Integrability of box-ball systems: crystals, Bethe ansatz and ultradiscretization

Atsuo Kuniba (Univ. Tokyo)

"*Box-Ball Systems from Integrable Systems and Probabilistic Perspectives*"

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Part 1. Combinatorial Bethe ansatz for n-color BBS (type A case)

Part 2. Generalizations from various perspectives

n-color Box-ball system (Takahashi-Satsuma '90, Takahashi '93)

$n = 3$ example.

$$\cdots 00000000\textcolor{red}{33211}0000000000000000000000000 \cdots$$
$$\cdots 0000000000000\textcolor{red}{33211}00000000000000000000 \cdots$$
$$\cdots 000000000000000000\textcolor{red}{33211}000000000000000 \cdots$$

0 =empty box, $\quad$ 1, 2, 3 = balls with colors

- **time evolution = (move 1) $\cdot$ (move 2) $\cdot$ (move 3)**

(move i) $\cdot$ **Pick the leftmost ball with color i and move it to the nearest right empty box.**

$\cdot$ **Do the same for the other color i balls.**

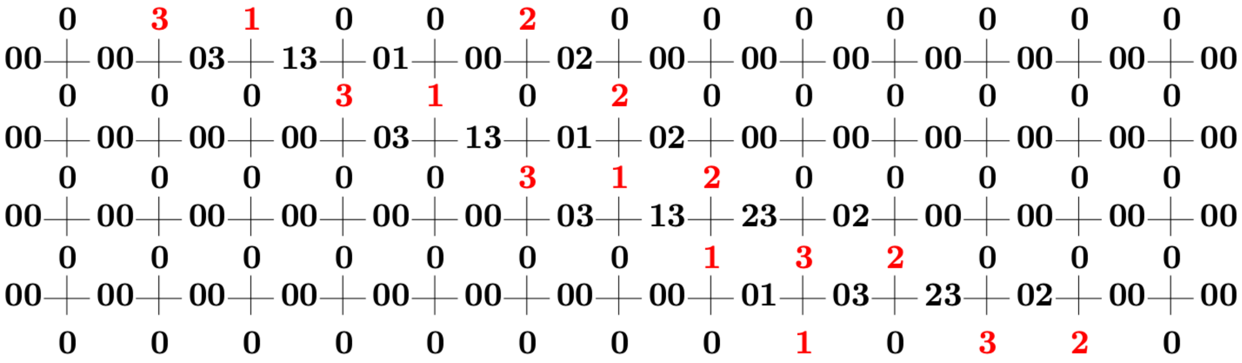
- **soliton=consecutive balls $i_1 \ldots i_a$ with color $i_1 \geq \cdots \geq i_a \geq 1$.**
- **velocity=amplitude.**

Solvable lattice model at “ Temperature 0 ” (Hatayama et al ’00)

Time evolution pattern

$\cdots 0310020000000 \cdots$
$\cdots 0003102000000 \cdots$
$\cdots 0000031200000 \cdots$
$\cdots 0000000132000 \cdots$
$\cdots 0000000010320 \cdots$

emerges from a configuration of a 2D lattice model in statistical mechanics



by forgetting the hidden variables on the horizontal edges.

n-color BBS = solvable vertex model associated with $U_q(\widehat{sl}_{n+1})$ at $q = 0$.

Time evolution = row transfer matrix at $q = 0$

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	$\xrightarrow{\text{UD}}$	Cellular automata Box-ball systems	$\xleftarrow{0 \leftarrow q}$	Lattice statistical models Solvable vertex models
Inverse scattering method		**Combinatorial Bethe ansatz**		**Bethe ansatz**

- Main ingredient = Kerov-Kirillov-Reshetikhin (KKR) bijection (1986).

$$\{\text{highest states}\} \xleftrightarrow{1:1} \{\text{rigged configurations}\}$$

- It asserts "formal completeness" of string solutions of Bethe eq. at a combinatorial level.
- Its unexpected connection to BBS was discovered in (K-Okado-Takagi-Yamada, RIMS-kôkyûroku '03).

