W_L$ $(W_i = W)$ and define

$$T(\lambda|\mu_1,\ldots,\mu_L) = \mathrm{Tr}_{W_0}\left(\mathcal{S}_{W_0,W_L}(\lambda,\mu_L)\cdots\mathcal{S}_{W_0,W_1}(\lambda,\mu_1)\right) \in \mathrm{End}(W^{\otimes L}).$$

To illustrate

$$T|\beta_1,\ldots,\beta_L\rangle = \sum_{\alpha_1,\ldots,\alpha_L} T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L}|\alpha_1,\ldots,\alpha_L\rangle \in W^{\otimes L},$$

$$\mathcal{S}(\lambda,\mu)^{\gamma,\delta}_{\alpha,\beta} = \begin{tikzcd} & \delta & \\ \alpha \arrow[r, dash] & {} \arrow[u] \arrow[r] & \gamma \\ & \beta \arrow[u, dash] & \end{tikzcd}$$

$$T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = \sum_{\gamma_1,\ldots,\gamma_L} \begin{tikzcd} & \alpha_1 & & \alpha_2 & & & \alpha_L & \\ \gamma_L \arrow[r, dash] & {} \arrow[u] \arrow[r] & \gamma_1 \arrow[r, dash] & {} \arrow[u] \arrow[r] & \gamma_2 \cdots \gamma_{L-1} \arrow[r, dash] & {} \arrow[r] & {} \arrow[u] \arrow[r] & \gamma_L. \\ & \beta_1 \arrow[u, dash] & & \beta_2 \arrow[u, dash] & & & \beta_L \arrow[u, dash] & \end{tikzcd}$$

Discrete time Markov Process

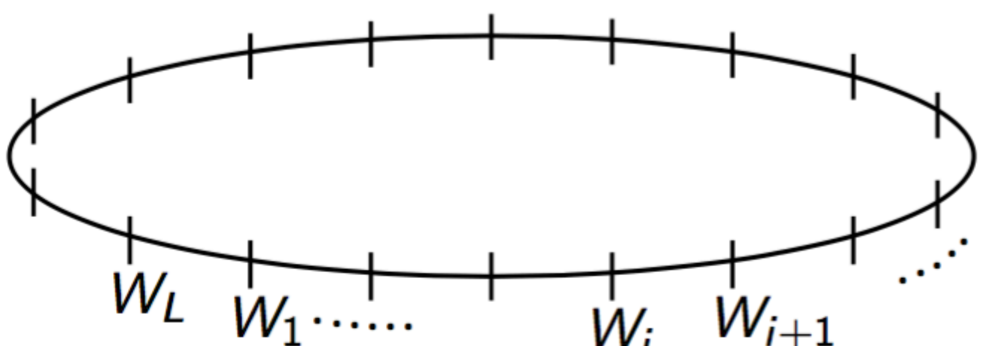
Proposition

1. *Sum-to-1:* $\sum_{\alpha_1,\ldots,\alpha_L} T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L} = 1$.
2. *Nonnegativity: Matrix elements of* $T(\lambda|\mu_1,\ldots,\mu_L) \in \mathbb{R}_{\geq 0}$ *when* $0 < \mu_i < \lambda < 1, 0 < q < 1$.
3. *YBE for* $\mathcal{S}(\lambda,\mu)$ *implies* $[T(\lambda|\mu_1,\ldots,\mu_L), T(\lambda'|\mu_1,\ldots,\mu_L)] = 0$.

Therefore

$$|P(t+1)\rangle = T(\lambda|\mu_1,\ldots,\mu_L)|P(t)\rangle \in W^{\otimes L}$$

defines a family of **discrete time Markov processes** that is simultaneously diagonalizable with respect to λ.



$(W_i = W)$

Continuous time Markov Process (1)

Set $\mu_1 = \cdots = \mu_L = \mu$, $T(\lambda|\mu) = T(\lambda|\mu, \ldots, \mu)$ and

$$H_+ = -\mu^{-1} \frac{\partial \log T(\lambda|\mu)}{\partial \lambda}\bigg|_{\lambda=1}, \qquad H_- = \mu \frac{\partial \log T(\lambda|\mu)}{\partial \lambda}\bigg|_{\lambda=\mu}.$$

Since $[T(\lambda|\mu), T(\lambda'|\mu)] = 0$, we have $[H_+, H_-] = 0$ and $T(\lambda|\mu), H_\pm$ all have common eigenvectors.

