New solutions to the tetrahedron equation associated with quantized six-vertex models

Atsuo Kuniba (Univ. Tokyo)

Joint work with Shuichiro Matsuike and Akihito Yoneyama

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1. Quantized (6V & YBE)
2. Solutions
3. Relation to quantized coordinate ring
4. Conjecture on RRRR=RRRR.

1. Quantized (6V & YBE)

YBE: $L_{12}L_{13}L_{23} = L_{23}L_{13}L_{12}$

Quantized YBE: $R_{456}L_{124}L_{135}L_{236} = L_{236}L_{135}L_{124}R_{456}$

⋯ YBE up to conjugation. Also called RLLL relation.

A version of the tetrahedron equation originally going back to A.B. Zamolodchikov ('80)

Appeared in several guises and studied from various viewpoints by
Maillet, Nijhoff, Korepanov, Bazhanov, Kashaev, Mangazeev, Sergeev,
Okado, Yoneyama, K,⋯

$L_{12} \longrightarrow L_{123}$

"auxiliary space"

$GV \longrightarrow$ quantized GV

$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$

$E_{ij} v_k = \delta_{jk} v_i$

$$L = \sum_{abij} E_{ai} \otimes E_{bj} \otimes \mathcal{L}^{ab}_{ij}$$

$$\in \mathrm{End}(V \otimes V) \otimes \begin{cases} \mathcal{W}_q & \cdots \ q\text{-Weyl alg} \\ \mathcal{O}_q & \cdots \ q\text{-oscillator alg} \end{cases}$$

$\boxed{W_q}$ q-Weyl alg. (generators $X^{\pm 1}, Z^{\pm 1}$)

$$XZ = qZX$$

Reps. π_X : $X|m\rangle = q^m|m\rangle$, $Z|m\rangle = |m+1\rangle$

"smart choice" π_Z : $X|m\rangle = |m-1\rangle$, $Z|m\rangle = q^m|m\rangle$

These are irreducible representations on

$$F = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}|m\rangle$$

π_X vs π_Z .... "coordinate" vs "momentum" reps.

$\boxed{\mathcal{O}_q}$ q - oscillator alg. (generators k, a^+, a^-)

$$k a^+ = q a^+ k \qquad a^+ a^- = 1 - k^2$$
$$k a^- = q^{-1} a^- k \qquad a^- a^+ = 1 - q^2 k^2$$

π_0 : $k|m\rangle = q^m |m\rangle$

$a^+|m\rangle = |m+1\rangle, \qquad a^- |m\rangle = (1-q^{2m})|m-1\rangle$

This is an irreducible representation on

$$F_+ = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle$$

<u>Remark</u> $\iota : \mathcal{O}_q \hookrightarrow W_q$

$$k \longmapsto X$$
$$a^+ \longmapsto Z$$
$$a^- \longmapsto Z^{-1}(1-X^2)$$

is an embedding.

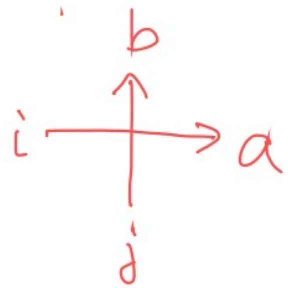
π_0 is a restriction of

$$\mathcal{O}_q \xhookrightarrow{\iota} W_q \xrightarrow{\pi_X} \mathrm{End}(F)$$

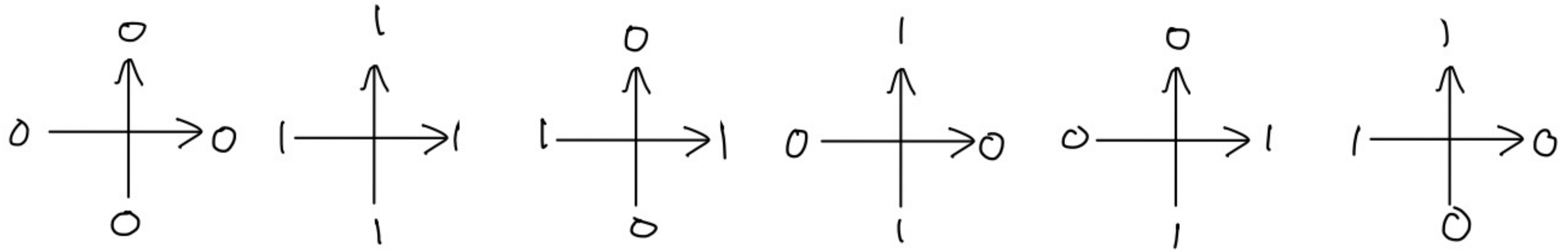
to $\mathrm{End}(F_+)$.

