Aspects of reflection equations

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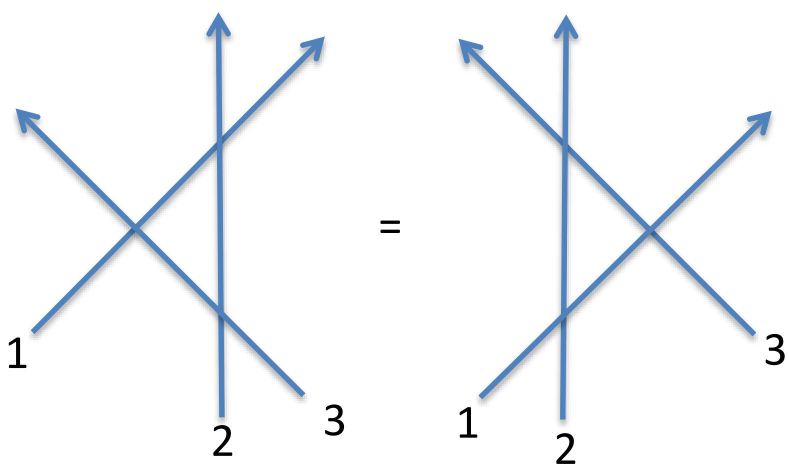
Key to quantum integrability

Yang-Baxter equation (YBE)

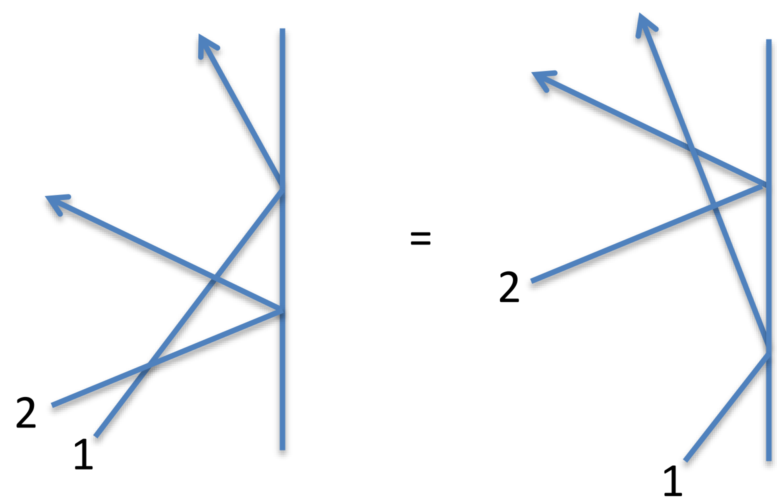
$$R(x)_{12}R(xy)_{13}R(y)_{23} = R(y)_{23}R(xy)_{13}R(x)_{12}$$



R: 2-body scattering

Reflection equation (RE)

$$K(y)_1R(xy)_{12}K(x)_2R(x/y)_{21} = R(x/y)_{12}K(x)_2R(xy)_{21}K(y)_1$$



K: reflection at boundary

Today's topic:

Part I

Quantized coordinate ring A_q
matrix product structure
3D integrability

Part II

Quantum affine algebra U_p
q-Onsager coideal B_q

This talk is mainly based on

[K-Pasquier]
Matrix product solutions to the RE from 3D integrability, arXiv:1802.09164,
[K-Okado-Yoneyama]
Matrix product solution to the RE associated with a coideal subalgebra of $U_q(A^{(1)}_{n-1})$, arXiv:1812.03767.

Contents

Part I
- Quantized coordinate ring A_q
- Intertwiner for $A_q(sp_4)$: 3D K
- Reformulation as quantized reflection equation
- Reduction of quantized RE to the ordinary RE with spectral parameter

Part II
- $U_p(A^{(1)}_{n-1})$ and fundamental representations
- q-Onsager coideal B_q

Generalizations

Quantized coordinate ring $A_q(g)$ (g = finite dim. simple Lie algebra)

- Quantum group corresponding to the dual to $U_q(g)$.
  Studied by Drinfeld (87), Vaksman-Soibelman (89,91), Reshetikhin-Takhtajan-Faddeev (90), Noumi-Yamada-Mimachi (92), Kashiwara (93), Geiss-Leclerc-Schröer (11), Saito (14), Tanisaki (14) etc.

- **Simplest example**

  $A_q(sl_2)$: generated by $t_{11}, t_{12}, t_{21}, t_{22}$ with the relations

$$t_{11}t_{21} = qt_{21}t_{11},\ t_{12}t_{22} = qt_{22}t_{12},\ t_{11}t_{12} = qt_{12}t_{11},\ t_{21}t_{22} = qt_{22}t_{21},$$
$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1.$$

  Hopf algebra with coproduct $\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$.

- **Fundamental representation** $\pi_1 : A_q(sl_2) \to \mathrm{End}(F_q)$

  $F_q = \oplus_{m\geq 0}\mathbb{C}|m\rangle$: q boson Fock space

$$\pi_1 : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{a}^- & -\mathbf{k} \\ \mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \qquad \mathbf{k} = q^{\mathbf{h}+\frac{1}{2}}$$
$$\mathbf{h}|m\rangle = m|m\rangle,\ \mathbf{a}^+|m\rangle = |m+1\rangle,\ \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle.$$

Theorem (Classification of irreducible representations. Soibelman 1991)

1. *Irreducible reps.* $\overset{1:1}{\longleftrightarrow}$ *elements of the Weyl group* $W(g)$ *(up to a "torus degree of freedom").*

   Set π_i := *the irreducible rep. for the simple reflection* $s_i \in W(g)$ *(*i *: a vertex of the Dynkin diagram of* g*).*

2. *The irreducible rep. corresponding to the reduced expression* $s_{i_1} \cdots s_{i_r} \in W(g)$ *is realized as the tensor product* $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$.

