New solutions to the tetrahedron equation associated with quantized six-vertex models

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1. Quantized (6V & YBE)
2. Solutions
3. Relation to quantized coordinate ring
4. Conjecture on RRRR=RRRR.

1. Quantized (6V & YBE)

YBE: $$L_{23}L_{13}L_{12} = L_{12}L_{13}L_{23}$$

Quantized YBE: $$R_{456}L_{236}L_{135}L_{124} = L_{124}L_{135}L_{236}R_{456}$$

$\cdots$ YBE up to conjugation. Also called RLLL relation.

A version of the tetrahedron equation originally going back to A.B. Zamolodchikov ('80).

Appeared in several guises and has been studied from various viewpoints by
Maillet, Nijhoff, Korepanov, Bazhanov, Kashaev, Mangazeev, Sergeev,
Okado, Yoneyama, K,$\cdots$.

$\exists$ intriguing application to stationary states in multispecies TASEP and TAZRP in 1D.

$$L_{12} \xrightarrow{\text{“quantization”}} L_{\underbrace{12}_{\mathbb{C}^2 \otimes \mathbb{C}^2}3}$$

“auxiliary space”

$\cdots$ 6V model whose Boltzmann weights are

$$\begin{cases} \mathcal{W}_q \text{ (q-Weyl algebra)-valued} \longrightarrow \left.\begin{matrix} L^X \\ L^Z \end{matrix}\right\} \text{Two representations of } \mathcal{W}_q \\[2ex] \mathcal{O}_q \text{ (q-Oscillator algebra)-valued} \longrightarrow L^O \end{cases}$$

We will consider L^X, L^Z, L^O.

$\mathcal{W}_q$ q-Weyl algebra (generators $X^{\pm 1},\ Z^{\pm 1}$)

$$XZ = qZX$$

$$\text{Representations} \begin{cases} \pi_X :\ X|m\rangle = q^m|m\rangle,\ \ Z|m\rangle = |m+1\rangle \\ \\ \pi_Z :\ X|m\rangle = |m-1\rangle,\ \ Z|m\rangle = q^m|m\rangle \end{cases}$$

These are irreducible representations on

$$F = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}|m\rangle$$

π_X vs π_Z $\cdots$ "coordinate" vs "momentum " representations of the q-canonical commutation relation.

$\mathcal{O}_q$ q-Oscillator algebra

(generators $\mathbf{k},\ \mathbf{a}^+, \mathbf{a}^-$)

$$\mathbf{k}\,\mathbf{a}^+ = q\mathbf{a}^+\mathbf{k}, \quad \mathbf{a}^+\mathbf{a}^- = 1 - \mathbf{k}^2$$

$$\mathbf{k}\,\mathbf{a}^- = q^{-1}\mathbf{a}^-\mathbf{k}, \quad \mathbf{a}^-\mathbf{a}^+ = 1 - q^2\mathbf{k}^2$$

$$\pi_O: \quad \mathbf{k}|m\rangle = q^m|m\rangle, \ \ \mathbf{a}^+|m\rangle = |m+1\rangle, \ \ \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle$$

π_O is an irreducible representation on

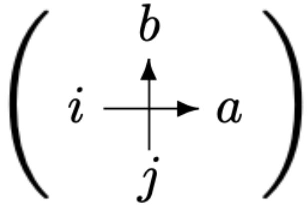
$$F_+ = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle$$

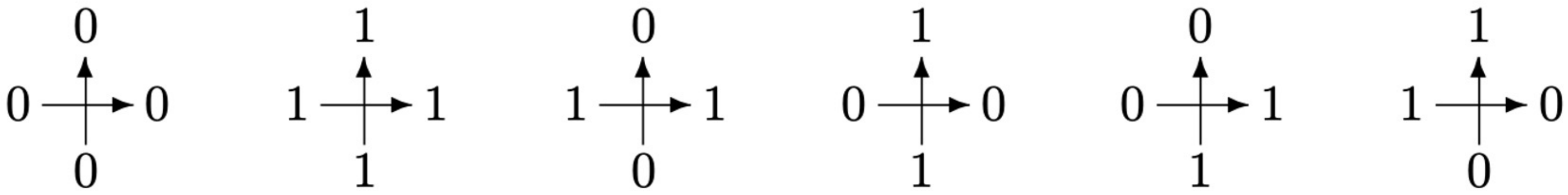
There is an embedding $\mathcal{O}_q \hookrightarrow \mathcal{W}_q$

$$\mathbf{k} \longmapsto X$$

$$\mathbf{a}^+ \longmapsto Z$$

$$\mathbf{a}^- \longmapsto Z^{-1}(1-X^2)$$

$L^A\,(A = X, Z, O)$ are defined by quantized 6V weights $\quad \pi_A$ 



$\left.\begin{array}{l} L^X \\ L^Z \end{array}\right\}$	r	s	twX	$-qtX$	Z	$Z^{-1}(rs - t^2wX^2)$
L^O	1	1	$\mu\mathbf{k}$	$-q\mu^{-1}\mathbf{k}$	$\mathbf{a}^+$	$\mathbf{a}^-$