We consider three L's. L^X, L^Z, L^0

$$L^A = \sum_{abij} E_{ai} \otimes E_{bj} \otimes \pi_A(\mathcal{L}^{ab}_{ij})$$



$(A = X, Z, 0)$



L^X, L^Z	r	s	twX	$-qtX$	Z	$Z^{-1}(rs - tw^2X)$
L^0	1	1	μk	$-q\mu^{-1}k$	a^+	a^-

r, s, t, w, μ are parameters.

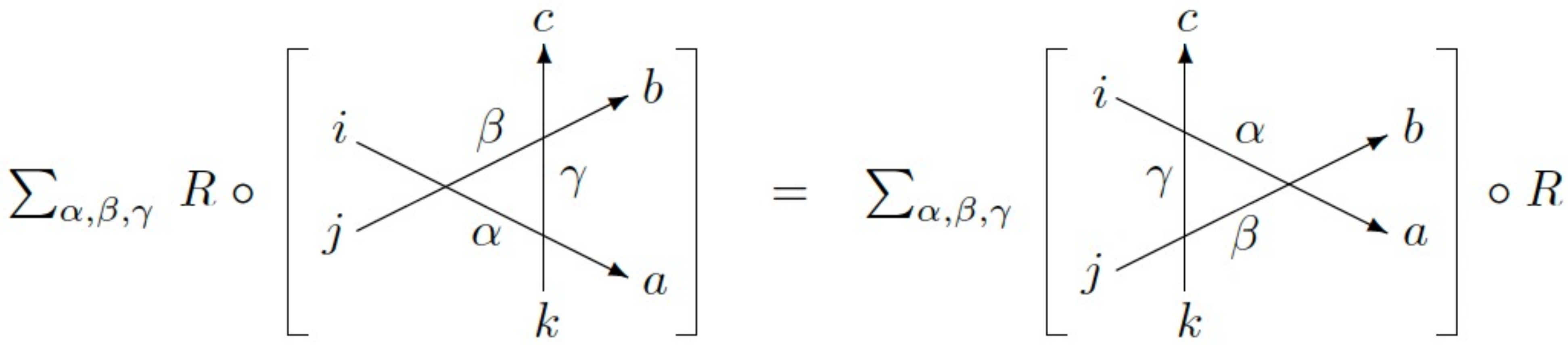
L^X, L^Z, L^0 … quantized 6V (free fermion type)

↑ Bazhanov - Sergeev ('06)

Quantized YBE = RLLL relation = a version of tetrahedron equation

$$R_{456}L_{236}L_{135}L_{124} = L_{124}L_{135}L_{236}R_{456}.$$

$$R \sum_{\alpha,\beta,\gamma} (\mathcal{L}_{ij}^{\alpha\beta} \otimes \mathcal{L}_{\alpha k}^{a\gamma} \otimes \mathcal{L}_{\beta\gamma}^{bc}) = \sum_{\alpha,\beta,\gamma} (\mathcal{L}_{\alpha\beta}^{ab} \otimes \mathcal{L}_{i\gamma}^{\alpha c} \otimes \mathcal{L}_{jk}^{\beta\gamma})\, R$$