- **Example. Spin $\frac{1}{2}$ periodic Heisenberg chain**

$$\boldsymbol{H} = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)$$

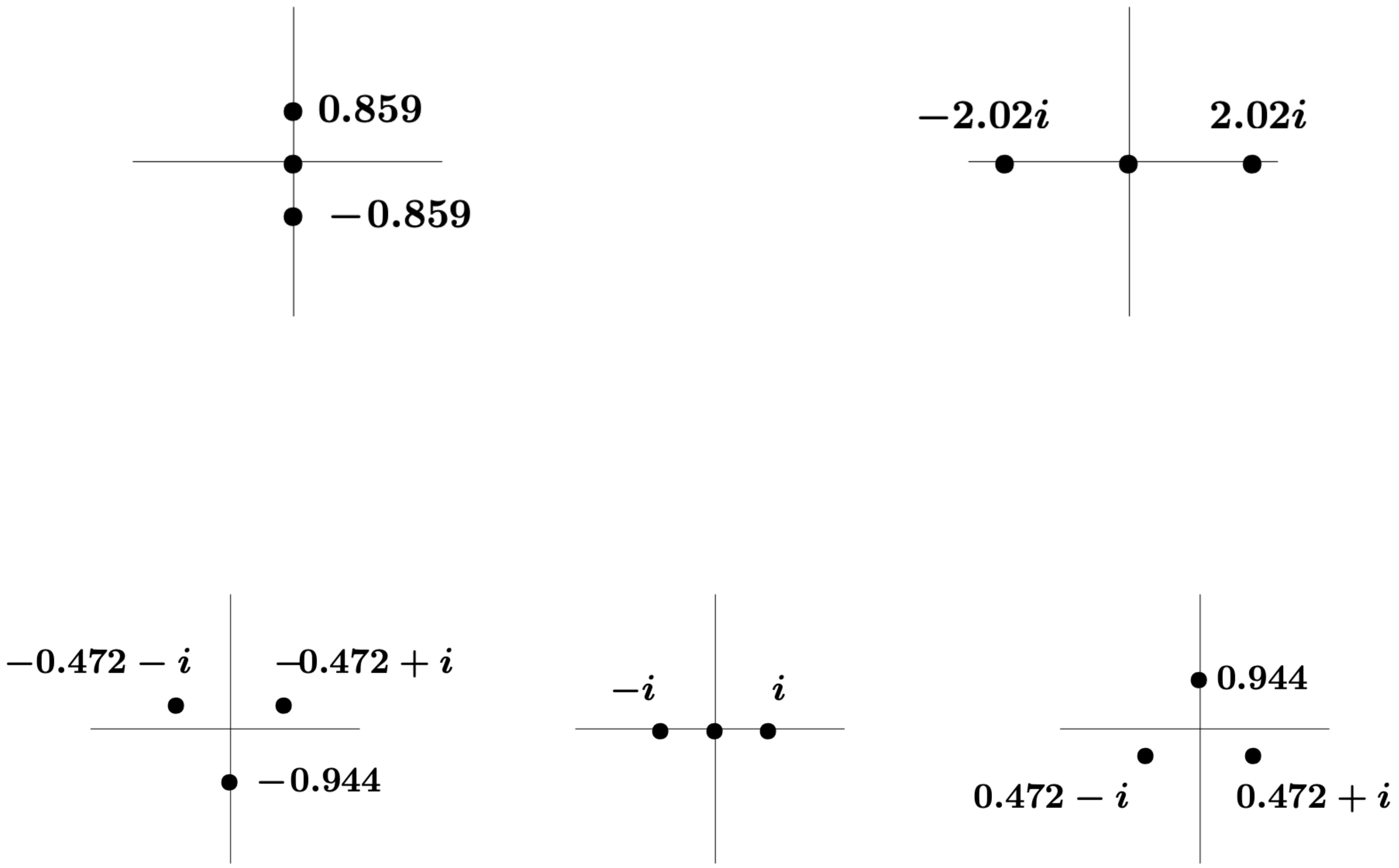
For $L = 6$ sites in 3 down-spin sector, the Bethe equation reads

$$\begin{aligned}
\left(\frac{u_1 + i}{u_1 - i}\right)^6 &= \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)}, \\
\left(\frac{u_2 + i}{u_2 - i}\right)^6 &= \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)}, \\
\left(\frac{u_3 + i}{u_3 - i}\right)^6 &= \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.
\end{aligned}$$