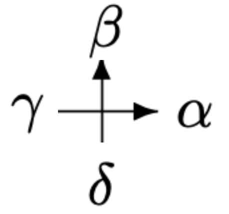
r, s, t, w, μ are parameters.

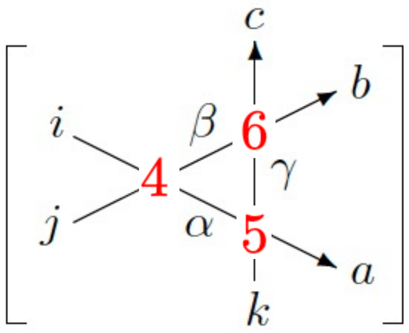
Relations of generators in $\mathcal{W}_q, \mathcal{O}_q$ may be viewed as quantizations of the "free-fermion condition" of 6V.

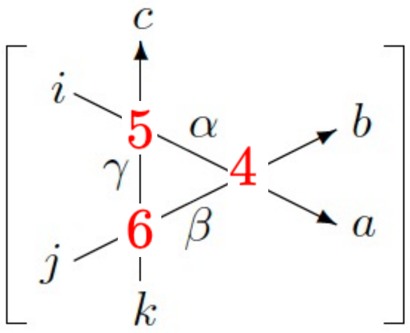
Quantized YBE = RLLL relation = a version of tetrahedron equation

$$R_{456}L_{236}L_{135}L_{124} = L_{124}L_{135}L_{236}R_{456}.$$

$$R\sum_{\alpha,\beta,\gamma}(\mathcal{L}^{\alpha\beta}_{ij}\otimes\mathcal{L}^{a\gamma}_{\alpha k}\otimes\mathcal{L}^{bc}_{\beta\gamma}) = \sum_{\alpha,\beta,\gamma}(\mathcal{L}^{ab}_{\alpha\beta}\otimes\mathcal{L}^{\alpha c}_{i\gamma}\otimes\mathcal{L}^{\beta\gamma}_{jk})\,R$$

$\mathcal{L}^{\alpha\beta}_{\gamma\delta} =$ 

$\sum_{\alpha,\beta,\gamma}$ $R_{456}\circ$ 

$=$ $\sum_{\alpha,\beta,\gamma}$ 

$\circ\, R_{456}$ ··· Linear equations on R.

R depends on the choice of the three kinds of L's as

$$RL^ZL^ZL^Z = L^ZL^ZL^ZR \longrightarrow R = R^{ZZZ}$$

$$RL^ZL^OL^O = L^OL^OL^ZR \longrightarrow R = R^{OOZ}$$

In general $$RL^CL^BL^A = L^AL^BL^CR \longrightarrow R = R^{ABC}$$ ··· We call it 3d R.