We have L^X, L^Z, L^O. So we have

$$R L^Z L^Z L^Z = L^Z L^Z L^Z R \longrightarrow R = R^{ZZZ}$$

$$R L^Z L^O L^O = L^O L^O L^Z R \longrightarrow R = R^{OOZ} \quad \text{etc.}$$

In general

$$R L^C L^B L^A = L^A L^B L^C R \longrightarrow R = R^{ABC}.$$

$A,B,C = Z,X,O$

RLLL relation for R^{ZZZ}

$$Y_\alpha = Z^{-1}(r_\alpha s_\alpha - t_\alpha^2 w_\alpha X^2)$$

$$
\begin{aligned}
&R(1 \otimes X \otimes X) = (1 \otimes X \otimes X)R,\\
&R\bigl(r_2 t_1 X \otimes 1 \otimes Y_3 + t_3 Z \otimes Y_2 \otimes X\bigr) = r_1 t_2 \bigl(1 \otimes X \otimes Y_3\bigr)R,\\
&R\bigl(-q t_1 t_3 w_1 X \otimes Y_2 \otimes X + r_2 Y_1 \otimes 1 \otimes Y_3\bigr) = r_1 r_3 \bigl(1 \otimes Y_2 \otimes 1\bigr)R,\\
&r_1 t_2 R(1 \otimes X \otimes Z) = \bigl(r_2 t_1 X \otimes 1 \otimes Z + t_3 Y_1 \otimes Z \otimes X\bigr)R,\\
&R\bigl(q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Z \otimes Y_2 \otimes Z\bigr) = \bigl(q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Y_1 \otimes Z \otimes Y_3\bigr)R,\\
&R\bigl(t_1 w_1 X \otimes Y_2 \otimes Z + r_2 t_3 w_3 Y_1 \otimes 1 \otimes X\bigr) = r_3 t_2 w_2 \bigl(Y_1 \otimes X \otimes 1\bigr)R,\\
&R(X \otimes X \otimes 1) = (X \otimes X \otimes 1)R,\\
&s_3 t_2 R\bigl(Y_1 \otimes X \otimes 1\bigr) = \bigl(t_1 X \otimes Y_2 \otimes Z + s_2 t_3 Y_1 \otimes 1 \otimes X\bigr)R,\\
&s_1 s_3 R\bigl(1 \otimes Y_2 \otimes 1\bigr) = \bigl(-q t_1 t_3 w_3 X \otimes Y_2 \otimes X + s_2 Y_1 \otimes 1 \otimes Y_3\bigr)R,\\
&r_1 r_3 R(1 \otimes Z \otimes 1) = (-q t_1 t_3 w_1 X \otimes Z \otimes X + r_2 Z \otimes 1 \otimes Z)R,\\
&r_3 t_2 w_2 R(Z \otimes X \otimes 1) = \bigl(t_1 w_1 X \otimes Z \otimes Y_3 + r_2 t_3 w_3 Z \otimes 1 \otimes X\bigr)R,\\
&R(X \otimes X \otimes 1) = (X \otimes X \otimes 1)R,\\
&R(t_1 X \otimes Z \otimes Y_3 + s_2 t_3 Z \otimes 1 \otimes X) = s_3 t_2 (Z \otimes X \otimes 1)R,\\
&R\bigl(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Y_1 \otimes Z \otimes Y_3\bigr) = \bigl(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Z \otimes Y_2 \otimes Z\bigr)R,\\
&R\bigl(s_1 t_2 w_2 1 \otimes X \otimes Y_3 = \bigl(s_2 t_1 w_1 X \otimes 1 \otimes Y_3 + t_3 w_3 Z \otimes Y_2 \otimes X\bigr)R,\\
&R(-q t_1 t_3 w_3 X \otimes Z \otimes X + s_2 Z \otimes 1 \otimes Z) = s_1 s_3 (1 \otimes Z \otimes 1)R,\\
&R\bigl(t_3 w_3 Y_1 \otimes Z \otimes X + s_2 t_1 w_1 X \otimes 1 \otimes Z\bigr) = s_1 t_2 w_2 (1 \otimes X \otimes Z)R,\\
&R(1 \otimes X \otimes X) = (1 \otimes X \otimes X)R.
\end{aligned}
$$

Main Result

(1) Obtained R^{ABC} explicitly for

ABC	$\sharp(Z)$	feature	locally finiteness
ZZZ	3	factorized	no
OZZ	2	${}_2\phi_1$	no
ZZO		${}_2\phi_1$	no
ZOZ		${}_3\phi_2$-like	no
OOZ	1	factorized	yes
ZOO		factorized	yes
OZO		factorized	no
OOO	0	${}_2\phi_1$	yes
XXZ	2	factorized	no
ZXX		factorized	no
XZX		factorized	no

New except for OOO.

(2) Relation to
quantized coordinate ring $Aq(sl_n)$

(3) Conjecture on tetrahedron equation
of type RRRR = RRRR

2. Solutions

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} R^{a,b,c}_{i,j,k} |a\rangle \otimes |b\rangle \otimes |c\rangle$$ ⋯ We call it 3d R

RLLL relation is translated into linear recursion relations for $R^{a,b,c}_{i,j,k}$

There always exists a unique solution up to normalization
in each sector specified by an appropriate parity condition.

We say that R is **locally finite** if $R^{a,b,c}_{i,j,k} = 0$
for all but finitely many (a,b,c) for any given (i,j,k).

$$(z;q)_m = \frac{(z;q)_\infty}{(zq^m;q)_\infty}, \quad (z;q)_\infty = \prod_{n\geq 0}(1-zq^n). \quad {}_2\phi_1\left(\begin{matrix}\alpha,\beta \\ \gamma\end{matrix};q,z\right) = \sum_{n\geq 0}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n$$

R^{ZZZ}

$$R^{zzz} \in \mathrm{End}(F \otimes F \otimes F)$$

(r_1, s_1, t_1, w_1) (r_2, s_2, t_2, w_2) (r_3, s_3, t_3, w_3)

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{t_1 t_3 w_1}\right)^{\frac{d_1}{2}} \left(\frac{s_2}{t_1 t_3 w_3}\right)^{\frac{d_2}{2}} \left(\frac{t_2}{s_1 t_3}\right)^{\frac{d_3}{2}} \left(\frac{t_2 w_2}{s_3 t_1 w_1}\right)^{\frac{d_4}{2}}$$

$$\times\, q^{\varphi} \frac{\Phi_{d_2}\left(\frac{s_1 s_3}{s_2}\right) \Phi_{d_3}\left(\frac{r_3 w_2}{s_3 w_1}\right) \Phi_{d_4}\left(\frac{r_1 w_3}{s_1 w_2}\right)}{\Phi_{-d_1}\left(\frac{q^2 r_1 r_3}{r_2}\right) \Phi_{d_3+d_4}\left(\frac{r_1 r_3 w_3}{s_1 s_3 w_1}\right)},$$