Baxter's formula works at **two** Hamiltonian points $\lambda = 1, \mu$.
$H_\pm$ are related by a daulity. Moreover, we have

1. Positivity; all the off-diagonal elements are nonnegative,
2. Sum-to-0; the sum of elements in any column is zero.

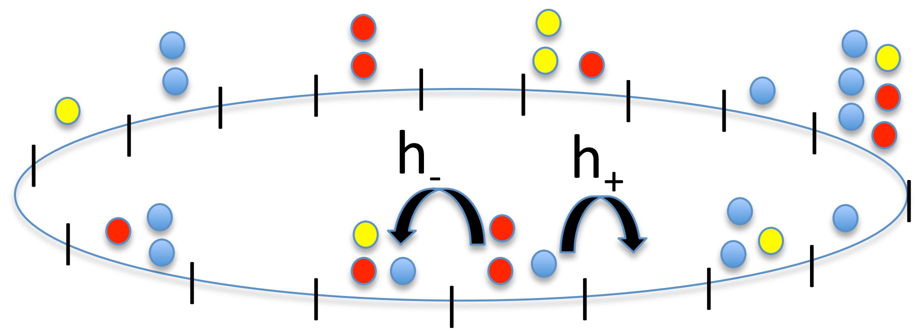
$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle \in W^{\otimes L}, \quad H = aH_+ + bH_- \ (a, b \in \mathbb{R}_{\geq 0})$$

defines a **continuous time Markov process**.

Continuous time Markov Process (2)

n=3 example

Particles 1 2 3



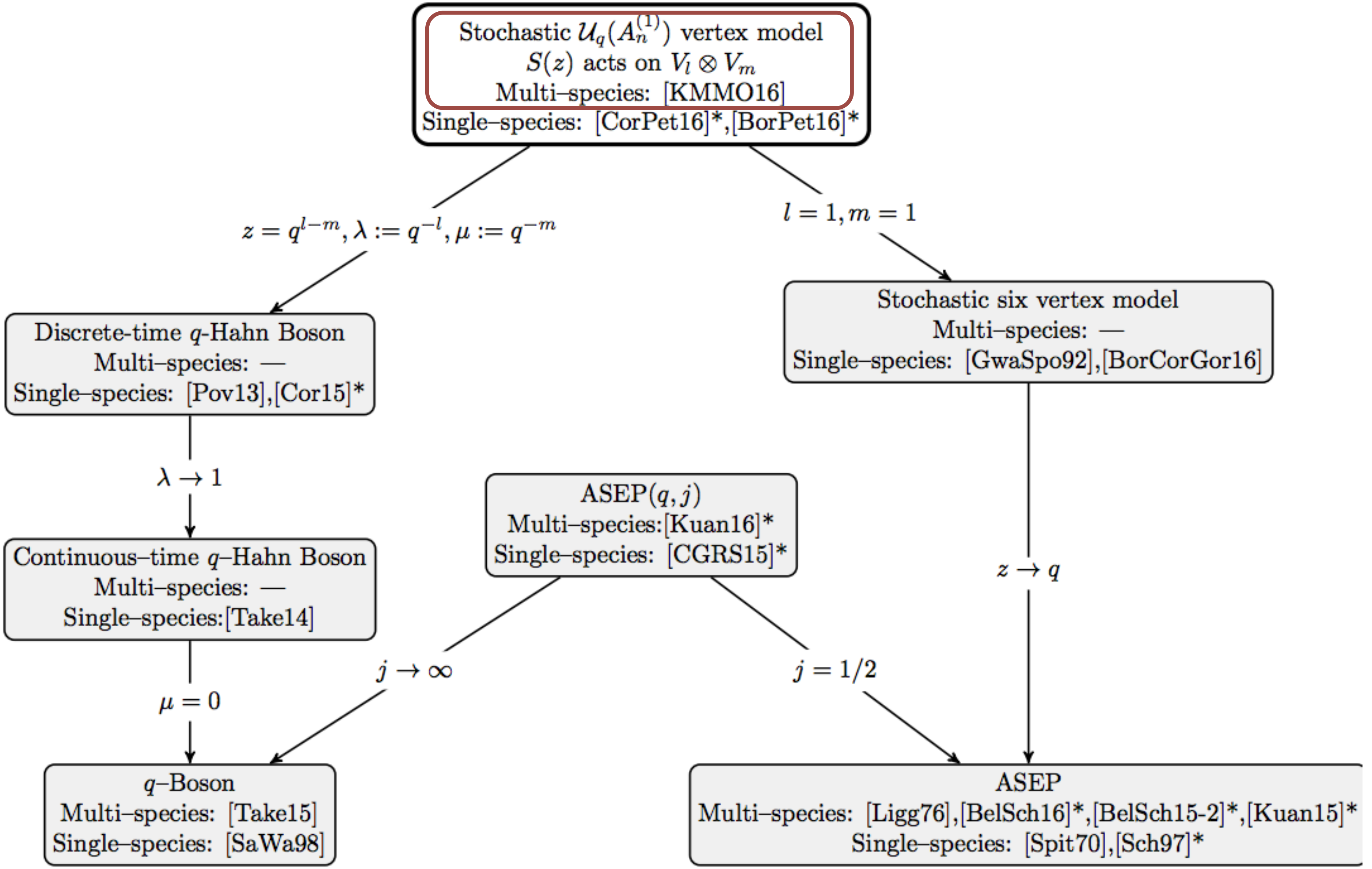
$H_\pm = \sum_{i \in \mathbb{Z}_L} h_{\pm,i,i+1}$ where $h_\pm$ is the **local** Markov matrix.