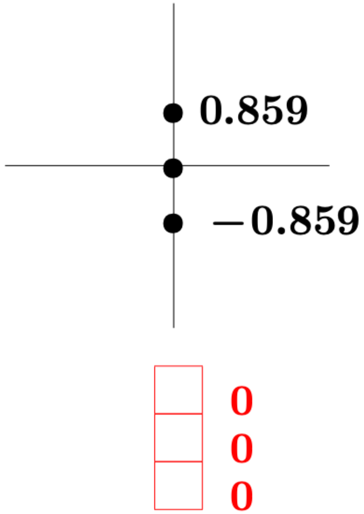
There should be **5** solutions for the completeness.

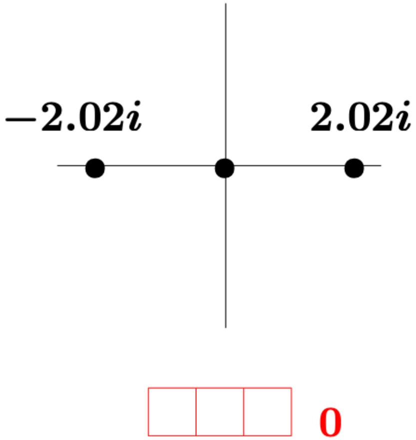
5 = Kostka number (multiplicity of the irreducible reps. in (spin 1/2)^L)

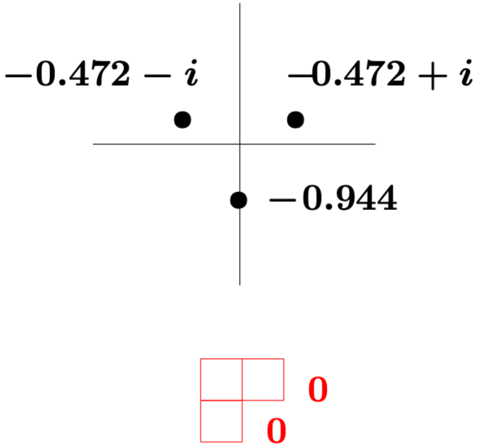
= number of highest states.

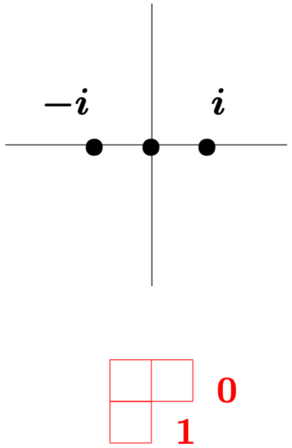


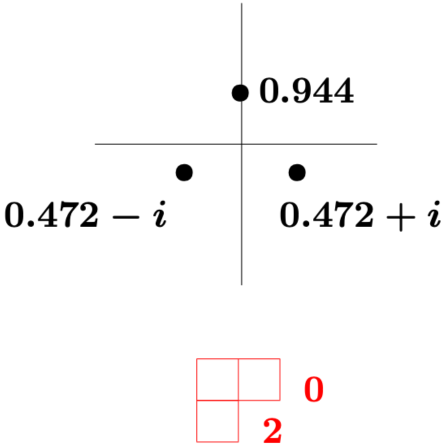
Indeed 5 solutions which are approximately described as collection of **strings**