$a, b, c, i, j, k \in \mathbb{Z}$

$$\varphi = \frac{1}{4}((d_1 - d_2)(d_1 + d_2 + d_3 + d_4) + d_3 d_4) - d_1,$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} a + c - j \\ b - i - k \end{pmatrix}, \quad \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -a - b + c + i + j - k \\ a - b - c - i + j + k \end{pmatrix}$$

$$\Phi_m(z) = \frac{1}{(zq^m; q^2)_\infty} \quad (m \in \mathbb{Z}),$$

Factorized
(thanks to smart choice ?)
Not locally finite

R^{OZZ}

$$R^{OZZ} \in \mathrm{End}(F_+ \otimes F \otimes F)$$

μ $\quad (r_2, s_2, t_2, w_2) \quad (r_3, s_3, t_3, w_3)$

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{r_3}\right)^a \left(\frac{s_3}{s_2}\right)^i \left(\frac{t_2 w_2}{\mu s_2}\right)^{-b+j} \left(-\frac{\mu t_3}{r_3}\right)^{-c+k} \frac{(z;q^2)_a}{(q^2;q^2)_a} q^{(a-b+j-1)c-(i-b+j-1)k-aj+bi}$$

$$\times\, {}_2\phi_1\left(\begin{matrix} q^{-2i}, z^{-1}q^2 \\ z^{-1}q^{-2a+2} \end{matrix}; q^2, yq^{2i+2j-2a-2b}\right).$$

$a, i \in \mathbb{Z}_{\geq 0}, \quad b, c, j, k \in \mathbb{Z}$

$$y = \frac{r_3 w_3}{\mu^2 s_3}, \qquad z = xq^{2k-2c+2}$$

q-hypergeometric, instead of factorization.
Not locally finite.

R^{OOZ}

$R^{OOZ} \in \mathrm{End}(F_+ \otimes F_+ \otimes F)$

$\mu_1 \quad \mu_2 \quad (r_3, s_3, t_3, w_3)$

$\exists$ unique solution iff $\frac{\mu_1}{\mu_2} =: q^d \quad d \in \mathbb{Z}$.

$$R^{a,b,c}_{i,j,k} = s_3^i (\mu_2 t_3)^{-a} \left(\frac{\mu_2 s_3}{t_3 w_3}\right)^j \left(\frac{t_3^2 w_3}{r_3 s_3}\right)^e q^{cj-bk} \frac{(q^{2+2e-2j};q^2)_j (q^{2a+2};q^2)_{i-a}}{(q^2;q^2)_f (q^{2a-2e};q^2)_{e-a}}$$

$$e = \frac{1}{2}(a-c+j+k+d), \quad f = \frac{1}{2}(b+c+i-k-d). \qquad a,b,i,j \in \mathbb{Z}_{\geq 0}, \quad c,k \in \mathbb{Z}$$

Factorized. Locally finite.

R^{000}

$$R^{000} \in \mathrm{End}(F_+ \otimes F_+ \otimes F_+)$$

$$\mu_1 \qquad \mu_2 \qquad \mu_3$$

$$R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i\left(-\frac{\mu_1}{\mu_3}\right)^b\left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}$$

$$\times \frac{(q^2;q^2)_{a+b}}{(q^2;q^2)_a(q^2;q^2)_b}\,{}_2\phi_1\left(\begin{matrix} q^{-2b}, q^{-2i} \\ q^{-2a-2b}\end{matrix};q^2, q^{-2c}\right)$$

$a, b, c, i, j, k \in \mathbb{Z}_{\geq 0}$

Locally finite.
For this 3D R, representation theoretical origin is known.

3. Relation to quantized coordinate ring

$A_q(sl_3)$ generated by t_{ij} $(1 \le i, j \le 3)$ with the relations

$$[t_{ik}, t_{jl}] = \begin{cases} 0 & (i<j, k>l), \\ (q-q^{-1})t_{jk}t_{il} & (i<j, k<l), \end{cases}$$
$$t_{ik}t_{jk} = qt_{jk}t_{ik} \ (i<j), \quad t_{ki}t_{kj} = qt_{kj}t_{ki} \ (i<j),$$
$$\sum_{\sigma \in \mathfrak{S}_3} (-q)^{l(\sigma)} t_{1\sigma_1} t_{2\sigma_2} t_{3\sigma_3} = 1,$$

algebra homomorphisms to the q-Weyl algebra

$$\rho_1 : A_q(sl_3) \to \mathcal{W}_q \qquad\qquad \rho_2 : A_q(sl_3) \to \mathcal{W}_q$$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \mapsto \begin{pmatrix} Z^{-1}(u_1 - g_1h_1X^2) & g_1X & 0 \\ -qh_1X & Z & 0 \\ 0 & 0 & u_1^{-1} \end{pmatrix} \qquad \begin{pmatrix} u_2^{-1} & 0 & 0 \\ 0 & Z^{-1}(u_2 - g_2h_2X^2) & g_2X \\ 0 & -qh_2X & Z \end{pmatrix}$$

u_i, g_i, h_i are parameters

$(u_i, g_i, h_i) = (1, \mu_i, \mu_i^{-1})$ q-osc.