RLLL relation for q-Weyl algebra case

$$Y_\alpha = Z^{-1}(r_\alpha s_\alpha - t_\alpha^2 w_\alpha X^2)$$

$$R(1 \otimes X \otimes X) = (1 \otimes X \otimes X)R,$$
$$R\big(r_2 t_1 X \otimes 1 \otimes Y_3 + t_3 Z \otimes Y_2 \otimes X\big) = r_1 t_2 \big(1 \otimes X \otimes Y_3\big)R,$$
$$R\big(-q t_1 t_3 w_1 X \otimes Y_2 \otimes X + r_2 Y_1 \otimes 1 \otimes Y_3\big) = r_1 r_3 \big(1 \otimes Y_2 \otimes 1\big)R,$$
$$r_1 t_2 R(1 \otimes X \otimes Z) = \big(r_2 t_1 X \otimes 1 \otimes Z + t_3 Y_1 \otimes Z \otimes X\big)R,$$
$$R\big(q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Z \otimes Y_2 \otimes Z\big) = \big(q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Y_1 \otimes Z \otimes Y_3\big)R,$$
$$R\big(t_1 w_1 X \otimes Y_2 \otimes Z + r_2 t_3 w_3 Y_1 \otimes 1 \otimes X\big) = r_3 t_2 w_2 \big(Y_1 \otimes X \otimes 1\big)R,$$
$$R(X \otimes X \otimes 1) = (X \otimes X \otimes 1)R,$$
$$s_3 t_2 R\big(Y_1 \otimes X \otimes 1\big) = \big(t_1 X \otimes Y_2 \otimes Z + s_2 t_3 Y_1 \otimes 1 \otimes X\big)R,$$
$$s_1 s_3 R\big(1 \otimes Y_2 \otimes 1\big) = \big(-q t_1 t_3 w_3 X \otimes Y_2 \otimes X + s_2 Y_1 \otimes 1 \otimes Y_3\big)R,$$
$$r_1 r_3 R(1 \otimes Z \otimes 1) = (-q t_1 t_3 w_1 X \otimes Z \otimes X + r_2 Z \otimes 1 \otimes Z)R,$$
$$r_3 t_2 w_2 R(Z \otimes X \otimes 1) = \big(t_1 w_1 X \otimes Z \otimes Y_3 + r_2 t_3 w_3 Z \otimes 1 \otimes X\big)R,$$
$$R(X \otimes X \otimes 1) = (X \otimes X \otimes 1)R,$$
$$R(t_1 X \otimes Z \otimes Y_3 + s_2 t_3 Z \otimes 1 \otimes X) = s_3 t_2 (Z \otimes X \otimes 1)R,$$
$$R\big(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Y_1 \otimes Z \otimes Y_3\big) = \big(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Z \otimes Y_2 \otimes Z\big)R,$$
$$R\big(s_1 t_2 w_2 1 \otimes X \otimes Y_3 = \big(s_2 t_1 w_1 X \otimes 1 \otimes Y_3 + t_3 w_3 Z \otimes Y_2 \otimes X\big)R,$$
$$R(-q t_1 t_3 w_3 X \otimes Z \otimes X + s_2 Z \otimes 1 \otimes Z) = s_1 s_3 (1 \otimes Z \otimes 1)R,$$
$$R\big(t_3 w_3 Y_1 \otimes Z \otimes X + s_2 t_1 w_1 X \otimes 1 \otimes Z\big) = s_1 t_2 w_2 (1 \otimes X \otimes Z)R,$$
$$R(1 \otimes X \otimes X) = (1 \otimes X \otimes X)R.$$

Main Result

(1) There always exists a unique R up to normalization in each sector specified by an appropriate parity condition.

The solution R^{ABC} is explicitly obtained for

ABC	feature	locally finiteness
ZZZ	factorized	no
OZZ	${}_2\phi_1$	no
ZZO	${}_2\phi_1$	no
ZOZ	${}_3\phi_2$-like	no
OOZ	factorized	yes
ZOO	factorized	yes
OZO	factorized	no
OOO	${}_2\phi_1$	yes
XXZ	factorized	no
ZXX	factorized	no
XZX	factorized	no

New except for OOO.

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} R^{a,b,c}_{i,j,k} |a\rangle \otimes |b\rangle \otimes |c\rangle$$

We say that R is **locally finite** if $R^{a,b,c}_{i,j,k} = 0$ for all but finitely many (a,b,c) for any given (i,j,k).

(2) Relation to quantized coordinate ring Aq(sl_n)

(3) Conjecture on tetrahedron equation of type RRRR = RRRR

$$(z;q)_m = \frac{(z;q)_\infty}{(zq^m;q)_\infty}, \qquad (z;q)_\infty = \prod_{n\geq 0}(1-zq^n)$$

$${}_2\phi_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};q,z\right) = \sum_{n\geq 0}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n$$

2. Solutions

R^{ZZZ}

$$R^{ZZZ} \in \mathrm{End}(F \otimes F \otimes F)$$

(r_1, s_1, t_1, w_1) (r_2, s_2, t_2, w_2) (r_3, s_3, t_3, w_3)

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \left(\frac{r_2}{t_1 t_3 w_1}\right)^{\frac{d_1}{2}} \left(\frac{s_2}{t_1 t_3 w_3}\right)^{\frac{d_2}{2}} \left(\frac{t_2}{s_1 t_3}\right)^{\frac{d_3}{2}} \left(\frac{t_2 w_2}{s_3 t_1 w_1}\right)^{\frac{d_4}{2}} \\ &\quad \times q^{\varphi} \frac{\Phi_{d_2}\left(\frac{s_1 s_3}{s_2}\right) \Phi_{d_3}\left(\frac{r_3 w_2}{s_3 w_1}\right) \Phi_{d_4}\left(\frac{r_1 w_3}{s_1 w_2}\right)}{\Phi_{-d_1}\left(\frac{q^2 r_1 r_3}{r_2}\right) \Phi_{d_3+d_4}\left(\frac{r_1 r_3 w_3}{s_1 s_3 w_1}\right)}, \\ \varphi &= \frac{1}{4}((d_1 - d_2)(d_1 + d_2 + d_3 + d_4) + d_3 d_4) - d_1, \\ \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= \begin{pmatrix} a + c - j \\ b - i - k \end{pmatrix}, \quad \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -a - b + c + i + j - k \\ a - b - c - i + j + k \end{pmatrix} \\ \Phi_m(z) &= \frac{1}{(zq^m; q^2)_\infty} \quad (m \in \mathbb{Z}), \end{aligned}$$