$$h_+|\alpha,\beta\rangle = \sum_{\gamma \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}} \frac{q^{\sum_{1\leq i<j\leq n}(\alpha_i-\gamma_i)\gamma_j}\mu^{|\gamma|-1}(q)_{|\gamma|-1}}{(\mu q^{|\alpha|-|\gamma|};q)_{|\gamma|}} \prod_{i=1}^n \binom{\alpha_i}{\gamma_i}_q |\alpha-\gamma,\beta+\gamma\rangle,$$

$$h_-|\alpha,\beta\rangle = \sum_{\gamma \in \mathbb{Z}^n_{\geq 0} \setminus \{0\}} \frac{q^{\sum_{1\leq i<j\leq n}\gamma_i(\beta_j-\gamma_j)}(q)_{|\gamma|-1}}{(\mu q^{|\beta|-|\gamma|};q)_{|\gamma|}} \prod_{i=1}^n \binom{\beta_i}{\gamma_i}_q |\alpha+\gamma,\beta-\gamma\rangle$$

up to diagonal terms.

Defines a Zero Range Process of n-species of particles where the transition rate depends on the occupancy of the departure site only.



Contains many integrable stochastic models known earlier (taken from Kuan ArXiv:1701.04468)

Contents.

I. Quantum/Stochastic R matrices

II. Integrable Markov process

III. Stationary states and matrix product formula

stationary states, example,
matrix product formula,
Zamolodchikov-Faddeev algebra,
q-boson realization,
final remarks.

Stationary states

Stationary states are those satisfying

$$|\overline{P}\rangle = T(\lambda|\mu_1, \dots, \mu_L)|\overline{P}\rangle \in W^{\otimes L}.$$

Because of the weight conservation

$$T^{\alpha_1,\dots,\alpha_L}_{\beta_1,\dots,\beta_L} = 0 \text{ unless } \alpha_1 + \dots + \alpha_L = \beta_1 + \dots + \beta_L \in \mathbb{Z}^n_{\geq 0},$$

T is a direct sum of matrices acting on finite-dimensional subspaces (**sectors**) of $W^{\otimes L}$ parametrized by $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 0}$.

$$S(m) = \{(\sigma_1, \dots, \sigma_L) \in (\mathbb{Z}^n_{\geq 0})^L \mid \sigma_1 + \dots + \sigma_L = m\},$$

$$|\overline{P}(m)\rangle = \sum_{(\sigma_1,\dots,\sigma_L)\in S(m)} \underbrace{\mathbb{P}(\sigma_1, \dots, \sigma_L)}|\sigma_1, \dots, \sigma_L\rangle.$$