$$\begin{tikzcd}
 & \mathcal{O}_q \arrow[r, "\pi_X"] & \mathrm{End}(F_+) \cdots \rho_{O,i} \\
A_q(sl_3) \arrow[ur, "\rho_i"] \arrow[dr] & & \\
 & W_q \arrow[r, "\pi_Z"', "\text{smart choice}"'] & \mathrm{End}(F) \cdots \rho_{Z,i}
\end{tikzcd}$$

(i=1,2)

(u_i, g_i, h_i) generic

q-Weyl

smart choice

Soibelman ('91) : $\rho_{0,1} \otimes \rho_{0,2} \otimes \rho_{0,1} \simeq \rho_{0,2} \otimes \rho_{0,1} \otimes \rho_{0,2}$

Kapranov - Voevodsky ('94) : $\Phi\,(\rho_{0,1} \otimes \rho_{0,2} \otimes \rho_{0,1}) = (\rho_{0,2} \otimes \rho_{0,1} \otimes \rho_{0,2})\,\Phi$

Bazhanov - Sergeev ('06) : $R^{000} L^0 L^0 L^0 = L^0 L^0 L^0 R^{000}$

+ Mangazeev ('10)

K - Okado ('12) : $R^{000} = \Phi \circ \sigma \quad \left(\sigma(|i\rangle \otimes |j\rangle \otimes |k\rangle) = |k\rangle \otimes |j\rangle \otimes |i\rangle\right)$

Prop.

$$\tilde{R}\,\big(\overset{(u_1,g_1,h_1)}{\rho_{z,1}} \otimes \overset{(u_2,g_2,h_2)}{\rho_{z,2}} \otimes \rho_{z,1}\big) = \big(\rho_{z,2} \otimes \rho_{z,1} \otimes \rho_{z,2}\big)\,\tilde{R}$$

$$\Longleftrightarrow \quad R^{zzz} L^z L^z L^z = \underset{(r_1,s_1,t_1,w_1)}{L^z}\ \underset{(r_2,s_2,t_2,w_2)}{L^z}\ \underset{(r_3,s_3,t_3,w_3)}{L^z}\ R^{zzz}$$

with $R^{zzz} = \tilde{R}\circ\sigma$ provided that

$$u_1 = u_2 (=: u) \qquad g_1 h_1 = g_2 h_2 (=: p)$$

$$\frac{r_1}{t_1} = \frac{r_2}{t_2}, \quad \frac{s_2}{t_2} = \frac{s_3}{t_3}, \quad \frac{r_2}{r_1 r_3} = u, \quad \frac{s_1 s_3}{s_2} = u^2, \quad \frac{t_1^2 w_1}{r_1 s_1} = \frac{t_2^2 w_2}{r_2 s_2} = \frac{t_3^2 w_3}{r_3 s_3} = \frac{p}{u}.$$