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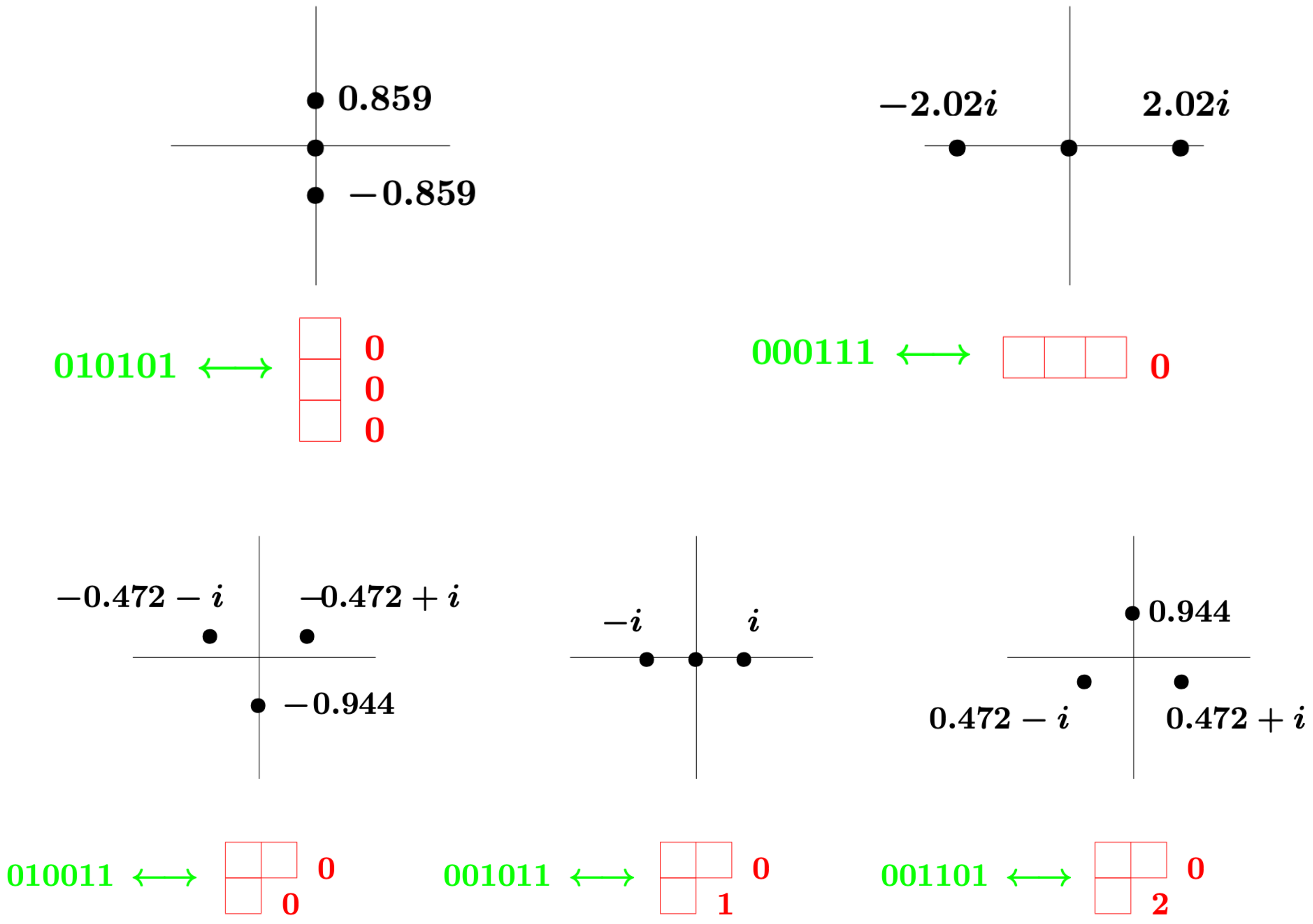


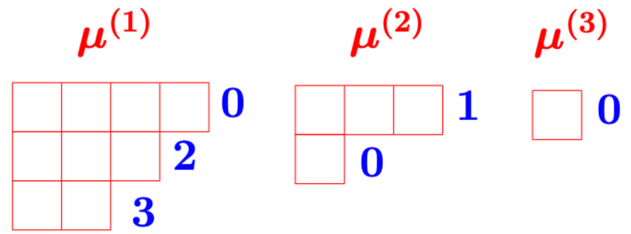
Indeed 5 solutions which are approximately described as collection of **strings**