Stationary probability

Example

$n = 2, m = (2,1), \mu_1 = \mu_2 = \mu_3 = \mu$. The stationary states for $L = 2, 3$ are:

$$|\overline{P}(2,1)\rangle = (1 - q^2\mu)(3 + q - \mu - 3q\mu)|\emptyset, 112\rangle$$
$$+ (1-\mu)(1 + q + 2q^2 - 2q\mu - q^2\mu - q^3\mu)|2, 11\rangle$$
$$+ (1+q)(1-\mu)(2 + q + q^2 - \mu - q\mu - 2q^2\mu)|1, 12\rangle + \text{cyclic.}$$

$$|\overline{P}(2,1)\rangle = 3(1 - q\mu)(1 - q^2\mu)(2 + q - (1+2q)\mu)|\emptyset, \emptyset, 112\rangle$$
$$+ (1-\mu)(1-q\mu)(3 + 3q + 3q^2 - (1 + 5q + 2q^2 + q^3)\mu)|\emptyset, 2, 11\rangle$$
$$+ (1+q)(1-\mu)(1-q\mu)(3 + 3q + 3q^2 - (2 + 2q + 5q^2)\mu)|\emptyset, 1, 12\rangle$$
$$+ (1+q)(1-\mu)(1-q\mu)(5 + 2q + 2q^2 - (3 + 3q + 3q^2)\mu)|\emptyset, 12, 1\rangle$$
$$+ (1-\mu)(1-q\mu)(1 + 2q + 5q^2 + q^3 - (3q + 3q^2 + 3q^3)\mu)|\emptyset, 11, 2\rangle$$
$$+ (1+q)(1+q+q^2)(1-\mu)^2(2 + q - (1+2q)\mu)|1, 1, 2\rangle + \text{cyclic.}$$

Conjecturally $\mathbb{P}(\sigma_1, \ldots, \sigma_L) \in \mathbb{Z}_{\geq 0}[q, -\mu_1, \ldots, -\mu_L]$ in a certain normalization.

Matrix product formula

- T is nonnegative and satisfies Sum-to-1.

$\xrightarrow{\text{Perron-Frobenius}}$ Stationary states are algebraic.

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algebraic $\cap$ transcendental $\simeq$ Matrix product structure

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(\underbrace{X_{\sigma_1}(\mu_1) \cdots X_{\sigma_L}(\mu_L)}_{\text{Operators acting on some } \textbf{\textit{auxiliary space}}}).$$

Zamolodchikov-Faddeev algebra (1)

Proposition (to be proved on the next page)

If the operators $X_\alpha(\mu)$ $(\alpha \in \mathbb{Z}^n_{\geq 0})$ satisfy the ZF relation

$$X_\alpha(\mu) X_\beta(\lambda) = \sum_{\gamma,\delta} \mathcal{S}(\lambda,\mu)^{\beta,\alpha}_{\gamma,\delta} X_\gamma(\lambda) X_\delta(\mu)$$

and the trace is nonzero, the matrix product formula holds.

Symbolically $\boxed{X(\mu) \otimes X(\lambda) = \check{\mathcal{S}}(\lambda,\mu)\left[X(\lambda) \otimes X(\mu)\right]}$

$P\mathcal{S}(\lambda,\mu) \quad (P(u \otimes v) = v \otimes u)$

Originally introduced in integrable quantum field theories in (1+1)-dimension.

Structure function in that context = Scattering matrix satisfying **Unitarity**

Present context: Local form of the stationary condition

Structure function = Stochastic R satisfying **Sum-to-1**

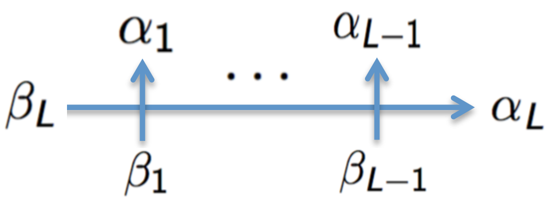
It is a part of so-called **RLL relation** $\left[L(\mu) \otimes L(\bar{\lambda})\right] \check{R}(\lambda,\mu) = \check{R}(\lambda,\mu)\left[L(\lambda) \otimes L(\mu)\right]$.

