Tetrahedron equation and matrix product method

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Joint work with S. Maruyama and M. Okado

Reference
Multispecies totally asymmetric zero range process I, II
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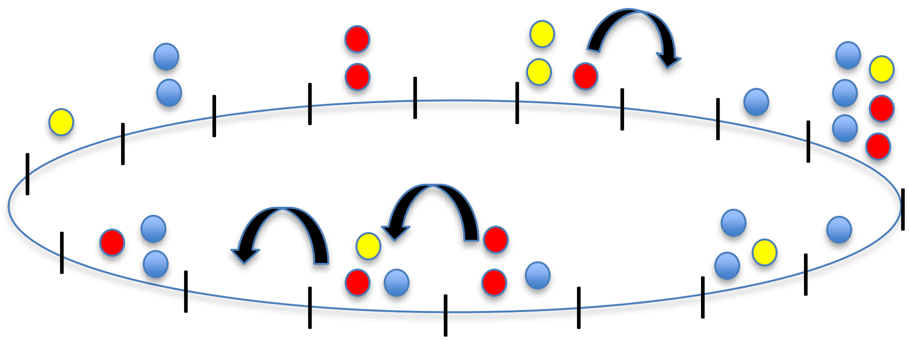
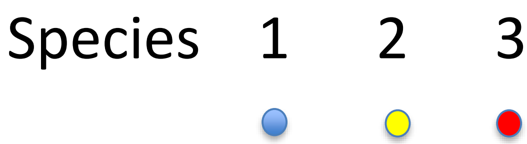
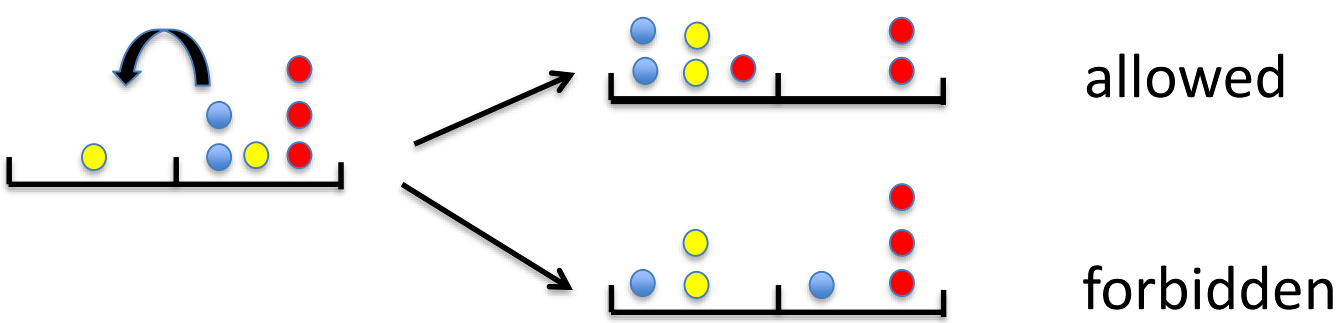
n-species Totally Asymmetric Zero-Rang Process (n-TAZRP)

n=3 example





Smaller species particles have *priority* to hop to the left neighbor site



Local state (site variable)

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n, \quad \alpha_a = \#(\text{species } a \text{ particles})$$

$$(\gamma, \delta) > (\alpha, \beta) \overset{\text{def}}{\Longleftrightarrow} \text{local transition } (\gamma, \delta) \to (\alpha, \beta) \text{ is allowed (priority rule obeyed)}$$

Local Markov matrix : $$h|\gamma, \delta\rangle = \sum_{(\gamma,\delta)>(\alpha,\beta)} \big(|\alpha, \beta\rangle - |\gamma, \delta\rangle\big)$$

Markov matrix : $$H = \sum_{i \in \mathbb{Z}_L} h_{i,i+1}$$

We consider n-TAZRP on 1D periodic chain, which is a Markov process governed by the master equation