KKR bijection for sl_{n+1}

$$\{\text{highest states}\} \overset{1:1}{\longleftrightarrow} \{\text{rigged configurations}\}$$

$n = 3$ example

000011102113220000 $\longleftrightarrow$



"Bethe vectors" — "Bethe roots"

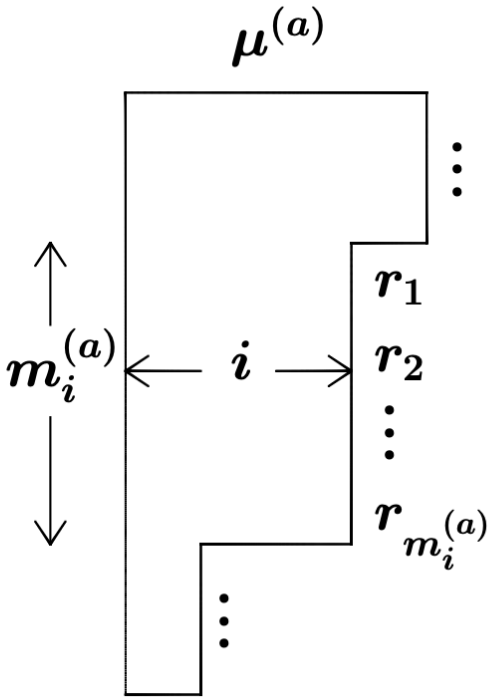
Solitons — Strings (bound states of magnons)

- highest states $= i_1 i_2 \dots i_L$ $(0 \le i_k \le n)$ satisfying the highest condition:

$$\#_0\{i_1, \dots, i_k\} \ge \#_1\{i_1, \dots, i_k\} \ge \cdots \ge \#_n\{i_1, \dots, i_k\} \quad (\forall k)$$

- rigged configuration: $((\mu^{(1)}, r^{(1)}), \dots, (\mu^{(n)}, r^{(n)}))$

$$\left.\begin{array}{l} \mu^{(1)}, \dots, \mu^{(n)} : \text{configuration} = n\text{-tuple of Young diagrams} \\ r^{(1)}, \dots, r^{(n)} : \text{rigging} = \text{integers assigned to each row} \end{array}\right\} + \text{selection rule (next page)}$$



$$m_i^{(a)} = \#(\text{length } i \text{ rows in } \mu^{(a)}), \ \ \textstyle\sum_{i\ge 1} i m_i^{(a)} = |\mu^{(a)}|$$

$$0 \le r_1 \le \cdots \le r_{m_i^{(a)}} \le p_i^{(a)}$$

$\cdots$ "Fermionic" selection rule

$$p_i^{(a)} = L\delta_{a,1} - \textstyle\sum_{b=1}^{n} C_{ab} \sum_{j\ge 1} \min(i,j) m_j^{(b)}$$

$\cdots$ vacancy for *holes*

$$C_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1}$$

(C_{ab}) $\cdots$ Cartan matrix of sl_{n+1}

$$\# \text{ of rigging choices for a fixed configuration} = \prod_{a=1}^{n} \prod_{i\ge 1} \binom{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}$$

$\leftarrow$ called "Fermionic formula"

This is an sl_{n+1} generalization of Bethe's formula for # of string solutions (1931).

hat also eine Möglichkeit weniger, die des letzten Komplexes von n Wellen, λ_{q_n}, kann schließlich nur noch

$$Q'_n - (q_n - 1) = Q_n + 1$$

verschiedene Werte annehmen, wo

$$Q_n(N, q_1 q_2 \ldots) = N - 2 \sum_{p<n} p\, q_p - 2 \sum_{p \geqq n} n\, q_p. \tag{44}$$

