Toward 3D integrability from quantum group

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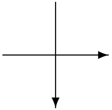
Integrability in 2D

2D R-matrix

$$R : V \otimes V \to V \otimes V \quad \text{i.e. } R \in \mathrm{End}(V^{\otimes 2})$$

$$V = \oplus_n \mathbb{C}|n\rangle = \begin{cases} \text{space of 1-particle states} \\ \text{space of local spin states} \end{cases}$$

$$R(|i\rangle \otimes |j\rangle) = \sum_{ab} R_{ij}^{ab} |a\rangle \otimes |b\rangle$$

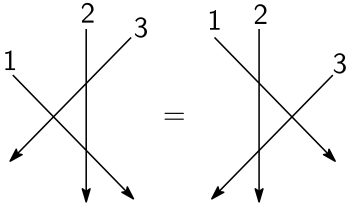
$R =$  $\cdots$ $\begin{cases} \text{2 particle scattering amplitude in (1+1)D} \\ \text{local Boltzmann weight of the vertex in 2D} \end{cases}$

Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ \in \mathrm{End}(V^{\otimes 3}),$$

where R_{ij} acts on the ith and jth components:

$$R_{12} : V \otimes V \otimes V, \quad R_{23} : V \otimes V \otimes V, \quad R_{13} : V \otimes V \otimes V$$



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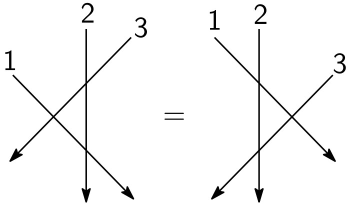

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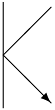
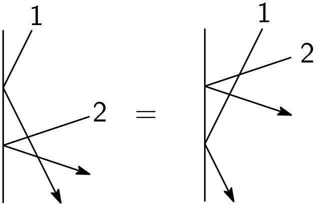
Key to quantum integrability in 2D

Integrability in the presence of boundary reflections

$K =$  $: V \to V$ (reflection amplitude matrix)

Reflection equation



$$R_{21}K_2R_{12}K_1 = K_1R_{21}K_2R_{12} \in \mathrm{End}(V^{\otimes 2})$$
$$(K_1 = K \otimes 1, \quad K_2 = 1 \otimes K)$$

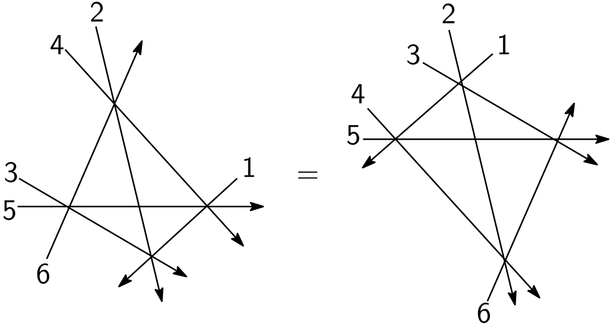
$\cdots$ Factorization condition at the boundary

What about 3D?

Tetrahedron equation (A.B. Zamolodchikov (1980))

$$R : V \otimes V \otimes V \to V \otimes V \otimes V \qquad \text{(3D } R\text{)}$$

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$$



$$R = \begin{cases} \text{3 string scattering amplitude in (2+1)D} \\ \text{local Boltzmann weight of the vertex in 3D} \end{cases}$$

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- One such approach is provided by $A_q(G)$ ($G =$ Lie group) called quantized algebra of functions on G.
- What is $A_q(G)$? It is another class of quantum group studied by Drinfeld (87), Vaksman-Soibelman (89,91), Reshetikhin-Takhatajan-Faddeev (90), Noumi-Yamada-Mimachi (92), Kashiwara (93), Geiss-Leclerc-Schröer (2011-) etc.

- **Simplest example:**

Recall $$\mathrm{SL}_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid [t_{ij}, t_{kl}] = 0,\ t_{11}t_{22} - t_{12}t_{21} = 1 \right\}.$$

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$A_q(\mathrm{SL}_2)$ is generated by $t_{11}, t_{12}, t_{21}, t_{22}$ with the relations

$$t_{11}t_{21} = qt_{21}t_{11},\ t_{12}t_{22} = qt_{22}t_{12},\ t_{11}t_{12} = qt_{12}t_{11},\ t_{21}t_{22} = qt_{22}t_{21},$$
$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1.$$

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$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1.$$