Crucial Corollary

If $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}$ are 2 different reduced expressions, then

$$\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}.$$

$\Longrightarrow$ Exists the unique map Φ called intertwiner such that

$$(\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}) \circ \Phi = \Phi \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_r})$$

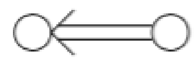
Intertwiner for $A_q(sp_4)$: 3D K

Generators $$T = \begin{pmatrix} t_{11} & \cdots & t_{14} \\ \vdots & \ddots & \vdots \\ t_{41} & \cdots & t_{44} \end{pmatrix}$$ (RTF presentation `90)

Relations
$$\begin{cases} RTT = TTR \qquad R = \text{Constant } R \text{ matrix for vector rep. of } U_q(sp_4) \\ TCT^tC^{-1} = CT^tC^{-1}T = I \qquad C = \begin{pmatrix} 0 & 0 & 0 & -q^{-2} \\ 0 & 0 & q^{-1} & 0 \\ 0 & -q & 0 & 0 \\ -q^2 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

Fundamental representations

π_1 π_2



F_q F_{q^2}

$$\pi_1(T) \qquad\qquad \pi_2(T)$$

$$\begin{pmatrix} \mathbf{a}^- & -\mathbf{k} & 0 & 0 \\ \mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & 0 & -\mathbf{k} & \mathbf{a}^+ \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & -\mathbf{K} & 0 \\ 0 & \mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\langle \mathbf{A}^{\pm}, \mathbf{K} \rangle = \langle \mathbf{a}^{\pm}, \mathbf{k} \rangle|_{q \to q^2}$$

$$W(sp_4) = \langle s_1, s_2 \rangle, \quad s_1^2 = s_2^2 = 1, \quad s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \ \ \text{(longest element)}$$

$\therefore\ {}^{\exists 1}$Intertwiner $\quad \Phi : F_q \otimes F_{q^2} \otimes F_q \otimes F_{q^2} \to F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q$

satisfying
$$\Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2)\Delta(g) = (\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1)\Delta(g) \circ \Phi,$$
$$\Phi(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \qquad (g \in A_q(sp_4)).$$

We use **3D K** defined by

$$\mathcal{K} = \Phi \circ \sigma \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q),$$
$$\sigma(u_1 \otimes u_2 \otimes u_3 \otimes u_4) = u_4 \otimes u_3 \otimes u_2 \otimes u_1$$

The intertwining relation for the 3D K is

$$\mathcal{K} \circ (\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1)\Delta^{\mathrm{op}}(g) = (\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1)\Delta(g) \circ \mathcal{K}.$$

$$(\Delta^{\mathrm{op}} := \sigma \Delta \sigma)$$

$$[\mathbf{1}\otimes\mathbf{a}^-\otimes\mathbf{1}\otimes\mathbf{a}^- - q\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{k},\, \mathcal{K}] = 0,$$

$$(\mathbf{1}\otimes\mathbf{a}^-\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+)\mathcal{K}$$

$$= \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^- - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k}),$$

$$(\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-)\mathcal{K} = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^-),$$

$$[\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k}, \mathcal{K}] = 0,$$

$$(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^- - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k})\mathcal{K}$$

$$= \mathcal{K}(\mathbf{1}\otimes\mathbf{a}^-\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+),$$

$$[\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{A}^-\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+,\, \mathcal{K}] = 0,$$

$$(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k})\mathcal{K}$$

$$= \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k}),$$

$$(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+)\mathcal{K} = \mathcal{K}(\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+),$$

$$(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^-)\mathcal{K} = \mathcal{K}(\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-),$$

$$(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k})\mathcal{K}$$

$$= \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k}),$$

$$[\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{A}^+\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^-,\, \mathcal{K}] = 0,$$

$$(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k})\mathcal{K}$$

$$= \mathcal{K}(\mathbf{1}\otimes\mathbf{a}^+\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-),$$

$$(\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+)\mathcal{K} = \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+),$$

$$(\mathbf{1}\otimes\mathbf{a}^+\otimes\mathbf{1}\otimes\mathbf{k} + \mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-)\mathcal{K}$$

$$= \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes\mathbf{1}\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k}),$$

$$[\mathbf{1}\otimes\mathbf{a}^+\otimes\mathbf{1}\otimes\mathbf{a}^+ - q\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{k},\, \mathcal{K}] = 0.$$

Properties of 3D K [K-Okado `12]

Weight conservation (used later)

$$[(xy^{-1})^{\mathbf{h}_1} x^{\mathbf{h}_2} (xy)^{\mathbf{h}_3} y^{\mathbf{h}_4}, \mathcal{K}] = 0. \qquad \mathbf{h}_1 = \mathbf{h} \otimes 1 \otimes 1 \otimes 1 \ \text{ etc.}$$

Explicit formula (not used later)

$$\mathcal{K}(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle) = \sum_{a,b,c,d \geq 0} \mathcal{K}^{a,b,c,d}_{i,j,k,l} |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle.$$

$$\mathcal{K}^{a,b,c,d}_{i,j,k,l} = \frac{(q^4)_i}{(q^4)_a} \sum_{\alpha,\beta,\gamma} \frac{(-1)^{\alpha+\gamma}}{(q^4)_{c-\beta}} q^{\phi_1} \mathcal{K}^{i,j+k-\alpha-\beta-\gamma,0,l+k-\alpha-\beta-\gamma}_{a,b+c-\alpha-\beta-\gamma,0,c+d-\alpha-\beta-\gamma} \left\{ \begin{matrix} k, c-\beta, j+k-\alpha-\beta, k+l-\alpha-\beta \\ \alpha, \beta, \gamma, b-\alpha, d-\alpha, k-\alpha-\beta, c-\beta-\gamma \end{matrix} \right\},$$

$$\mathcal{K}^{a,b,0,d}_{i,j,0,l} = \sum_{\lambda} (-1)^{b+\lambda} \frac{(q^4)_{a+\lambda}}{(q^4)_a} q^{\phi_2} \left\{ \begin{matrix} j, l \\ \lambda, l-\lambda, b-\lambda, j-b+\lambda \end{matrix} \right\},$$

$$\phi_1 = \alpha(\alpha + 2c - 2\beta - 1) + (2\beta - c)(b + c + d) + \gamma(\gamma - 1) - k(j + k + l),$$

$$\phi_2 = (i + a + 1)(b + l - 2\lambda) + b - l,$$

$$(q)_k = \prod_{i=1}^{k} (1 - q^i), \qquad \left\{ \begin{matrix} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{matrix} \right\} = \begin{cases} \dfrac{\prod_{k=1}^{r} (q^2)_{i_k}}{\prod_{k=1}^{s} (q^2)_{j_k}} & \forall i_k, j_k \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{K}^{1,1,1,1}_{0,2,1,0} = q^5(1+q^2)(1-q^2-q^6), \qquad \mathcal{K}^{1,1,1,1}_{2,0,1,2} = q(1+q^2)(1-q^8)(1-q^4-q^8+q^{10}+q^{14}) \quad \text{etc.}$$