$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle$$

$$|P(t)\rangle = \sum_{(\sigma_1,\ldots,\sigma_L)\in S(m)} \overbrace{\mathbb{P}(\sigma_1,\ldots,\sigma_L;t)}^{\text{Probability of the configuration at time t}} |\sigma_1,\ldots,\sigma_L\rangle$$

Set of configurations in the *sector* $m=(m_1,\ldots,m_n)$

m_a = #(species *a* particles)

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- Problem in non-equilibrium statistical mechanics
- Stochastic dynamics of n-species particles with priority constraint within the same departure site (zero-range interaction)

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- Problem in non-equilibrium statistical mechanics
- Stochastic dynamics of n-species particles with priority constraint within the same departure site (zero-range interaction)
- Example of *Integrable Probability*:

Today's main topic

$\boxed{\text{Matrix product construction of Steady State} \leftarrow \text{tetrahedron equation}}$

Associated with Stochastic R matrix for $U_q(A^{(1)}_n)$ (arXiv:1604:08304)

These features become most manifest for multispecies setting n>1

Steady states: H |P>=0

Each sector m has the unique steady state

$$|\bar{P}_L(\mathbf{m})\rangle = |\xi_L(\mathbf{m})\rangle + C|\xi_L(\mathbf{m})\rangle + \cdots + C^{L-1}|\xi_L(\mathbf{m})\rangle$$

(L = chain length, m = sector, C = cyclic shift)

Example from 3-TAZRP

$$\begin{aligned}
|\xi_2(1,1,1)\rangle &= 2|1,23\rangle + |2,13\rangle + 3|3,12\rangle + 6|\emptyset,123\rangle, \\
|\xi_3(1,1,1)\rangle &= 5|1,2,3\rangle + |1,3,2\rangle + 9|\emptyset,1,23\rangle + 3|\emptyset,2,13\rangle + 6|\emptyset,3,12\rangle + 12|\emptyset,12,3\rangle \\
&\quad + 3|\emptyset,13,2\rangle + 3|\emptyset,23,1\rangle + 18|\emptyset,\emptyset,123\rangle, \\
|\xi_4(1,1,1)\rangle &= 17|\emptyset,1,2,3\rangle + 3|\emptyset,1,3,2\rangle + 12|\emptyset,1,\emptyset,23\rangle + 3|\emptyset,2,1,3\rangle + 7|\emptyset,2,3,1\rangle + 8|\emptyset,2,\emptyset,13\rangle \\
&\quad + 9|\emptyset,3,1,2\rangle + |\emptyset,3,2,1\rangle + 20|\emptyset,3,\emptyset,12\rangle + 24|\emptyset,\emptyset,1,23\rangle + 6|\emptyset,\emptyset,2,13\rangle + 10|\emptyset,\emptyset,3,12\rangle \\
&\quad + 30|\emptyset,\emptyset,12,3\rangle + 6|\emptyset,\emptyset,13,2\rangle + 4|\emptyset,\emptyset,23,1\rangle + 40|\emptyset,\emptyset,\emptyset,123\rangle, \\
|\xi_2(2,1,1)\rangle &= 2|1,123\rangle + |2,113\rangle + 3|3,112\rangle + 2|11,23\rangle + |12,13\rangle + 6|\emptyset,1123\rangle, \\
|\xi_3(2,1,1)\rangle &= 3|1,1,23\rangle + 2|1,2,13\rangle + |1,3,12\rangle + 5|1,12,3\rangle + |1,13,2\rangle + 5|2,3,11\rangle + |2,11,3\rangle \\
&\quad + 9|\emptyset,1,123\rangle + 3|\emptyset,2,113\rangle + 6|\emptyset,3,112\rangle + 9|\emptyset,11,23\rangle + 3|\emptyset,12,13\rangle + 3|\emptyset,13,12\rangle \\
&\quad + 3|\emptyset,23,11\rangle + 12|\emptyset,112,3\rangle + 3|\emptyset,113,2\rangle + 3|\emptyset,123,1\rangle + 18|\emptyset,\emptyset,1123\rangle,
\end{aligned}$$

n=1-TAZRP has trivial (uniform) steady states

Main Result

Steady state probability | Matrix product formula

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}(X_{\sigma_1} \cdots X_{\sigma_L})$$