$a, b, c, i, j, k \in \mathbb{Z}$

Factorized
Not locally finite

R^{OZZ}

$$R^{OZZ} \in \mathrm{End}(F_+ \otimes F \otimes F)$$

$\mu \quad (r_2, s_2, t_2, w_2) \quad (r_3, s_3, t_3, w_3)$

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{r_3}\right)^a \left(\frac{s_3}{s_2}\right)^i \left(\frac{t_2 w_2}{\mu s_2}\right)^{-b+j} \left(-\frac{\mu t_3}{r_3}\right)^{-c+k} \frac{(z;q^2)_a}{(q^2;q^2)_a} q^{(a-b+j-1)c-(i-b+j-1)k-aj+bi} ;$$
$$\times\, {}_2\phi_1 \left(\begin{matrix} q^{-2i}, z^{-1}q^2 \\ z^{-1}q^{-2a+2} \end{matrix} ; q^2, yq^{2i+2j-2a-2b} \right).$$

$$a, i \in \mathbb{Z}_{\geq 0}, \ b, c, j, k \in \mathbb{Z}$$

$$x = \frac{\mu^2 s_2}{r_2 w_2}, \qquad y = \frac{r_3 w_3}{\mu^2 s_3}, \qquad z = xq^{2k-2c+2}$$

q-hypergeometric, instead of factorization.
Not locally finite.

R^{OOZ}

$$R^{OOZ} \in \mathrm{End}(\underset{\underset{\mu_1}{\uparrow}}{F_+} \otimes \underset{\underset{\mu_2}{\uparrow}}{F_+} \otimes \underset{\underset{(r_3, s_3, t_3, w_3)}{\uparrow}}{F})$$

$\exists$ unique solution iff $\frac{\mu_1}{\mu_2} = q^d$ for some $d \in \mathbb{Z}$.

$$R^{a,b,c}_{i,j,k} = s_3^i (\mu_2 t_3)^{-a} \left(\frac{\mu_2 s_3}{t_3 w_3}\right)^j \left(\frac{t_3^2 w_3}{r_3 s_3}\right)^e q^{cj-bk} \frac{(q^{2+2e-2j};q^2)_j (q^{2a+2};q^2)_{i-a}}{(q^2;q^2)_f (q^{2a-2e};q^2)_{e-a}}$$

$$e = \frac{1}{2}(a - c + j + k + d), \quad f = \frac{1}{2}(b + c + i - k - d). \qquad a, b, i, j \in \mathbb{Z}_{\geq 0},\ c, k \in \mathbb{Z}$$

Factorized. Locally finite.

R^{OOO}

$$R^{OOO} \in \mathrm{End}(\underset{\mu_1}{F_+} \otimes \underset{\mu_2}{F_+} \otimes \underset{\mu_3}{F_+})$$

$$R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i\left(-\frac{\mu_1}{\mu_3}\right)^b\left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}$$

$$\times \frac{(q^2;q^2)_{a+b}}{(q^2;q^2)_a(q^2;q^2)_b}\,{}_2\phi_1\left(\begin{matrix} q^{-2b}, q^{-2i} \\ q^{-2a-2b} \end{matrix}; q^2, q^{-2c}\right)$$

$$a, b, c, i, j, k \in \mathbb{Z}_{\geq 0}$$

Locally finite.
For this 3d R, representation theoretical origin
(quantized coordinate ring, PBW bases,…) is known.