Reformulation as quantized reflection equation

The intertwining relation of the 3D K for generator t_{ij} (i,j = 1,…,4)

$$(\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1)\Delta(t_{ij}) \circ \mathcal{K} = \mathcal{K} \circ (\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1)\Delta^{\mathrm{op}}(t_{ij})$$

$\Longleftrightarrow$

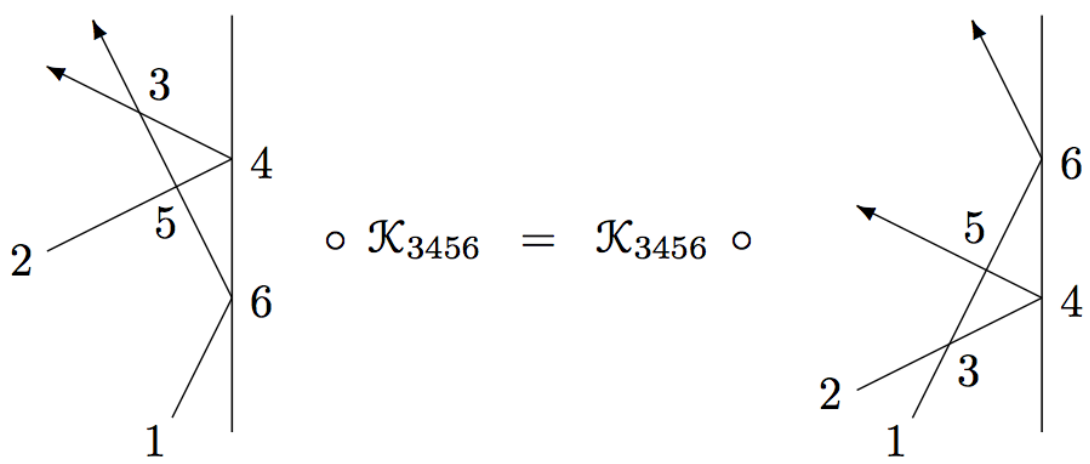
Quantized reflection equation := RE up to conjugation:

$$(L_{12}G_2L_{21}G_1)\mathcal{K} = \mathcal{K}(G_1L_{12}G_2L_{21}).$$

Here ***G*** and ***L*** are *``3D" operators* that act on $V := \mathbb{C}v_0 \otimes \mathbb{C}v_1$ and also on the auxiliary Fock spaces of q-boson. If they are indexed by 3,4,5,6, it means

$$L_{123}G_{24}L_{215}G_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}G_{16}L_{125}G_{24}L_{213}$$

$$\in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{F}_{q^2} \otimes \overset{4}{F}_{q} \otimes \overset{5}{F}_{q^2} \otimes \overset{6}{F}_{q})$$

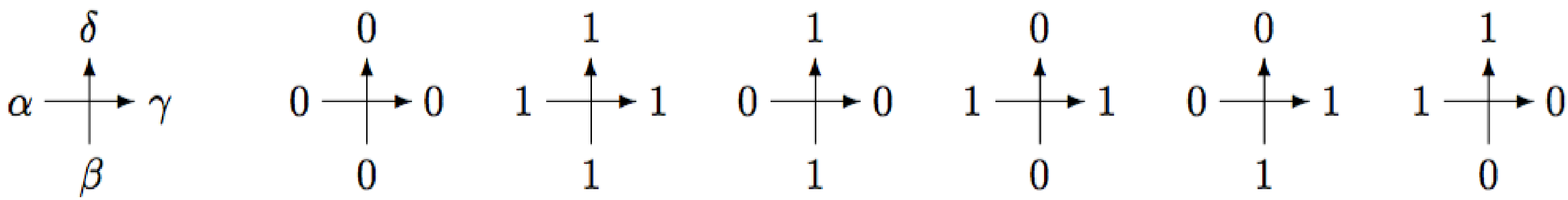


3D L and G operators

$$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$$

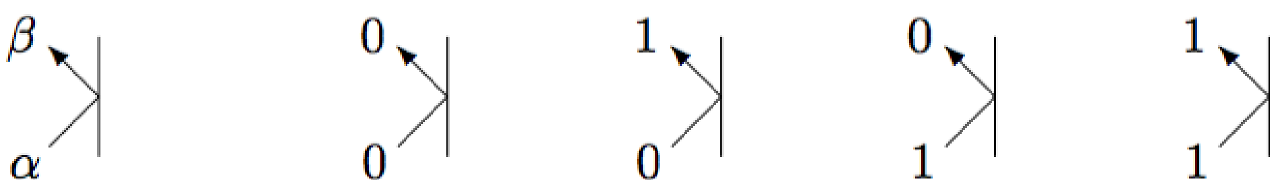
$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{K} & \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ & -\mathbf{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}(V \otimes V \otimes F_{q^2}), \qquad G = \begin{pmatrix} \mathbf{a}^+ & -\mathbf{k} \\ \mathbf{k} & \mathbf{a}^- \end{pmatrix} \in \mathrm{End}(V \otimes F_q),$$

$$L(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum_{\gamma,\delta} v_\gamma \otimes v_\delta \otimes L^{\gamma,\delta}_{\alpha,\beta}|m\rangle, \qquad G(v_\alpha \otimes |m\rangle) = \sum_\beta v_\beta \otimes G^\beta_\alpha |m\rangle.$$



$L^{\gamma,\delta}_{\alpha,\beta}$ $\quad 1 \quad 1 \quad \mathbf{K} \quad -\mathbf{K} \quad \mathbf{A}^+ \quad \mathbf{A}^-$