F = Fock space of *q*-boson at q=0

TAZRP operators

Piece of *layer transfer matrices* of 3D lattice model satisfying Tetrahedron equation

q-boson and Fock space

$\mathcal{A}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \rangle$ acts on $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ by

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = q^m|m\rangle$$

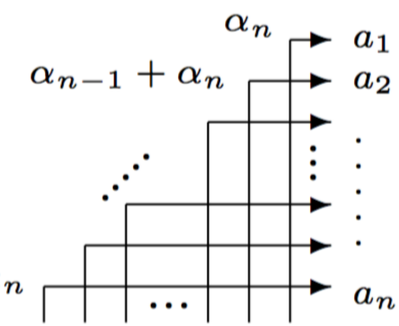
TAZRP operators

= $[(q=0)\text{-boson}]^{\otimes n(n-1)/2}$ valued partition function of region

$$X_\alpha = X_\alpha(z=1)$$

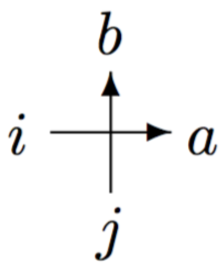
$$X_\alpha(z) = \sum z^{a_1+\cdots+a_n}$$

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$



$$\in \mathcal{A}_0^{\otimes n(n-1)/2}$$

Edges summed over $\mathbb{Z}_{\geq 0}$

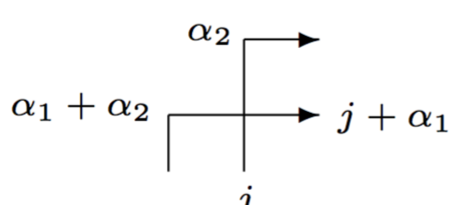


$$= \delta_{i+j}^{a+b}\,\theta(a \geq j)(\mathbf{a}^+)^j \mathbf{k}^{a-j}(\mathbf{a}^-)^b \in \mathcal{A}_{q=0}$$

$$\theta(\text{true}) = 1,\ \theta(\text{false}) = 0, \quad \delta_{i+j}^{a+b} = \theta(a+b = i+j)$$

n=2 Example

$$X_{\alpha_1,\alpha_2}(z) = \sum_{j \geq 0} z^{\alpha_1+\alpha_2+j}$$



$$= z^{\alpha_1+\alpha_2} \sum_{j \geq 0} z^j (\mathbf{a}^+)^j \mathbf{k}^{\alpha_1} (\mathbf{a}^-)^{\alpha_2}$$

To show:

the steady state condition for TAZRP operators

$$x^{|\beta|} \sum_{(\gamma,\delta)\geq(\alpha,\beta)} X_\gamma(x) X_\delta(y) = (x \leftrightarrow y) \qquad (|\beta| = \beta_1 + \cdots + \beta_n)$$

. . . Highly non-local relation

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. . . Highly non-local relation

$q \longrightarrow 0$

Strategy of the proof

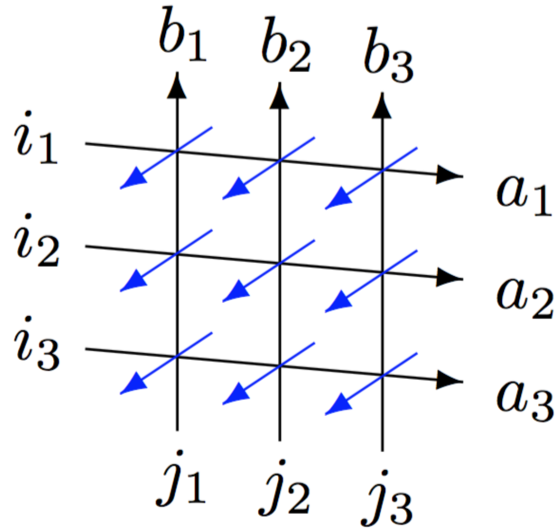
Bilinear relations of
layer transfer matrices

Tetrahedron equation
(Single local relation)