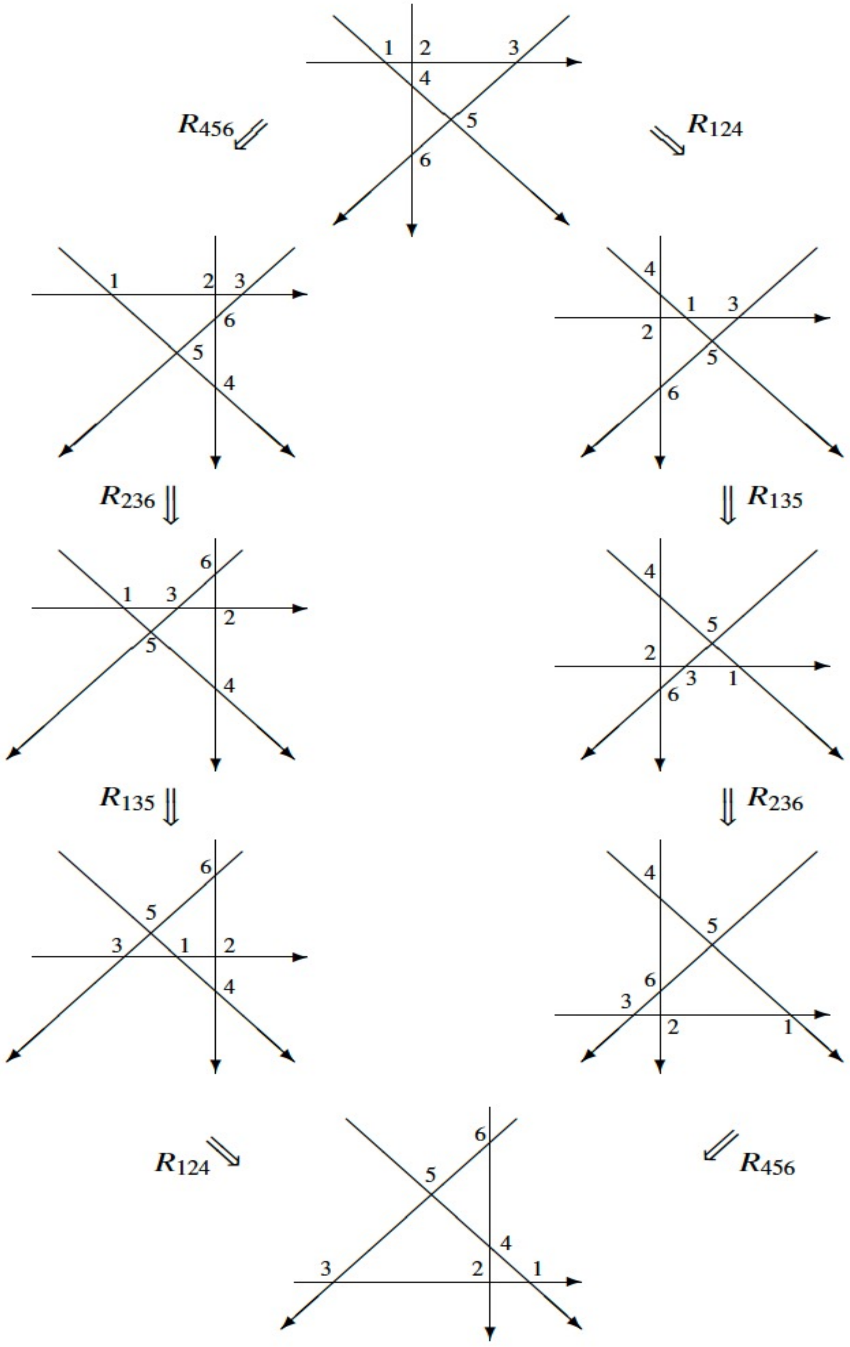
4. Conjecture on RRRR=RRRR

$$
\begin{aligned}
&R_{124}R_{135}R_{236}R_{456}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}\\
&=R_{124}R_{135}R_{236}L_{\beta\gamma4}L_{\alpha\gamma5}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\gamma\delta1}R_{456}\\
&=R_{124}R_{135}L_{\beta\gamma4}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}R_{236}R_{456}\\
&=R_{124}R_{135}L_{\beta\gamma4}L_{\beta\delta2}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\alpha\beta6}R_{236}R_{456}\\
&=R_{124}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}L_{\alpha\delta3}L_{\alpha\gamma5}L_{\alpha\beta6}R_{135}R_{236}R_{456}\\
&=L_{\gamma\delta1}L_{\beta\delta2}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},\\
&=L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},
\end{aligned}
$$

$$
\begin{aligned}
&R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\beta\delta2}L_{\gamma\delta1}\\
&=R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}L_{\alpha\delta3}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}\\
&=R_{456}R_{236}R_{135}L_{\alpha\beta6}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\beta\delta2}L_{\beta\gamma4}R_{124}\\
&=R_{456}R_{236}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}L_{\alpha\delta3}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\beta\gamma4}R_{135}R_{124}\\
&=R_{456}R_{236}L_{\gamma\delta1}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\alpha\gamma5}L_{\beta\gamma4}R_{135}R_{124}\\
&=R_{456}L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}R_{236}R_{135}R_{124}\\
&=L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{456}R_{236}R_{135}R_{124}.
\end{aligned}
$$

$(R_{124}R_{135}R_{236}R_{456})^{-1}R_{456}R_{236}R_{135}R_{124}$ commutes with $L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}$.

$\longrightarrow$ $R_{456}R_{236}R_{135}R_{124}=R_{124}R_{135}R_{236}R_{456}$ if irreducible.



In our case, a similar procedure starting from $L^{F}_{\alpha\beta 6}L^{E}_{\alpha\gamma 5}L^{D}_{\beta\gamma 4}L^{C}_{\alpha\delta 3}L^{B}_{\beta\delta 2}L^{A}_{\gamma\delta 1}$ suggests

$$R^{DEF}_{456}R^{BCF}_{236}R^{ACE}_{135}R^{ABD}_{124} = R^{ABD}_{124}R^{ACE}_{135}R^{BCF}_{236}R^{DEF}_{456}$$

where, A, B, C, D, E, F are Z or O or X.