Zamolodchikov-Faddeev algebra (2)

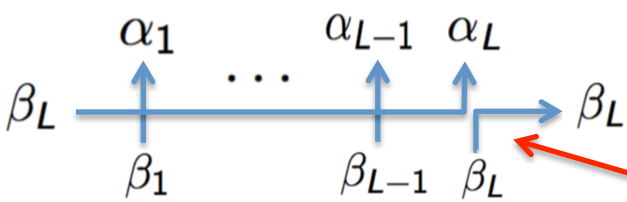
The proof of $|\overline{P}\rangle = T|\overline{P}\rangle$ with $T = T(\lambda = \mu_L | \mu_1, \ldots, \mu_L)$ goes as

$$\mathrm{Tr}(X_{\alpha_1}(\mu_1)\cdots X_{\alpha_L}(\mu_L))$$

$$= \sum_{\beta_1,\ldots,\beta_L} \Big(\quad \Big)\ \mathrm{Tr}(X_{\beta_L}(\mu_L)X_{\beta_1}(\mu_1)\cdots X_{\beta_{L-1}}(\mu_{L-1}))$$



$$= \sum_{\beta_1,\ldots,\beta_L} \Big(\quad \Big)\ \mathrm{Tr}(X_{\beta_1}(\mu_1)\cdots X_{\beta_L}(\mu_L))$$



$$= \sum_{\beta_1,\ldots,\beta_L} T^{\alpha_1,\ldots,\alpha_L}_{\beta_1,\ldots,\beta_L}\ \mathrm{Tr}(X_{\beta_1}(\mu_1)\cdots X_{\beta_L}(\mu_L)),$$

$\boxed{\check{S}(\mu_L, \mu_L) = \mathrm{id}}$

which is a standard maneuver in dealing with *quantum Knizhnik-Zamolodchikov type equation*.

q-Boson realization (1)

Consider the Fock space $F = \bigoplus_{m\geq 0} \mathbb{Q}(q)|m\rangle$ and the operators $\mathbf{b}_+, \mathbf{b}_-, \mathbf{k}$ acting on them as

$$\mathbf{b}_+|m\rangle = |m+1\rangle, \qquad \mathbf{b}_-|m\rangle = (1-q^m)|m-1\rangle, \qquad \mathbf{k}|m\rangle = q^m|m\rangle.$$

They satisfy the q-boson relation

$$\mathbf{k}\mathbf{b}_\pm = q^{\pm 1}\mathbf{b}_\pm\mathbf{k}, \qquad \mathbf{b}_+\mathbf{b}_- = 1-\mathbf{k}, \qquad \mathbf{b}_-\mathbf{b}_+ = 1-q\mathbf{k}.$$

Proposition ($n=2$)

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$, the operator on F

$$X_\alpha(\mu) = \frac{\mu^{-\mu_1-\mu_2}(\mu)_{\alpha_1+\alpha_2}}{(q)_{\alpha_1}(q)_{\alpha_2}} \frac{(\mathbf{b}_+;q)_\infty}{(\mu^{-1}\mathbf{b}_+;q)_\infty}\mathbf{k}^{\alpha_2}\mathbf{b}_-^{\alpha_1}$$

satisfies the ZF relation.

q-Boson realization (2)

General n case: $X_\alpha(\mu) = (q\text{-boson})^{\otimes n(n-1)/2} \in \mathrm{End}(F^{\otimes n(n-1)/2})$.

Recursive structure in rank n (reminiscent of **Nested Bethe ansatz**).

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, set

$$X_\alpha(\mu) = X^{(n)}_\alpha(\mu) = \frac{\mu^{-\alpha_0^+}(\mu)_{\alpha_0^+}}{\prod_{i=1}^n (q)_{\alpha_i}} Z^{(n)}_\alpha(\mu), \qquad \alpha_i^+ = \alpha_{i+1} + \cdots + \alpha_n,$$

Theorem (KO)

The following recursive construction yields $X_\alpha(\mu)$ satisfying ZF relation:

$$Z^{(n)}_\alpha(\zeta) = \sum_{\beta=(\beta_1,\ldots,\beta_{n-1})\in\mathbb{Z}^{n-1}_{\geq 0}} X^{(n-1)}_\beta(\zeta) \otimes \mathbf{b}_+^{\beta_1}\mathbf{k}^{\alpha_1^+}\mathbf{b}_-^{\alpha_1} \otimes \cdots \otimes \mathbf{b}_+^{\beta_{n-1}}\mathbf{k}^{\alpha_{n-1}^+}\mathbf{b}_-^{\alpha_{n-1}}$$