- **Fock representation** $\quad \boldsymbol{\pi_1 : A_q(\mathrm{SL}_2) \to \mathrm{End}(F_q)}$

$F_q = \oplus_{m \geq 0} \mathbb{C}|m\rangle$: q-oscillator Fock space

$$\pi_1 : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{a}^- & \mathbf{k} \\ -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}$$
$$\mathbf{k}|m\rangle = q^m|m\rangle,\ \mathbf{a}^+|m\rangle = |m+1\rangle,\ \mathbf{a}^-|m\rangle = (1 - q^{2m})|m-1\rangle.$$

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3. *The irreducible rep. corresponding to the* *reduced expression* $s_{i_1} \cdots s_{i_r} \in W(G)$ *is realized as the tensor product* $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$.

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Crucial Corollary

If $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}$ are 2 different reduced expressions, then

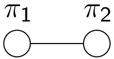
$$\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}.$$

$\Longrightarrow$ Exists the unique map Φ called intertwiner such that

$$(\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}) \circ \Phi = \Phi \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_r})$$

Example

$$A_q(\mathrm{SL}_3) = \langle t_{ij} \rangle_{i,j=1}^{3}$$



Fock representations

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}}$$

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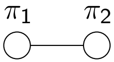

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$W(\mathrm{SL}_3) = \langle s_1, s_2 \rangle$. $\quad s_2 s_1 s_2 = s_1 s_2 s_1$ (Coxeter relation)

$\Longrightarrow \quad \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_1 \otimes \pi_2 \otimes \pi_1$ as representations on $(F_q)^{\otimes 3}$

Exists the intertwiner $\Phi : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}$ such that

$$(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi = \Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1).$$

Explicit form

$$R := \Phi P_{13}, \quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x,$$

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R_{ijk}^{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle.$$

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$$R^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c} \sum_{\lambda,\mu \geq 0, \lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \times \begin{bmatrix} i, j, c+\mu \\ \mu, \lambda, i-\mu, j-\lambda, c \end{bmatrix}.$$

$$(q)_m = \prod_{j=1}^{m}(1-q^j), \quad \begin{bmatrix} i_1, \dots, i_r \\ j_1, \dots, j_s \end{bmatrix} = \frac{\prod_{m=1}^{r}(q^2)_{i_m}}{\prod_{m=1}^{s}(q^2)_{j_m}}$$

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R satisfies the tetrahedron eq. $R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$.

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Essence of proof. Consider $A_q(\mathrm{SL}_4)$ and $W(\mathrm{SL}_4) = \langle s_1, s_2, s_3\rangle$.

$s_2s_1s_2 = s_1s_2s_1, \;\; s_3s_2s_3 = s_2s_3s_2, \;\; s_1s_2s_3s_1s_2s_1 = s_3s_2s_3s_1s_2s_3$ (longest el.)

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The intertwiner for the last one is constructed in 2 different ways as

$123\underline{121}$	Φ_{456}	$12\underline{31}21$	P_{34}
$1\underline{232}12$	Φ_{234}	$\underline{121}321$	Φ_{123}
$\underline{13}2\underline{31}2$	$P_{12}P_{45}$	$21\underline{232}1$	Φ_{345}
$3\underline{121}32$	Φ_{234}	$2\underline{13}2\underline{31}$	$P_{23}P_{56}$
$321\underline{232}$	Φ_{456}	$23\underline{121}3$	Φ_{345}
$32\underline{13}23$	P_{34}	$\underline{232}123$	Φ_{123}
323123		323123	

Equate the 2 sides, substitute $\Phi_{ijk} = R_{ijk}P_{ik}$ and cancel P_{ij}'s. □

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Recent developments

K-Sergeev [CMP in press], K-Okado [JPA (2012) & arXiv:1210.6430], K-Okado-Yamada [arXiv:1302.6298]

1. Type SO, Sp, F_4 cases
2. Explicit intertwiner K for the quartic Coxeter relation.
3. 3D analogue of reflection equation involving R and K.
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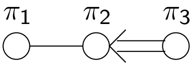
4. Reduction of the 3D R to infinitely many 2D R's.
5. Alternative characterization by Poincaré-Birkhoff-Witt basis of $U_q^+(g)$.