For A=⋯=F=O, the irreducibility is known from the representation theory of A_q. Therefore

$$R^{OOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOO}_{124} = R^{OOO}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{OOO}_{456}$$

holds. (Kapranov-Voevodsky '94)

Other cases are yet elusive, entailing also the convergence issue for the composition of the locally non-finite 3D R's.

$$\sum_{u,v,w,x,y,z} R^{d,e,f}_{x,y,z}R^{b,c,z}_{v,w,n}R^{a,w,y}_{u,k,m}R^{u,v,x}_{i,j,l} = \sum_{u,v,w,x,y,z} R^{a,b,d}_{u,v,x}R^{u,c,e}_{i,w,y}R^{v,w,f}_{j,k,z}R^{x,y,z}_{l,m,n}$$

ABC	$\sharp$(Z)	feature	locally finiteness
ZZZ	3	factorized	no
OZZ	2	${}_2\phi_1$	no
ZZO		${}_2\phi_1$	no
ZOZ		${}_3\phi_2$-like	no
OOZ	1	factorized	yes
ZOO		factorized	yes
OZO		factorized	no
OOO	0	${}_2\phi_1$	yes
XXZ	2	factorized	no
ZXX		factorized	no
XZX		factorized	no

$$\forall a,b,c,d,e,f,i,j,k,l,m,n$$

$$\sum_{u,v,w,x,y,z} R^{d,e,f}_{x,y,z} R^{b,c,z}_{v,w,n} R^{a,w,y}_{u,k,m} R^{u,v,x}_{i,j,l}$$
$$= \sum_{u,v,w,x,y,z} R^{a,b,d}_{u,v,x} R^{u,c,e}_{i,w,y} R^{v,w,f}_{j,k,z} R^{x,y,z}_{l,m,n}$$

"*immensely more complicated*" (R.J. Baxter '83)

We have listed up the cases (without X) in which the above sums with prescribed a,b,c,d,e,f, I,j,k,l,m,n become finite sums due to locally-finiteness.

They are given in the next page.

$$
\begin{aligned}
R^{OOO}_{456} R^{OOO}_{236} R^{ZOO}_{135} R^{ZOO}_{124} &= R^{ZOO}_{124} R^{ZOO}_{135} R^{OOO}_{236} R^{OOO}_{456},\\
R^{ZOO}_{456} R^{OOO}_{236} R^{OOO}_{135} R^{OOZ}_{124} &= R^{OOZ}_{124} R^{OOO}_{135} R^{OOO}_{236} R^{ZOO}_{456},\\
R^{OOZ}_{456} R^{OOZ}_{236} R^{OOO}_{135} R^{OOO}_{124} &= R^{OOO}_{124} R^{OOO}_{135} R^{OOZ}_{236} R^{OOZ}_{456},\\
R^{OOZ}_{456} R^{OOZ}_{236} R^{ZOO}_{135} R^{ZOO}_{124} &= R^{ZOO}_{124} R^{ZOO}_{135} R^{OOZ}_{236} R^{OOZ}_{456}.
\end{aligned}
$$

$$
\begin{aligned}
R^{OOO}_{456} R^{ZOO}_{236} R^{OOO}_{135} R^{OZO}_{124} &= R^{OZO}_{124} R^{OOO}_{135} R^{ZOO}_{236} R^{OOO}_{456},\\
R^{OZO}_{456} R^{OOO}_{236} R^{OOZ}_{135} R^{OOO}_{124} &= R^{OOO}_{124} R^{OOZ}_{135} R^{OOO}_{236} R^{OZO}_{456},\\
R^{OOO}_{456} R^{ZOO}_{236} R^{ZOO}_{135} R^{ZZO}_{124} &= R^{ZZO}_{124} R^{ZOO}_{135} R^{ZOO}_{236} R^{OOO}_{456},\\
R^{ZOO}_{456} R^{OOO}_{236} R^{ZOO}_{135} R^{ZOZ}_{124} &= R^{ZOZ}_{124} R^{ZOO}_{135} R^{OOO}_{236} R^{ZOO}_{456},\\
R^{ZOO}_{456} R^{ZOO}_{236} R^{OOO}_{135} R^{OZZ}_{124} &= R^{OZZ}_{124} R^{OOO}_{135} R^{ZOO}_{236} R^{ZOO}_{456},\\
R^{ZZO}_{456} R^{OOO}_{236} R^{OOZ}_{135} R^{OOZ}_{124} &= R^{OOZ}_{124} R^{OOZ}_{135} R^{OOO}_{236} R^{ZZO}_{456},\\
R^{ZOZ}_{456} R^{OOZ}_{236} R^{OOO}_{135} R^{OOZ}_{124} &= R^{OOZ}_{124} R^{OOO}_{135} R^{OOZ}_{236} R^{ZOZ}_{456},\\
R^{OZZ}_{456} R^{OOZ}_{236} R^{OOZ}_{135} R^{OOO}_{124} &= R^{OOO}_{124} R^{OOZ}_{135} R^{OOZ}_{236} R^{OZZ}_{456},\\
R^{ZOO}_{456} R^{ZOO}_{236} R^{ZOO}_{135} R^{ZZZ}_{124} &= R^{ZZZ}_{124} R^{ZOO}_{135} R^{ZOO}_{236} R^{ZOO}_{456},\\
R^{ZZZ}_{456} R^{OOZ}_{236} R^{OOZ}_{135} R^{OOZ}_{124} &= R^{OOZ}_{124} R^{OOZ}_{135} R^{OOZ}_{236} R^{ZZZ}_{456}.
\end{aligned}
$$