3D L oeprator = q-boson valued 6V model



G^β_α $\quad \mathbf{a}^+ \quad \mathbf{k} \quad -\mathbf{k} \quad \mathbf{a}^-$

Reduction of quantized RE to the ordinary RE with spectral parameter

Introduce the n-fold tensor product with the following label:

$$\overset{\mathbf{1}}{\mathbf{V}} := \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V} \;\simeq\; \overset{\mathbf{2}}{\mathbf{V}} := \overset{2_1}{V} \otimes \cdots \otimes \overset{2_n}{V} \;\simeq (\mathbb{C}^2)^{\otimes n}$$

Start from the n-copies of quantized RE for each layer i=1,…,n:

$$(L_{1_i 2_i 3} G_{2_i 4} L_{2_i 1_i 5} G_{1_i 6}) \mathcal{K}_{3456} = \mathcal{K}_{3456} (G_{1_i 6} L_{1_i 2_i 5} G_{2_i 4} L_{2_i 1_i 3}).$$

Their composition along the common auxiliary spaces 3,4,5,6 gives

$$\begin{aligned}&(L_{1_1 2_1 3} G_{2_1 4} L_{2_1 1_1 5} G_{1_1 6}) \cdots (L_{1_n 2_n 3} G_{2_n 4} L_{2_n 1_n 5} G_{1_n 6}) \mathcal{K}_{3456} \\ &= \mathcal{K}_{3456} (G_{1_1 6} L_{1_1 2_1 5} G_{2_1 4} L_{2_1 1_1 3}) \cdots (G_{1_n 6} L_{1_n 2_n 5} G_{2_n 4} L_{2_n 1_n 3}).\end{aligned}$$

Factors can be separated into ***L***-only and ***G***-only segments without changing the order of operators sharing common indices as

$$\begin{aligned}&(L_{1_1 2_1 3} \cdots L_{1_n 2_n 3})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 5} \cdots L_{2_n 1_n 5})(G_{1_1 6} \cdots G_{1_n 6}) \mathcal{K}_{3456} \\ &= \mathcal{K}_{3456} (G_{1_1 6} \cdots G_{1_n 6})(L_{1_1 2_1 5} \cdots L_{1_n 2_n 5})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 3} \cdots L_{2_n 1_n 3}).\end{aligned}$$

$$
\begin{aligned}
&(L_{1_1 2_1 3}\cdots L_{1_n 2_n 3})(G_{2_1 4}\cdots G_{2_n 4})(L_{2_1 1_1 5}\cdots L_{2_n 1_n 5})(G_{1_1 6}\cdots G_{1_n 6})\mathcal{K}_{3456} \\
&\quad = \mathcal{K}_{3456}(G_{1_1 6}\cdots G_{1_n 6})(L_{1_1 2_1 5}\cdots L_{1_n 2_n 5})(G_{2_1 4}\cdots G_{2_n 4})(L_{2_1 1_1 3}\cdots L_{2_n 1_n 3}).
\end{aligned}
$$

The weight conservation of 3D K implies

$$
\mathcal{K}_{3456}^{-1}(xy^{-1})^{\mathbf{h}_3}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6} = (xy^{-1})^{\mathbf{h}_3}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}\mathcal{K}_{3456}^{-1}.
$$

Multiply this to the above relation from the left and take the Trace over the auxiliary spaces 3,4,5,6.

$$
\begin{aligned}
&\mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3}L_{1_1 2_1 3}\cdots L_{1_n 2_n 3}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4}G_{2_1 4}\cdots G_{2_n 4}\big)\times \\
&\qquad\qquad \times \mathrm{Tr}_5\big((xy)^{\mathbf{h}_5}L_{2_1 1_1 5}\cdots L_{2_n 1_n 5}\big)\mathrm{Tr}_6\big(y^{\mathbf{h}_6}G_{1_1 6}\cdots G_{1_n 6}\big) \\
&= \mathrm{Tr}_6\big(y^{\mathbf{h}_6}G_{1_1 6}\cdots G_{1_n 6}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5}L_{1_1 2_1 5}\cdots L_{1_n 2_n 5}\big)\times \\
&\qquad\qquad \times \mathrm{Tr}_4\big(x^{\mathbf{h}_4}G_{2_1 4}\cdots G_{2_n 4}\big)\mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3}L_{2_1 1_1 3}\cdots L_{2_n 1_n 3}\big).
\end{aligned}
$$

This is the RE with spectral parameters x, y. It holds in

$$
\mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}\otimes\overset{\mathbf{2}}{\mathbf{V}}) \quad (\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V}\otimes\cdots\otimes\overset{1_n}{V},\ \overset{\mathbf{2}}{\mathbf{V}} = \overset{2_1}{V}\otimes\cdots\otimes\overset{2_n}{V}).
$$

$$
\begin{aligned}
&\mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3} L_{1_1 2_1 3}\cdots L_{1_n 2_n 3}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4} G_{2_1 4}\cdots G_{2_n 4}\big)\times\\
&\qquad\qquad \times\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} L_{2_1 1_1 5}\cdots L_{2_n 1_n 5}\big)\mathrm{Tr}_6\big(y^{\mathbf{h}_6} G_{1_1 6}\cdots G_{1_n 6}\big)\\
&=\mathrm{Tr}_6\big(y^{\mathbf{h}_6} G_{1_1 6}\cdots G_{1_n 6}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} L_{1_1 2_1 5}\cdots L_{1_n 2_n 5}\big)\times\\
&\qquad\qquad \times\mathrm{Tr}_4\big(x^{\mathbf{h}_4} G_{2_1 4}\cdots G_{2_n 4}\big)\mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3} L_{2_1 1_1 3}\cdots L_{2_n 1_n 3}\big).
\end{aligned}
$$

$\Longleftrightarrow$

Reflection equation

$$R(xy^{-1})_{\mathbf{1},\mathbf{2}}\, K(x)_{\mathbf{2}}\, R(xy)_{\mathbf{2},\mathbf{1}}\, K(y)_{\mathbf{1}} = K(y)_{\mathbf{1}}\, R(xy)_{\mathbf{1},\mathbf{2}}\, K(x)_{\mathbf{2}}\, R(xy^{-1})_{\mathbf{2},\mathbf{1}}.$$

for the R and K-matrices defined by the matrix product forms:

$$R(z)_{\mathbf{1},\mathbf{2}} = \mathrm{Tr}_a(z^{\mathbf{h}_a} L_{1_1 2_1 a}\cdots L_{1_n 2_n a}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}\otimes\overset{\mathbf{2}}{\mathbf{V}}),$$

$$K(z)_{\mathbf{1}} = \mathrm{Tr}_b(z^{\mathbf{h}_b} G_{1_1 b}\cdots G_{1_n b}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}),$$

$$K(z)_{\mathbf{2}} = \mathrm{Tr}_b(z^{\mathbf{h}_b} G_{2_1 b}\cdots G_{2_n b}) \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}}),$$