Layer transfer matrix with boundary condition **b**, **i**

n = 3 Example

$$\mathbb{T}(z)^{\mathbf{b}}_{\mathbf{i}} = \chi'_{\mathbf{b}}\,\chi_{\mathbf{i}} \sum_{\mathbf{a},\mathbf{j}} \chi'_{\mathbf{a}}\,\chi_{\mathbf{j}}$$



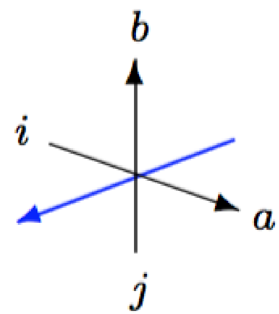
$$\in \mathcal{A}_q^{\otimes n^2}$$

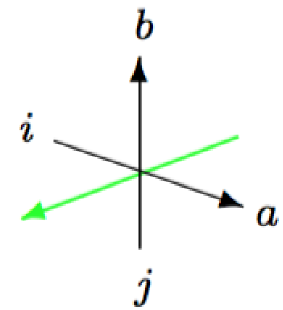
$$\mathbf{a} = (a_1, \ldots, a_n), \quad \chi'_{\mathbf{a}} = \prod_{l=1}^{n} (-q;q)_{a_l}, \quad \chi_{\mathbf{j}} = \prod_{l=1}^{n} (q;q)^{-1}_{j_l} \quad \text{etc}$$

All black edges except **b**, **i** are summed over $\mathbb{Z}_{\geq 0}$

Each 3D vertex is a *q*-boson acting on the Fock space on the blue lines

3D R-operators

$$\hat{\mathcal{R}}^{ab}_{ij}(z) =$$ 

$$\hat{\mathcal{S}}^{ab}_{ij}(z) =$$ 

$$\hat{\mathcal{R}}^{ab}_{ij}(z) = \hat{\mathcal{S}}^{ba}_{ji}(z^{-1}) = \delta^{a+b}_{i+j}\, z^{j-b} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^{\lambda+\mu^2-ib} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2} (\mathbf{a}^-)^{\mu} (\mathbf{a}^+)^{j-\lambda} \mathbf{k}^{i+\lambda-\mu}$$

Define 3D R-operators $\mathcal{R}(z)$, $\mathcal{S}(z) : F \otimes F \otimes F \to F \otimes F \otimes F$ by

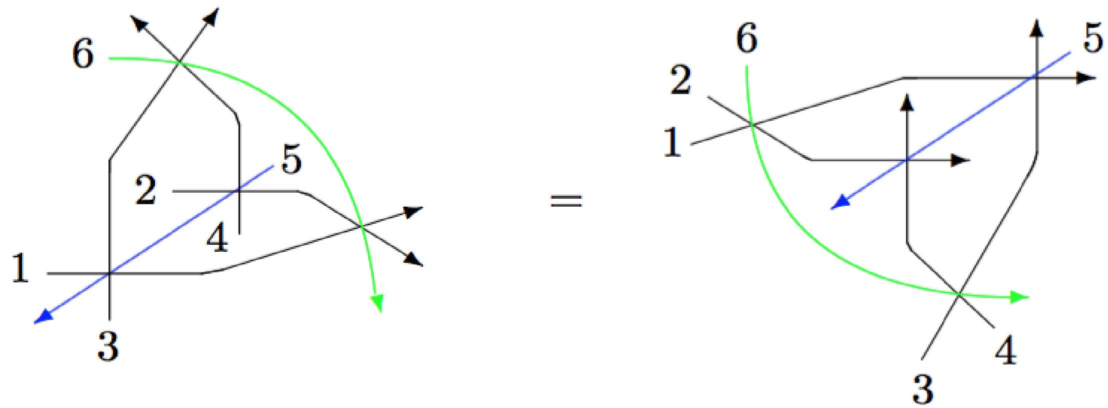
$$\mathcal{R}(z)\big(|i\rangle \otimes |j\rangle \otimes |k\rangle\big) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes \hat{\mathcal{R}}^{ab}_{ij}(z)|k\rangle$$

$$\mathcal{S}(z)\big(|i\rangle \otimes |j\rangle \otimes |k\rangle\big) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes \hat{\mathcal{S}}^{ab}_{ij}(z)|k\rangle$$