3. Relation to quantized coordinate ring

$A_q(sl_3)$: Hopf algebra dual to $U_q(sl_3)$ generated by t_{ij} $(1 \le i, j \le 3)$ obeying

$$[t_{ik}, t_{jl}] = \begin{cases} 0 & (i < j, k > l), \\ (q - q^{-1}) t_{jk} t_{il} & (i < j, k < l), \end{cases}$$

$$t_{ik} t_{jk} = q t_{jk} t_{ik} \ (i < j), \quad t_{ki} t_{kj} = q t_{kj} t_{ki} \ (i < j),$$

$$\sum_{\sigma \in \mathfrak{S}_3} (-q)^{l(\sigma)} t_{1\sigma_1} t_{2\sigma_2} t_{3\sigma_3} = 1,$$

There are two algebra homomorphisms to the q-Weyl algebra

$$\rho_1 : A_q(sl_3) \to \mathcal{W}_q \qquad\qquad \rho_2 : A_q(sl_3) \to \mathcal{W}_q$$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \mapsto \begin{pmatrix} Z^{-1}(u_1 - g_1 h_1 X^2) & g_1 X & 0 \\ -q h_1 X & Z & 0 \\ 0 & 0 & u_1^{-1} \end{pmatrix} \qquad \begin{pmatrix} u_2^{-1} & 0 & 0 \\ 0 & Z^{-1}(u_2 - g_2 h_2 X^2) & g_2 X \\ 0 & -q h_2 X & Z \end{pmatrix}$$

u_i, g_i, h_i are parameters.

When $(u_i, g_i, h_i) = (1, \mu_i, \mu_i^{-1})$, they may be viewed as $\rho_i : A_q(sl_3) \longrightarrow \mathcal{O}_q$.

$$
\begin{tikzcd}
 & \mathcal{O}_q \arrow[r, "\pi_O"] & \mathrm{End}(F_+) \cdots \rho_{O,i} \\
A_q(sl_3) \arrow[ur, "{(u_i,g_i,h_i) = (1,\mu_i,\mu_i^{-1})}"] \arrow[dr, "{(u_i,g_i,h_i)\ \text{generic}}"'] \arrow[r, phantom, "\rho_i"] & & & (i=1,2) \\
 & \mathcal{W}_q \arrow[r, "\pi_Z"] & \mathrm{End}(F) \cdots \rho_{Z,i}
\end{tikzcd}
$$

—————— Facts about R^{OOO} ——————

Soilbelman ('91): $\rho_{O,1} \otimes \rho_{O,2} \otimes \rho_{O,1} \simeq \rho_{O,2} \otimes \rho_{O,1} \otimes \rho_{O,2}$

Kapranov-Voevodsky ('94): $\Phi(\rho_{O,1} \otimes \rho_{O,2} \otimes \rho_{O,1}) = (\rho_{O,2} \otimes \rho_{O,1} \otimes \rho_{O,2})\Phi$

Bazhanov-Serveev ('06): $R^{OOO}L^O L^O L^O = L^O L^O L^O R^{OOO}$
(+ Mangazeev ('10))

K-Okado ('12): $R^{OOO} = \Phi \circ \sigma \quad \big(\sigma(|i\rangle \otimes |j\rangle \otimes |k\rangle) = (i \leftrightarrow k)\big)$

Proposition (R^{ZZZ} case)

$$\tilde{\Phi}(\rho_{Z,1} \otimes \rho_{Z,2} \otimes \rho_{Z,1}) = (\rho_{Z,2} \otimes \overset{(u_1,g_1,h_1)}{\rho_{Z,1}} \otimes \overset{(u_2,g_2,h_2)}{\rho_{Z,2}})\tilde{\Phi}$$

$$\Longleftrightarrow \qquad R^{ZZZ}L^Z L^Z L^Z = \underset{(r_1,s_1,t_1,w_1)}{L^Z}\,\underset{(r_2,s_2,t_2,w_2)}{L^Z}\,\underset{(r_3,s_3,t_3,w_3)}{L^Z} R^{ZZZ}$$

with $\quad R^{ZZZ} = \tilde{\Phi} \circ \sigma$ provided that

$$u_1 = u_2 (=: u) \qquad\qquad g_1 h_1 = g_2 h_2 (=: p)$$

$$\frac{r_1}{t_1} = \frac{r_2}{t_2}, \quad \frac{s_2}{t_2} = \frac{s_3}{t_3}, \quad \frac{r_2}{r_1 r_3} = u, \quad \frac{s_1 s_3}{s_2} = u^2, \quad \frac{t_1^2 w_1}{r_1 s_1} = \frac{t_2^2 w_2}{r_2 s_2} = \frac{t_3^2 w_3}{r_3 s_3} = \frac{p}{u}.$$