Explcit factorized form available. For instance for $n = 3$,

$$X_{0,0,0}(\zeta) = \frac{(\mathbf{b}_+ \otimes 1 \otimes 1)_\infty}{(\zeta^{-1}\mathbf{b}_+ \otimes 1 \otimes 1)_\infty} \frac{(\mathbf{b}_- \otimes \mathbf{b}_+ \otimes 1)_\infty (\mathbf{k} \otimes 1 \otimes \mathbf{b}_+)_\infty}{(\zeta^{-1}\mathbf{b}_- \otimes \mathbf{b}_+ \otimes 1)_\infty (\zeta^{-1}\mathbf{k} \otimes 1 \otimes \mathbf{b}_+)_\infty}.$$

Final Remark (1)

FZ relation and (Sum-to-1) for $\mathcal{S}(\lambda,\mu)$ imply

$$[A(\lambda|w), A(\mu|w)] = 0 \quad \text{for} \quad A(\lambda|w) = \sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} X_\alpha(\lambda) w_1^{\alpha_1}\cdots w_n^{\alpha_n}.$$

\`\`Grand canonical partition function" \`\`Canonical partition function"

$$\mathrm{Tr}\Big(A(z_1|w)\cdots A(z_L|w)\Big) = \sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} \underbrace{F_\alpha(z_1,\ldots,z_L;q)}_{\text{Symmetric rational function of } z_1,\ldots,z_L} w_1^{\alpha_1}\cdots w_n^{\alpha_n}$$

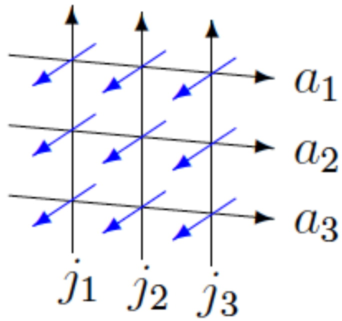
Similar constructions for the simplest stochastic R matrix $S^{1,1}(z)$ with boundary twist have led to generalizations of Macdonald polynomials and their matrix product formulas.

cf. Cantini-de Gier-Wheeler, Borodin-Petrov, …

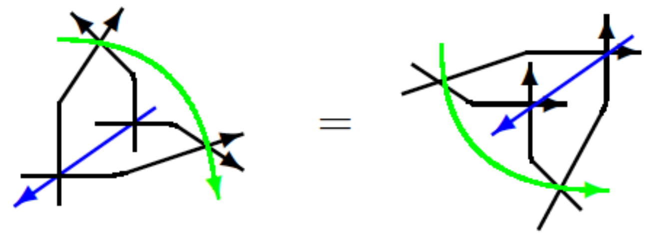
Final Remark (2)

Layer transfer matrix

$$\mathbb{T}(z|\mathbf{a},\mathbf{j}) =$$

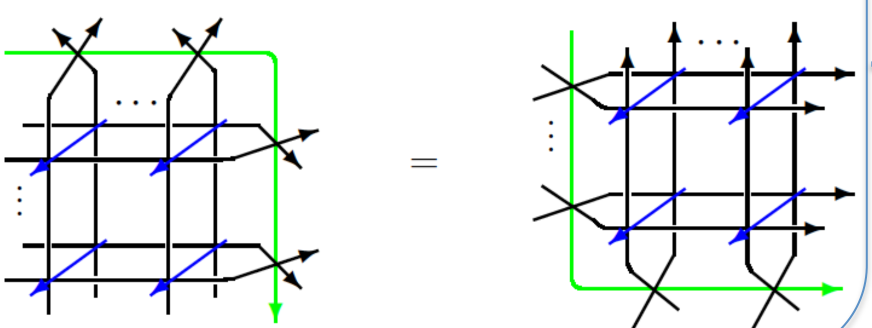


Tetrahedron equation



Bilinear relation of Layer transfer matrices

$$\sum x^* y^{**}\, \mathbb{T}(x|\mathbf{a},\mathbf{j})\mathbb{T}(y|\mathbf{a}',\mathbf{j}') = (x \leftrightarrow y)$$



$q \to 0$ limit

$$\mathbb{T}(z|\mathbf{a},\mathbf{j}) \to \sum X_\alpha(z) \otimes (\text{frozen part})$$

ZF relation at q=0

$$X(x) \otimes X(y) = (\check{S}_{q=0})\big[X(y) \otimes X(x)\big]$$

Leads to combinatorial algorithm for stationary probability related to **crystals**.
Relation of this X(x) at q=0 and the previous one is yet to be clarified.