Fact (reducible to Kapranov-Voevodsky 1994)

As operators on $F^{\otimes 6}$ the tetrahedron eq. holds: $(z_{ij} = z_i/z_j)$

$$\mathcal{S}(z_{12})_{126}\mathcal{S}(z_{34})_{346}\mathcal{R}(z_{13})_{135}\mathcal{R}(z_{24})_{245} = \mathcal{R}(z_{24})_{245}\mathcal{R}(z_{13})_{135}\mathcal{S}(z_{34})_{346}\mathcal{S}(z_{12})_{126}$$



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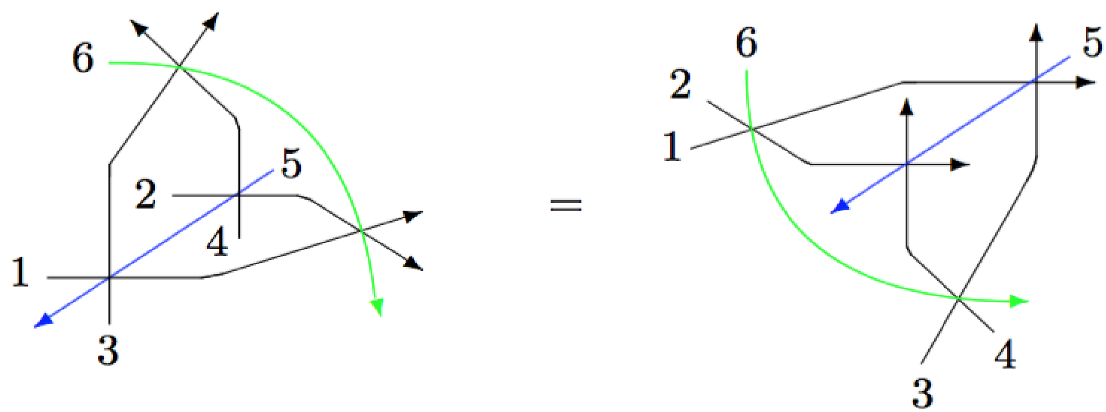


Background and relevant topics:

- Intertwiner of Soibelman's represetations of quantized coordinate ring of SL4
- Transition coefficients of the PBW bases of $U_q^+(sl_3)$
- Quantum geometry interpretation (Bazhanov-Mangazeev-Sergeev 2008)