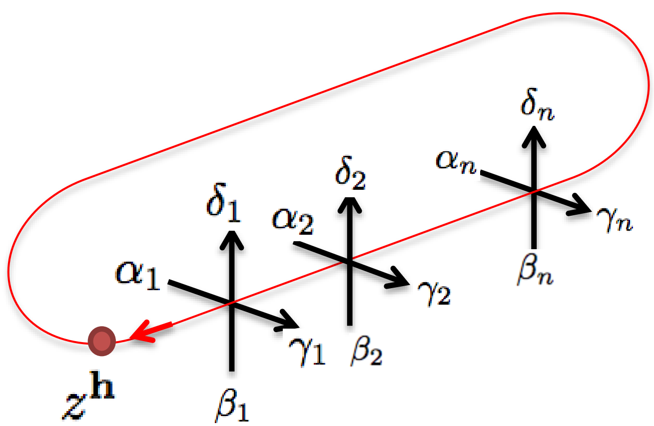
a, b : dummy labels for the auxiliary q-boson Fock spaces $\overset{a}{F}_{q^2}$, $\overset{b}{F}_q$.

★ (Dummy) space indices **1**, **2** here should not be confused with the `level' indices l,m etc in $R_{l,m}(z)$, $K_m(z)$ which will be introduced later.

R-matrix

$$R(z)(\mathbf{v}_\alpha \otimes \mathbf{v}_\beta) = \sum_{\gamma,\delta} R(z)^{\gamma,\delta}_{\alpha,\beta} \mathbf{v}_\gamma \otimes \mathbf{v}_\delta$$

$$R(z)^{\gamma,\delta}_{\alpha,\beta} = \mathrm{Tr}\left(z^{\mathbf{h}} L^{\gamma_1,\delta_1}_{\alpha_1,\beta_1} \cdots L^{\gamma_n,\delta_n}_{\alpha_n,\beta_n}\right)$$

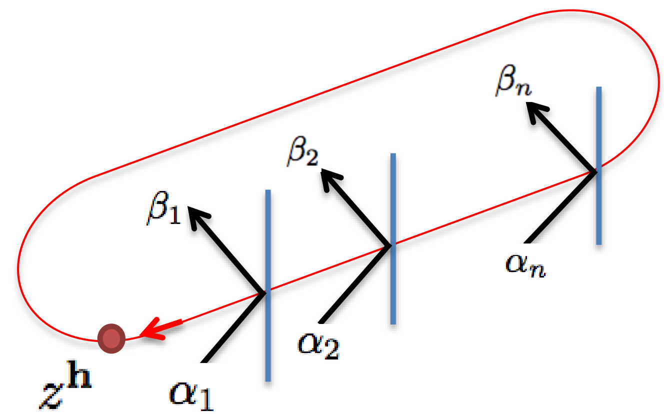


$$\mathrm{End}(\mathbf{V} \otimes \mathbf{V})$$

K-matrix

$$K(z)\mathbf{v}_\alpha = \sum_{\beta} K(z)^{\beta}_{\alpha} \mathbf{v}_\beta$$

$$K(z)^{\beta}_{\alpha} = \mathrm{Tr}\left(z^{\mathbf{h}} G^{\beta_1}_{\alpha_1} \cdots G^{\beta_n}_{\alpha_n}\right)$$



$$\mathrm{End}(\mathbf{V})$$

$$\mathbf{V} = V^{\otimes n} = \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\{0,1\}^n} \mathbb{C}\mathbf{v}_\alpha, \quad \mathbf{v}_\alpha = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}$$

Example: n=3

Write a base v_{010} as $|010\rangle$ etc. Then a suitably normalized K(z) acts as

$$
\begin{aligned}
&|000\rangle \mapsto |111\rangle, \qquad |111\rangle \mapsto |000\rangle,\\
&|001\rangle \mapsto -\frac{(-1+q^2)z|011\rangle}{q(-1+qz)} - \frac{(-1+q^2)z|101\rangle}{-1+qz} + |110\rangle,\\
&|010\rangle \mapsto -\frac{(-1+q^2)z|011\rangle}{-1+qz} + |101\rangle - \frac{(-1+q^2)|110\rangle}{q(-1+qz)},\\
&|011\rangle \mapsto -\frac{(-1+q^2)z|001\rangle}{q(-1+qz)} - \frac{(-1+q^2)z|010\rangle}{-1+qz} + |100\rangle,\\
&|100\rangle \mapsto |011\rangle - \frac{(-1+q^2)|101\rangle}{q(-1+qz)} - \frac{(-1+q^2)|110\rangle}{-1+qz},\\
&|101\rangle \mapsto -\frac{(-1+q^2)z|001\rangle}{-1+qz} + |010\rangle - \frac{(-1+q^2)|100\rangle}{q(-1+qz)},\\
&|110\rangle \mapsto |001\rangle - \frac{(-1+q^2)|010\rangle}{q(-1+qz)} - \frac{(-1+q^2)|100\rangle}{-1+qz}.
\end{aligned}
$$

∗ K(z) is trigonometric and **dense** (*all* elements are non-vanishing).
∗ Swaps #0 and #1, hence splits into *irreducible components* according to #1.
∗ The component [#1=1]->[#1=n-1] reproduces [Gandenberger, hep-th/9911178] obtained in the context of affine Toda field theory with boundaries.