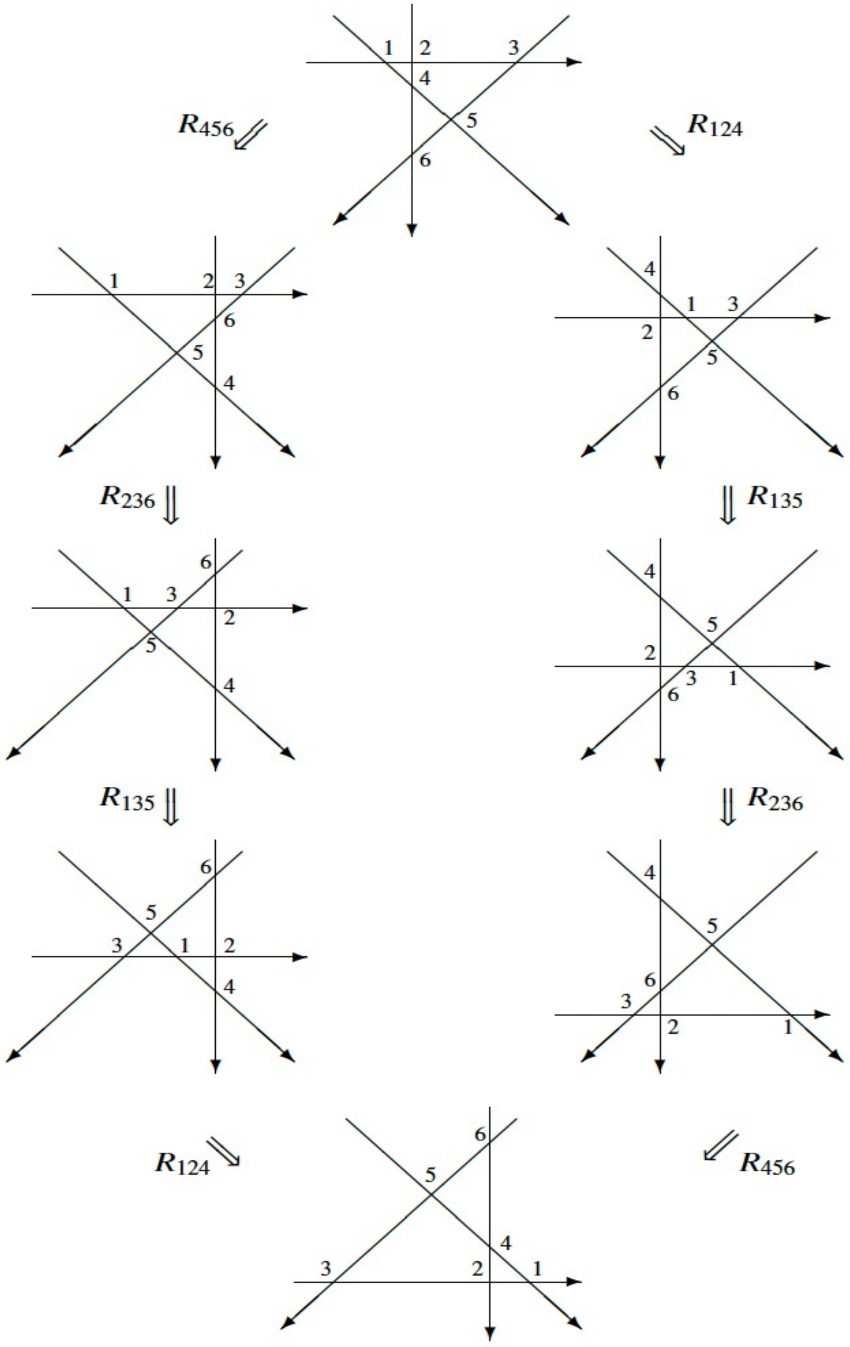
4. Conjecture on RRRR=RRRR

$$
\begin{aligned}
&R_{124}R_{135}R_{236}R_{456}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}\\
&=R_{124}R_{135}R_{236}L_{\beta\gamma4}L_{\alpha\gamma5}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\gamma\delta1}R_{456}\\
&=R_{124}R_{135}L_{\beta\gamma4}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}R_{236}R_{456}\\
&=R_{124}R_{135}L_{\beta\gamma4}L_{\beta\delta2}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\alpha\beta6}R_{236}R_{456}\\
&=R_{124}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}L_{\alpha\delta3}L_{\alpha\gamma5}L_{\alpha\beta6}R_{135}R_{236}R_{456}\\
&=L_{\gamma\delta1}L_{\beta\delta2}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},\\
&=L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},
\end{aligned}
$$

$$
\begin{aligned}
&R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\beta\delta2}L_{\gamma\delta1}\\
&=R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}L_{\alpha\delta3}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}\\
&=R_{456}R_{236}R_{135}L_{\alpha\beta6}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\beta\delta2}L_{\beta\gamma4}R_{124}\\
&=R_{456}R_{236}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}L_{\alpha\delta3}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\beta\gamma4}R_{135}R_{124}\\
&=R_{456}R_{236}L_{\gamma\delta1}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\alpha\gamma5}L_{\beta\gamma4}R_{135}R_{124}\\
&=R_{456}L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}R_{236}R_{135}R_{124}\\
&=L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{456}R_{236}R_{135}R_{124}.
\end{aligned}
$$

$(R_{124}R_{135}R_{236}R_{456})^{-1}R_{456}R_{236}R_{135}R_{124}$ commutes with $L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}$.