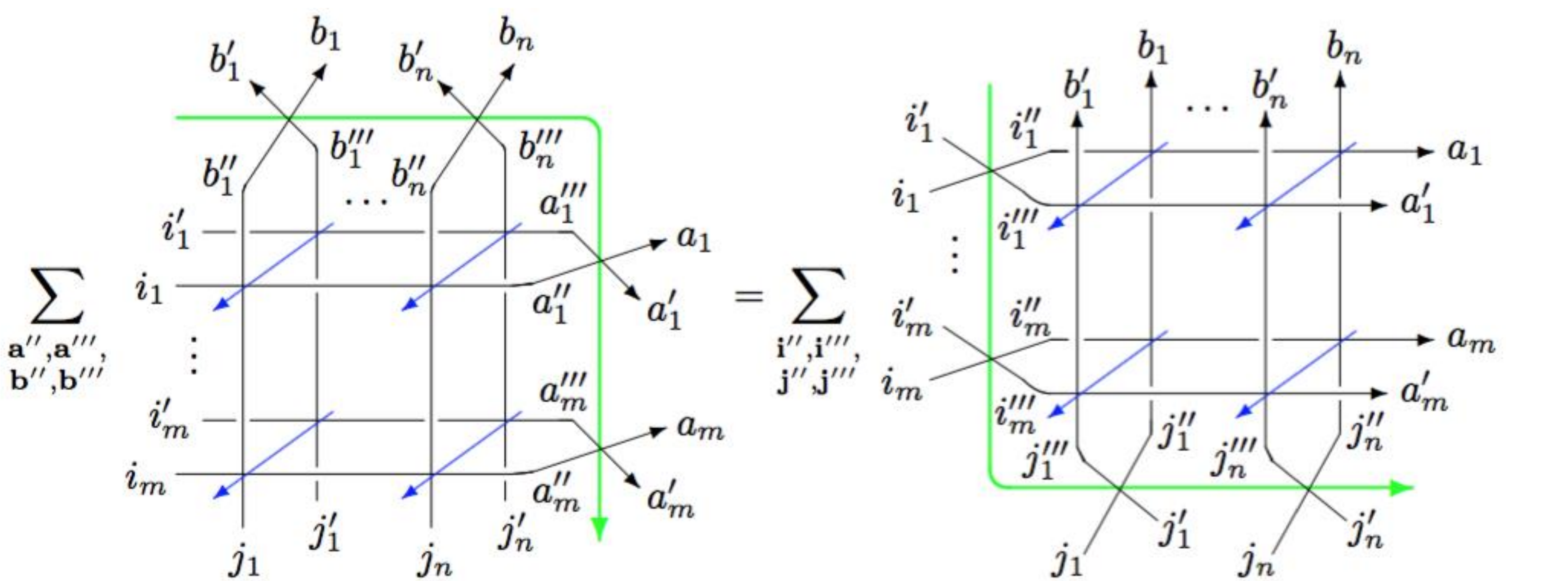
Theorem (Bilinear relation of layer transfer matrices)

$$\sum_{\substack{\mathbf{b},\mathbf{b}',\mathbf{i},\mathbf{i}' \\ \mathbf{b}+\mathbf{b}'=\mathbf{s},\, \mathbf{i}+\mathbf{i}'=\mathbf{r}}} x^{|\mathbf{b}|+|\mathbf{i}|} y^{|\mathbf{b}'|+|\mathbf{i}'|}\, \mathbb{T}(x)^{\mathbf{b}}_{\mathbf{i}}\, \mathbb{T}(y)^{\mathbf{b}'}_{\mathbf{i}'} = (x \leftrightarrow y)$$

$$\forall \mathbf{s}, \mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$$

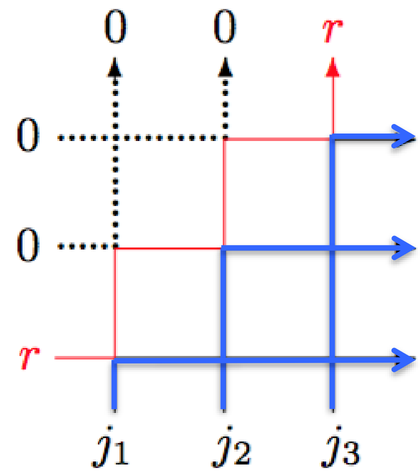
Generalizes the commutativity corresponding to **s** = **r** = (0,…,0)

Follows from repeated applications of the tetrahedron eq.



At q=0, Layer transfer matrix is frozen to TAZRP operators

$$\mathbb{T}(z)^{0,\dots,0,r}_{0,\dots,0,r} = z^{-r}\sum_{\mathbf{a},\mathbf{j}} z^{|\mathbf{j}|}$$



(n=3 example)

$$= z^{-r}\sum_{\alpha\in(\mathbb{Z}_{\geq 0})^n} \overbrace{X_\alpha(z)}^{\mathcal{A}_0^{\otimes n(n-1)/2}} \otimes \overbrace{\text{simply described operators}}^{\mathcal{A}_0^{\otimes n(n+1)/2}}$$

TAZRP operator

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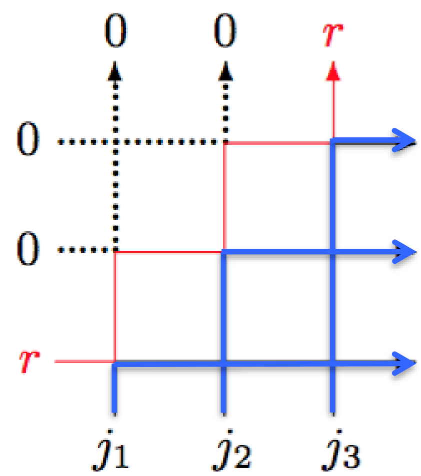

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TAZRP operator

Stationary condition for TAZRP operators $\xleftarrow{q=0}$ Bilinear relation of layer transfer matrices $\longleftarrow$ Tetrahedron eq.