Part II

Part I
- Quantized coordinate ring A_q
- Intertwiner for $A_q(sp_4)$: 3D K
- Reformulation as quantized reflection equation
- Reduction of quantized RE to the ordinary RE with spectral parameter

Part II
- $U_p(A^{(1)}_{n-1})$ and fundamental representations
- q-Onsager coideal B_q
- Generalizations

$U_p(A^{(1)}_{n-1})$ and fundamental representations

Generators : $e_i, f_i, k_i^{\pm 1}$ $(i \in \mathbb{Z}_n)$, Cartan matrix : $(a_{ij})_{i,j\in\mathbb{Z}_n}$, $a_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$

Relations : $k_i k_i^{-1} = k_i^{-1} k_i = 1$, $[k_i, k_j] = 0$, $k_i e_j k_i^{-1} = p^{2a_{ij}} e_j$, $k_i f_j k_i^{-1} = p^{-2a_{ij}} f_j$,

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{p^2 - p^{-2}}, \quad + \text{ Serre type relations,}$$

Coproduct : $\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}$, $\Delta e_i = e_i \otimes 1 + k_i \otimes e_i$, $\Delta f_i = 1 \otimes f_i + f_i \otimes k_i^{-1}$

$\mathbf{V} = V^{\otimes n}$ has the decomposition

$$\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_n, \qquad \mathbf{V}_k = \bigoplus_{\boldsymbol{\alpha} \in \{0,1\}^n,\ \alpha_1 + \cdots + \alpha_n = k} \mathbb{C}\mathbf{v}_{\boldsymbol{\alpha}},$$

Each component $\mathbf{V}_k$ becomes an irreducible U_p module by

$$e_j \mathbf{v}_{\boldsymbol{\alpha}} = z^{\delta_{j,0}} \mathbf{v}_{\boldsymbol{\alpha} - \mathbf{d}_j + \mathbf{d}_{j+1}}, \qquad \mathbf{d}_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0) \qquad (j \in \mathbb{Z}_n)$$

$$f_j \mathbf{v}_{\boldsymbol{\alpha}} = z^{-\delta_{j,0}} \mathbf{v}_{\boldsymbol{\alpha} + \mathbf{d}_j - \mathbf{d}_{j+1}},$$

$$k_j \mathbf{v}_{\boldsymbol{\alpha}} = p^{2(\alpha_{j+1} - \alpha_j)} \mathbf{v}_{\boldsymbol{\alpha}},$$

Write it as $(\rho_{k,z}, \mathbf{V}_{k,z})$ (z: spectral parameter)

$\cdots$ fundamental representation.

Characterization of the matrix product constructed $R(z)$ by U_p

Proposition (Bazhanov-Sergeev `06, up to convention)

$$R(z) = \bigoplus_{0 \le l,m \le n} R_{l,m}(z), \qquad R_{l,m}(x/y) : \mathbf{V}_{l,x} \otimes \mathbf{V}_{m,y} \to \mathbf{V}_{l,x} \otimes \mathbf{V}_{m,y},$$

$R_{l,m}(z)$ is the quantum R matrix for the fundamental representation of $U_p(A^{(1)}_{n-1})$ with $p = \pm iq$.

* [BS] derives the 3D L operator by a `quantum geometry' argument.
* It is equivalent to the approach based on $A_q(sl_3)$ like $A_q(sp_4)$ discussed here.
* The intertwining relation for $A_q(sl_3)$ is presented as Quantized YBE (= YBE up to conjugation).
* Its intertwiner, called 3D R, satisfies the Tetrahedron equation.

For later use, introduce the *checked* R-matrix

$$\check{R}(x/y) = R(x/y)P : \bigoplus_{0 \le l,m \le n} \left(\mathbf{V}_{l,x} \otimes \mathbf{V}_{m,y} \to \mathbf{V}_{m,y} \otimes \mathbf{V}_{l,x} \right) \qquad (P(u \otimes v) = v \otimes u)$$

$$\Delta_{(m,y),(l,x)} \check{R}(x/y) = \check{R}(x/y) \Delta_{(l,x),(m,y)}, \qquad \Delta_{(l,x),(m,y)} := (\rho_{l,x} \otimes \rho_{m,y}) \circ \Delta$$

q-Onsager coideal B_q

Set q = ±ip in the rest. Let B_q be the subalgebra of U_p generated by

$$b_j = e_j - q^2 k_j f_j + \frac{q}{1+q^2} k_j \quad (j \in \mathbb{Z}_n)$$

$$\Delta b_j = k_j \otimes b_j + (e_j - q^2 f_j) \otimes 1 \quad \Longrightarrow \quad \Delta B_q \subset U_p \otimes B_q$$

B_q is a coideal subalgebra

* $e_j + s_j k_j f_j + t_j k_j$ for any coefficients s_j, t_j generates a coideal subalgebra.

* The above B_q is the special case that becomes isomorphic to the type A q-Onsager algebra in [Baseilhac-Belliard `09].

Proposition

The matrix product constructed K(z) is characterized as the intertwiner of B_q.

$$K(z) = \bigoplus_{0 \le k \le n} K_k(z), \qquad K_k(z) : \mathbf{V}_{k,z} \to \mathbf{V}_{n-k,z^{-1}}.$$

$$\rho_{n-k,z^{-1}}(b) K_k(z) = K_k(z) \rho_{k,z}(b) \qquad (\forall b \in B_q)$$

This is due to the quadratic relation of the matrix product operators for K:

$$G_{\alpha_1}^{\beta_1+1} G_{\alpha_2}^{\beta_2-1} + p^{2(\beta_2-\beta_1+1)} G_{\alpha_1}^{\beta_1-1} G_{\alpha_2}^{\beta_2+1} \pm \frac{\mathrm{i}p^{2(\beta_2-\beta_1)}}{1-p^2} G_{\alpha_1}^{\beta_1} G_{\alpha_2}^{\beta_2}$$
$$= G_{\alpha_1-1}^{\beta_1} G_{\alpha_2+1}^{\beta_2} + p^{2(\beta_2-\beta_1-1)} G_{\alpha_1+1}^{\beta_1} G_{\alpha_2-1}^{\beta_2} \pm \frac{\mathrm{i}p^{2(\alpha_2-\alpha_1)}}{1-p^2} G_{\alpha_1}^{\beta_1} G_{\alpha_2}^{\beta_2}.$$

$\mathbf{V}_{l,x} \otimes \mathbf{V}_{m,y}$ is irreducible as a B_q module for generic x,y.