$\longrightarrow$ $R_{456}R_{236}R_{135}R_{124}=R_{124}R_{135}R_{236}R_{456}$ if irreducible.



In our case, a similar procedure starting from $L^F_{\alpha\beta 6}L^E_{\alpha\gamma 5}L^D_{\beta\gamma 4}L^C_{\alpha\delta 3}L^B_{\beta\delta 2}L^A_{\gamma\delta 1}$ suggests

$$R^{DEF}_{456}R^{BCF}_{236}R^{ACE}_{135}R^{ABD}_{124} = R^{ABD}_{124}R^{ACE}_{135}R^{BCF}_{236}R^{DEF}_{456}$$

where, A, B, C, D, E, F are Z or O or X.

For A=⋯=F=O, the irreducibility is known from the representation theory of A_q. Therefore

$$R^{OOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOO}_{124} = R^{OOO}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{OOO}_{456}$$ holds. (Kapranov-Voevodsky '94)

Other cases are yet elusive, also due to convergence issue for composition of the locally non-finite 3D R's.

$|i\rangle\otimes|j\rangle\otimes|k\rangle\otimes|l\rangle\otimes|m\rangle\otimes|n\rangle \mapsto |a\rangle\otimes|b\rangle\otimes|c\rangle\otimes|d\rangle\otimes|e\rangle\otimes|f\rangle$ component of $RRRR = RRRR$:

$$\sum_{u,v,w,x,y,z} R^{d,e,f}_{x,y,z}R^{b,c,z}_{v,w,n}R^{a,w,y}_{u,k,m}R^{u,v,x}_{i,j,l} = \sum_{u,v,w,x,y,z} R^{a,b,d}_{u,v,x}R^{u,c,e}_{i,w,y}R^{v,w,f}_{j,k,z}R^{x,y,z}_{l,m,n}$$

We have listed up the cases (without X) in which the above sums with prescribed a,b,c,d,e,f, I,j,k,l,m,n become finite sums due to locally-finiteness. They are given in the next page.

$$R_{456}^{OOO}R_{236}^{OOO}R_{135}^{ZOO}R_{124}^{ZOO}=R_{124}^{ZOO}R_{135}^{ZOO}R_{236}^{OOO}R_{456}^{OOO},$$
$$R_{456}^{ZOO}R_{236}^{OOO}R_{135}^{OOO}R_{124}^{OOZ}=R_{124}^{OOZ}R_{135}^{OOO}R_{236}^{OOO}R_{456}^{ZOO},$$
$$R_{456}^{OOZ}R_{236}^{OOZ}R_{135}^{OOO}R_{124}^{OOO}=R_{124}^{OOO}R_{135}^{OOO}R_{236}^{OOZ}R_{456}^{OOZ},$$
$$R_{456}^{OOZ}R_{236}^{OOZ}R_{135}^{ZOO}R_{124}^{ZOO}=R_{124}^{ZOO}R_{135}^{ZOO}R_{236}^{OOZ}R_{456}^{OOZ}.$$