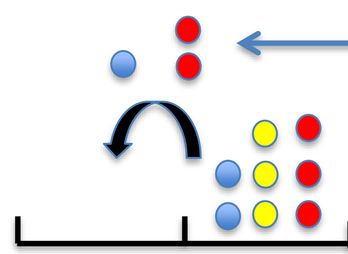
Integrable origin of n-TAZRP

n-TAZRP $\xleftarrow[\mu = 0]{q=0}$ $U_q(A^{(1)}_n)$-Zero Range Process associated with *Stochastic R matrix*

(K-Mangazeev-Maruyama-Okado, arXiv:1604.08304)

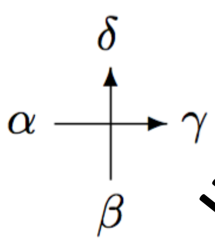
Quantum R matrix in a special gauge satisfying the axioms of Markov matrix

Matrix elements generalize Povolotsky's transition rate for *q-Hahn process* (n=1)



γ =(1,0,2)

β=(2,3,3)



$$q^{\sum_{1\le i<j\le n}(\beta_i-\gamma_i)\gamma_j}\left(\frac{\mu}{\lambda}\right)^{\gamma_1+\cdots+\gamma_n}\frac{(\lambda;q)_{\gamma_1+\cdots+\gamma_n}(\frac{\mu}{\lambda};q)_{\beta_1+\cdots+\beta_n-\gamma_1-\cdots-\gamma_n}}{(\mu;q)_{\beta_1+\cdots+\beta_n}}\prod_{i=1}^{n}\frac{(q;q)_{\beta_i}}{(q;q)_{\gamma_i}(q;q)_{\beta_i-\gamma_i}}$$

Matrix product formula for $U_q(A^{(1)}_2)$-ZRP (K-Okado, arXiv:1608.02779)

(Discrete time Markov process with inhomogeneity $\mu_1, \ldots, \mu_L$)

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(X_{\sigma_1}(\mu_1) \cdots X_{\sigma_L}(\mu_L)),$$

$$X_\alpha(\mu) = \mu^{-\alpha_1-\alpha_2} \frac{(\mu; q)_{\alpha_1+\alpha_2}}{(q;q)_{\alpha_1}(q;q)_{\alpha_2}} \frac{(\mathbf{a}^+; q)_\infty}{(\mu^{-1}\mathbf{a}^+; q)_\infty} \frac{\mathbf{k}^{\alpha_2}}{(-q\mathbf{k}; q)_{\alpha_1}} (\mathbf{a}^-)^{\alpha_1}$$

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↑

A q-boson representation of *Zamolodchikov-Faddeev algebra*

$$X(\mu) \otimes X(\lambda) = \underbrace{\check{S}(\lambda, \mu)}_{U_q(A^{(1)}_2)\text{ Stochastic R matrix}} [X(\lambda) \otimes X(\mu)]$$

satisfying an *auxiliary condition*

q=0, μ_i =0 case agrees with the tetrahedron result for 2-TAZRP

Concluding remarks

At q=0, a parallel story holds for
n-species *Totally Asymmetric Simple Exclusion Process* (n-TASEP)

TAZRP and TASEP correspond to the *two* situations in which type A quantum R matrices are factorized into solutions of the tetrahedron equation as follows:

Tetrahedron:	3D R operator	3D L operator
Yang-Baxter:	R matrix for symm. tensor rep.	R matrix for anti-symm. tensor rep.
Markov process:	n-TAZRP (today's talk)	n-TASEP (arXiv:1506.04490, 1509.09018)