Therefore the following two maps $\mathbf{V}_{l,x} \otimes \mathbf{V}_{m,y} \to \mathbf{V}_{n-l,x^{-1}} \otimes \mathbf{V}_{n-m,y^{-1}}$ commuting with ΔB_q must coincide (with proper normalizations).

$$
\begin{tikzcd}
\mathbf{V}_x \otimes \mathbf{V}_{y^{-1}} \arrow[r, "\check{R}(xy)"] & \mathbf{V}_{y^{-1}} \otimes \mathbf{V}_x \arrow[r, "1 \otimes K(x)"] & \mathbf{V}_{y^{-1}} \otimes \mathbf{V}_{x^{-1}} \arrow[d, "\check{R}(x/y)"] \\
\mathbf{V}_x \otimes \mathbf{V}_y \arrow[u, "1 \otimes K(y)"] \arrow[d, "\check{R}(x/y)"'] & & \mathbf{V}_{x^{-1}} \otimes \mathbf{V}_{y^{-1}} \\
\mathbf{V}_y \otimes \mathbf{V}_x \arrow[r, "1 \otimes K(x)"] & \mathbf{V}_y \otimes \mathbf{V}_{x^{-1}} \arrow[r, "\check{R}(xy)"] & \mathbf{V}_{x^{-1}} \otimes \mathbf{V}_y \arrow[u, "1 \otimes K(y)"']
\end{tikzcd}
$$

(l,m indices suppressed)

This implies the RE:

$$\check{R}(x/y)(1 \otimes K(x))\check{R}(xy)(1 \otimes K(y)) = (1 \otimes K(y))\check{R}(xy)(1 \otimes K(x))\check{R}(x/y)$$

・・・ Implements the approach to K(**z**) by coideal subalgebras by [Delius-MacKay, `01] etc.

Summary so far:

* Intertwining relation of $A_q(sp_4)$ is identified/reformulated as a Quantized RE.
* (Quantized RE)^n leads to a family of matrix product solutions to ordinary RE.
* They are associated with fundamental representations of $U_p(A^{(1)}_{n-1})$.
* The K-matrix is characterized as the intertwiner of the q-Onsager coideal.

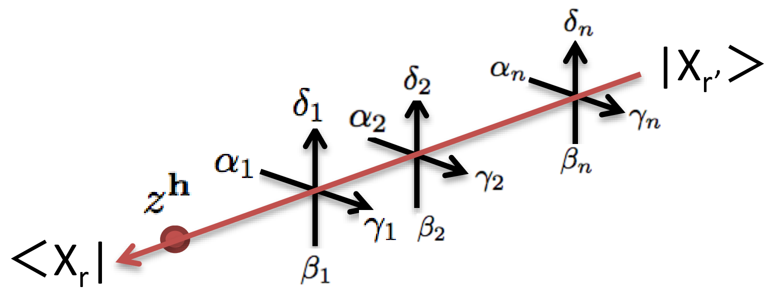
Generalizations (brief ending remarks)

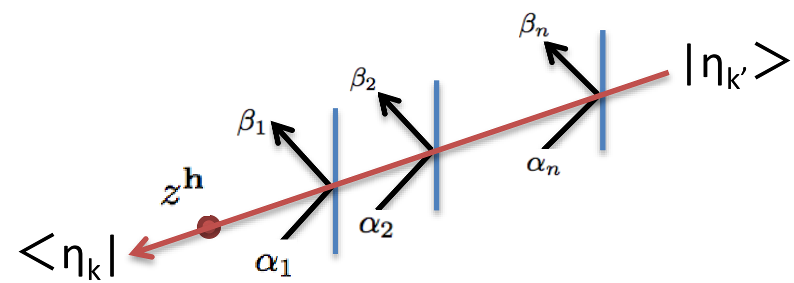
1. $U_p(\hat{g})$ with some non-exceptional $\hat{g}$
2. Case of symmetric tensor representation of $U_p(A^{(1)}_{n-1})$
3. $A_q(G_2)$

1. $U_p(\hat{g})$ with some non-exceptional $\hat{g}$

There are boundary vectors $|\chi_r\rangle$, $|\eta_k\rangle$ (r,k=1,2) in the Fock spaces such that

$$R^{r,r'}(z) := \langle \chi_r | z^{\mathbf{h}} LL \cdots L | \chi_{r'} \rangle \qquad K^{k,k'}(z) := \langle \eta_k | z^{\mathbf{h}} GG \cdots G | \eta_{k'} \rangle$$





conjecturally satisfy the RE for the following cases:

$\hat{\mathfrak{g}}$	R matrix	K matrix
$D^{(2)}_{n+1}$	$R^{1,1}(z)$	$K^{1,1}(z)$, $K^{1,2}(z)$, $K^{2,1}(z)$, $K^{2,2}(z)$
$B^{(1)}_n$	$R^{2,1}(z)$	$K^{2,1}(z)$, $K^{2,2}(z)$
$D^{(1)}_n$	$R^{2,2}(z)$	$K^{2,2}(z)$

These R-matrices on $2^n \times 2^n$ dim. space are associated with the spin representations of $U_p(\hat{g})$.

2. Case of symmetric tensor representation of $U_p(A^{(1)}_{n-1})$

Degree s symmetric tensor rep.

$$W_{s,z} := \bigoplus_{\substack{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n \\ \alpha_1+\cdots+\alpha_n=s}} \mathbb{C}w_\alpha, \qquad W^*_{s,z} := \text{antipode dual representation of } W_{s,z},$$

$$K(z): W_{s,z} \to W^*_{s,z^{-1}}, \qquad K(z)w_\alpha = \sum_\beta K(z)^\beta_\alpha w_\beta,$$

$$K(z)^\beta_\alpha = (\text{scalar})\,\mathrm{Tr}\big(z^{-\mathbf{h}} G^{\beta_1}_{\alpha_1}\cdots G^{\beta_n}_{\alpha_n}\big), \qquad G^j_i \in \mathrm{End}(F_q)$$

∗ K(z) characterized by essentially the same coideal B_q satisfies RE.