$$R_{456}^{OOO}R_{236}^{ZOO}R_{135}^{OOO}R_{124}^{OZO}=R_{124}^{OZO}R_{135}^{OOO}R_{236}^{ZOO}R_{456}^{OOO},$$
$$R_{456}^{OZO}R_{236}^{OOO}R_{135}^{OOZ}R_{124}^{OOO}=R_{124}^{OOO}R_{135}^{OOZ}R_{236}^{OOO}R_{456}^{OZO},$$
$$R_{456}^{OOO}R_{236}^{ZOO}R_{135}^{ZOO}R_{124}^{ZZO}=R_{124}^{ZZO}R_{135}^{ZOO}R_{236}^{ZOO}R_{456}^{OOO},$$
$$R_{456}^{ZOO}R_{236}^{OOO}R_{135}^{ZOO}R_{124}^{ZOZ}=R_{124}^{ZOZ}R_{135}^{ZOO}R_{236}^{OOO}R_{456}^{ZOO},$$
$$R_{456}^{ZOO}R_{236}^{ZOO}R_{135}^{OOO}R_{124}^{OZZ}=R_{124}^{OZZ}R_{135}^{OOO}R_{236}^{ZOO}R_{456}^{ZOO},$$
$$R_{456}^{ZZO}R_{236}^{OOO}R_{135}^{OOZ}R_{124}^{OOZ}=R_{124}^{OOZ}R_{135}^{OOZ}R_{236}^{OOO}R_{456}^{ZZO},$$
$$R_{456}^{ZOZ}R_{236}^{OOZ}R_{135}^{OOO}R_{124}^{OOZ}=R_{124}^{OOZ}R_{135}^{OOO}R_{236}^{OOZ}R_{456}^{ZOZ},$$
$$R_{456}^{OZZ}R_{236}^{OOZ}R_{135}^{OOZ}R_{124}^{OOO}=R_{124}^{OOO}R_{135}^{OOZ}R_{236}^{OOZ}R_{456}^{OZZ},$$
$$R_{456}^{ZOO}R_{236}^{ZOO}R_{135}^{ZOO}R_{124}^{ZZZ}=R_{124}^{ZZZ}R_{135}^{ZOO}R_{236}^{ZOO}R_{456}^{ZOO},$$
$$R_{456}^{ZZZ}R_{236}^{OOZ}R_{135}^{OOZ}R_{124}^{OOZ}=R_{124}^{OOZ}R_{135}^{OOZ}R_{236}^{OOZ}R_{456}^{ZZZ}.$$

$$R_{456}^{OOO}R_{236}^{OZO}R_{135}^{OZO}R_{124}^{OOO}=R_{124}^{OOO}R_{135}^{OZO}R_{236}^{OZO}R_{456}^{OOO},$$
$$R_{456}^{OOO}R_{236}^{OZO}R_{135}^{ZZO}R_{124}^{ZOO}=R_{124}^{ZOO}R_{135}^{ZZO}R_{236}^{OZO}R_{456}^{OOO},$$
$$R_{456}^{OZO}R_{236}^{OOO}R_{135}^{ZOZ}R_{124}^{ZOO}=R_{124}^{ZOO}R_{135}^{ZOZ}R_{236}^{OOO}R_{456}^{OZO},$$
$$R_{456}^{OOO}R_{236}^{ZZO}R_{135}^{OZO}R_{124}^{OZO}=R_{124}^{OZO}R_{135}^{OZO}R_{236}^{ZZO}R_{456}^{OOO},$$
$$R_{456}^{OOZ}R_{236}^{ZOZ}R_{135}^{OOO}R_{124}^{OZO}=R_{124}^{OZO}R_{135}^{OOO}R_{236}^{ZOZ}R_{456}^{OOZ},$$
$$R_{456}^{ZOO}R_{236}^{OZO}R_{135}^{OZO}R_{124}^{OOZ}=R_{124}^{OOZ}R_{135}^{OZO}R_{236}^{OZO}R_{456}^{ZOO},$$
$$R_{456}^{OZO}R_{236}^{OZO}R_{135}^{OZZ}R_{124}^{OOO}=R_{124}^{OOO}R_{135}^{OZZ}R_{236}^{OZO}R_{456}^{OZO},$$
$$R_{456}^{OOZ}R_{236}^{OZZ}R_{135}^{OZO}R_{124}^{OOO}=R_{124}^{OOO}R_{135}^{OZO}R_{236}^{OZZ}R_{456}^{OOZ},$$
$$R_{456}^{OZO}R_{236}^{OZO}R_{135}^{ZZZ}R_{124}^{ZOO}=R_{124}^{ZOO}R_{135}^{ZZZ}R_{236}^{OZO}R_{456}^{OZO},$$
$$R_{456}^{OOZ}R_{236}^{ZZZ}R_{135}^{OZO}R_{124}^{OZO}=R_{124}^{OZO}R_{135}^{OZO}R_{236}^{ZZZ}R_{456}^{OOZ}.$$

Conjecture.
They are all valid (based on computer check).

Claim [M. Noumi]
They should be reducible to a duality of his and Ito's multiple series generalization of the q-hypergeometric.

Outlook

$$\mathrm{YBE} \ \underset{\textbf{Reduction via 3D}}{\overset{\text{quantization}}{\rightleftarrows}} \ \mathrm{QYBE}$$

generates infinite family of quantum R matrices.

$q^N = 1$: Beyond generalized Chiral Potts models ?

$R^{OOO} =$ transition coefficient of PBW basses of U_q^+. $R^{ZZZ} = ?$

BCG type:

Quantized reflection equation $\quad K(LGLG) = (GLGL)K$

Quantized G_2 reflection equation $\quad F(LJLJLJ) = (JLJLJL)F$