$$
\begin{aligned}
R^{OOO}_{456} R^{OZO}_{236} R^{OZO}_{135} R^{OOO}_{124} &= R^{OOO}_{124} R^{OZO}_{135} R^{OZO}_{236} R^{OOO}_{456},\\
R^{OOO}_{456} R^{OZO}_{236} R^{ZZO}_{135} R^{ZOO}_{124} &= R^{ZOO}_{124} R^{ZZO}_{135} R^{OZO}_{236} R^{OOO}_{456},\\
R^{OZO}_{456} R^{OOO}_{236} R^{ZOZ}_{135} R^{ZOO}_{124} &= R^{ZOO}_{124} R^{ZOZ}_{135} R^{OOO}_{236} R^{OZO}_{456},\\
R^{OOO}_{456} R^{ZZO}_{236} R^{OZO}_{135} R^{OZO}_{124} &= R^{OZO}_{124} R^{OZO}_{135} R^{ZZO}_{236} R^{OOO}_{456},\\
R^{OOZ}_{456} R^{ZOZ}_{236} R^{OOO}_{135} R^{OZO}_{124} &= R^{OZO}_{124} R^{OOO}_{135} R^{ZOZ}_{236} R^{OOZ}_{456},\\
R^{ZOO}_{456} R^{OZO}_{236} R^{OZO}_{135} R^{OOZ}_{124} &= R^{OOZ}_{124} R^{OZO}_{135} R^{OZO}_{236} R^{ZOO}_{456},\\
R^{OZO}_{456} R^{OZO}_{236} R^{OZZ}_{135} R^{OOO}_{124} &= R^{OOO}_{124} R^{OZZ}_{135} R^{OZO}_{236} R^{OZO}_{456},\\
R^{OOZ}_{456} R^{OZZ}_{236} R^{OZO}_{135} R^{OOO}_{124} &= R^{OOO}_{124} R^{OZO}_{135} R^{OZZ}_{236} R^{OOZ}_{456},\\
R^{OZO}_{456} R^{OZO}_{236} R^{ZZZ}_{135} R^{ZOO}_{124} &= R^{ZOO}_{124} R^{ZZZ}_{135} R^{OZO}_{236} R^{OZO}_{456},\\
R^{OOZ}_{456} R^{ZZZ}_{236} R^{OZO}_{135} R^{OZO}_{124} &= R^{OZO}_{124} R^{OZO}_{135} R^{ZZZ}_{236} R^{OOZ}_{456}.
\end{aligned}
$$

Conjecture.
They are all valid (based on computer check).

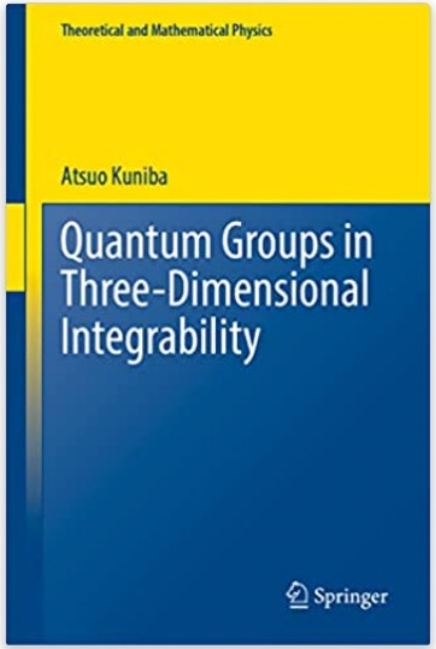
Claim [Noumi (野海), 28 September 2022]
They should be reducible to a duality of (his and Ito's?) multiple series generalization of the q-hypergeometric.

Outlook

Reduction: RLLL=LLLR, RRRR = RRRR → infinitely many solutions to YBE.

Smart choice: Factorized Boltzmann weights.
q^N =1. Beyond generalized Chiral Potts models ?

R^{OOO} = transition coefficient of PBW base of U^+_q. R^{ZZZ} = ?



BCG_2 type: Application to
quantized reflection equation K(LGLG)=(GLGL)K.
quantized G_2 reflection equation F(LJLJLJ)=(JLJLJL)F