∗ Matrix product operators are (terminating) q-hypergeometric series of q-boson.

$$G^j_i = {}_2\phi_1\left({q^{-j}, -q^{-j} \atop -q^{-i-j}}; q, q\mathbf{k}\right)(\mathbf{a}^-)^{i-j} \quad (i \ge j),$$

$$= (\mathbf{a}^+)^{j-i}\,{}_2\phi_1\left({q^{-i}, -q^{-i} \atop -q^{-i-j}}; q, q\mathbf{k}\right) \quad (i \le j),$$

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^m)|m-1\rangle, \quad \mathbf{k}|m\rangle = q^m|m\rangle.$$

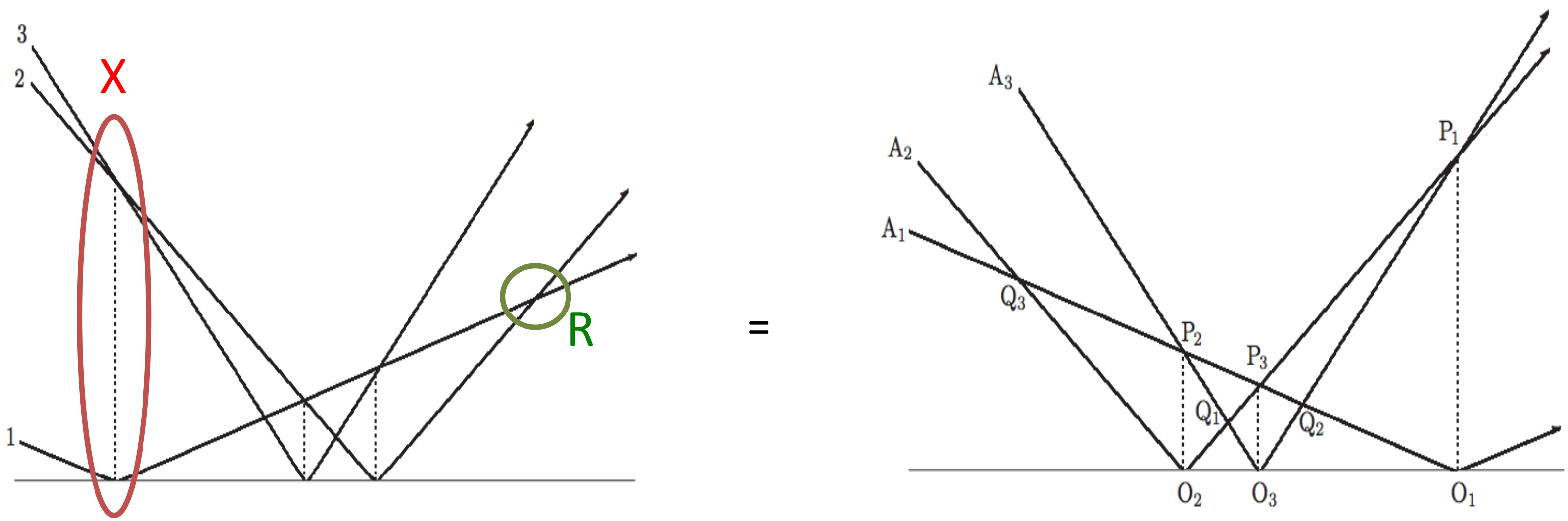
∗ The crystal limit q=0 reproduces the set-theoretical solution to the RE related to the box-ball system with reflecting end [K-O-Yamada, `05].

Open problem: Formulation by A_q and 3D picture.

3. $A_q(G_2)$

G_2 reflection equation (essentially due to [Cherednik `84], appears also in `root algebra')

$$R_{12}(x)X_{132}(xy)R_{23}(x^2y^3)X_{213}(xy^2)R_{31}(xy^3)X_{321}(y)$$
$$= X_{231}(y)R_{13}(xy^3)X_{123}(xy^2)R_{32}(x^2y^3)X_{312}(xy)R_{21}(x).$$



* Special 3-body scattering matching the geometry of Desargues-Pappus theorem.
* Intertwining relation of $A_q(G_2)$ = quantized G_2 RE (:= G_2 RE up to conjugation).
* Matrix product solutions have been constructed [K, arXiv:1804.04305] :

$$X(z) = \mathrm{Tr}(z^{\mathbf{h}} JJ \cdots J) \quad \text{or} \quad X(z) = \langle \xi | z^{\mathbf{h}} JJ \cdots J | \xi \rangle.$$

(This one is yet conjectural)

Open problem: Find a coideal-like object that characterizes X(z) as an intertwiner.

Schematic comparison of $A_q(sp_4)$ and $A_q(sl_3)$

$$A_q(sp_4): s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$$

$$\Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2) = (\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1) \circ \Phi$$

$$\text{3D K}: \ \mathcal{K} = \Phi \circ \sigma$$

$$\text{quantized RE}: \ \mathcal{K}(GLGL) = (LGLG)\mathcal{K}$$

$$\text{matrix product}: \ K(z) = \mathrm{Tr}(z^{\mathbf{h}} GG \cdots G)$$

$$A_q(sl_6): s_1 s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_3 = \text{reverse}$$

3D reflection equation (proposed by Isaev-Kulish '97)

$$\mathcal{R}_{456}\mathcal{R}_{489}\mathcal{K}_{3579}\mathcal{R}_{269}\mathcal{R}_{258}\mathcal{K}_{1678}\mathcal{K}_{1234} = \mathcal{K}_{1234}\mathcal{K}_{1678}\mathcal{R}_{258}\mathcal{R}_{269}\mathcal{K}_{3579}\mathcal{R}_{489}\mathcal{R}_{456}$$

$$A_q(sl_3): s_1 s_2 s_1 = s_2 s_1 s_2$$

$$\Psi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1) = (\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Psi$$

$$\text{3D R}: \ \mathcal{R} = \Psi \circ \sigma$$

$$\text{quantized YBE}: \ \mathcal{R}(LLL) = (LLL)\mathcal{R}$$

$$\text{matrix product}: \ R(z) = \mathrm{Tr}(z^{\mathbf{h}} LL \cdots L)$$

$$A_q(sl_4): s_1 s_2 s_3 s_1 s_2 s_1 = s_3 s_2 s_3 s_1 s_2 s_3$$

Tetrahedron equation (proposed by Zamolodchikov '80)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$