$A_q(\mathrm{Sp}_6) = \langle t_{jk} \rangle_{j,k=1}^{6}$: [Reshetikhin-Takhatajan-Faddeev 1990]



Fock representations $\pi_i(t_{jk})$ are given by

$$\pi_1 : \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} \\ 0 & 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 \\ 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} & 0 \\ 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \langle \mathbf{A}^{\pm}, \mathbf{K} \rangle = \langle \mathbf{a}^{\pm}, \mathbf{k} \rangle |_{q \to q^2}.$$

$W(\mathrm{Sp}_6) = \langle s_1, s_2, s_3 \rangle$

$$s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2.$$

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Write $\pi_{i_1,\ldots,i_r}$ to mean $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$ to save space.

Equivalence	Intertwiner	
$\pi_{121} \simeq \pi_{212}$,	$\Phi = RP_{13}$	(same as SL_3 case),
$\pi_{2323} \simeq \pi_{3232}$,	$\Psi = KP_{14}P_{23}$	(New object).

$$K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q), \quad R \in \mathrm{End}((F_q)^{\otimes 3}).$$

Matrix elements

$$K|i,j,k,l\rangle = \sum_{a,b,c,d} K^{abcd}_{ijkl}|a,b,c,d\rangle \quad (|i,j,k,l\rangle = |i\rangle\otimes|j\rangle\otimes|k\rangle\otimes|l\rangle)$$

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Theorem (Explicit form)

$$K^{abcd}_{ijkl} = \frac{(q^4)_i}{(q^4)_a}\sum_{\alpha,\beta,\gamma}\frac{(-1)^{\alpha+\gamma}}{(q^4)_{c-\beta}}q^{\phi_1}K^{i,j+k-\alpha-\beta-\gamma,0,k+l-\alpha-\beta-\gamma}_{a,b+c-\alpha-\beta-\gamma,0,c+d-\alpha-\beta-\gamma}$$

$$\times\begin{bmatrix} k, c-\beta, j+k-\alpha-\beta, k+l-\alpha-\beta \\ \alpha,\beta,\gamma, b-\alpha, d-\alpha, k-\alpha-\beta, c-\beta-\gamma\end{bmatrix},$$

$$\phi_1 = \alpha(\alpha+2c-2\beta-1)+(2\beta-c)(b+c+d)+\gamma(\gamma-1)-k(j+k+l).$$

$$K^{a,b,0,d}_{i,j,0,l} = \sum_{\lambda}(-1)^{m+\lambda}\frac{(q^4)_{a+\lambda}}{(q^4)_a}q^{\phi_2}\begin{bmatrix} j,l \\ \lambda, l-\lambda, b-\lambda, j-b+\lambda\end{bmatrix},$$

$$\phi_2 = (a+i+1)(b+l-2\lambda)+b-l.$$

Theorem (JPA **45** (2012) 465206)

R and K yield the first nontrivial solution to the **3D reflection equation** *proposed by Isaev-Kulish in 1997:*

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

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  R : Scattering amplitude of 3 strings.

  K: Reflection amplitude.
- Diagrams (corresponding geometric objects)

  World sheet of reflecting string = open book (spine=reflection line)

  Reflecting boundary plane = desk surface

  K = crossing of the spines of 2 books

A natural geometric configuration for 3D reflection equation

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is the "3 open books placed on a desk".

Elementary geometry exercise: Draw the projection onto the desk surface.

A natural geometric configuration for 3D reflection equation

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is the "3 open books placed on a desk".

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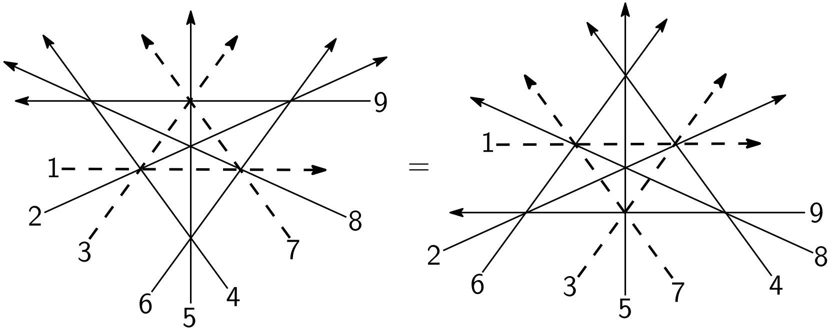
Answer: (dotted lines 1,3,7 correspond to the spines of the 3 books)



Remarks

1. Another solution to the 3D reflection eq. is available from SO case.
2. Higher rank cases are reducible to rank 2 cases.
3. Exists decent classical and combinatorial ($q = 0$) analogue. (Elements of R and K are actually polynomials(!) in q.)