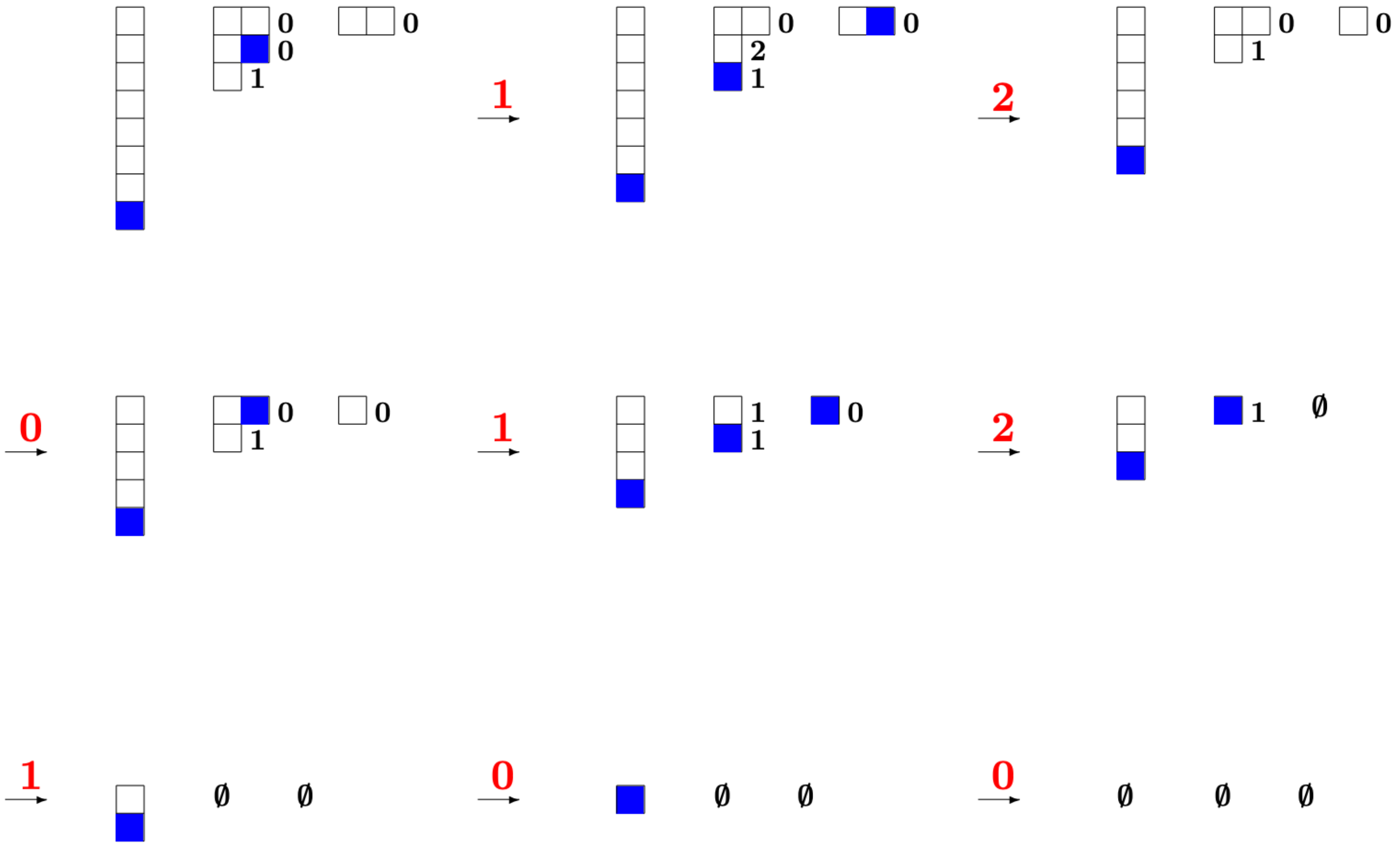
Schließlich ist zu berücksichtigen, daß Vertauschung der λ der verschiedenen Wellenkomplexe mit gleicher Anzahl n von Wellen nicht zu neuen Lösungen führt. Die gesamte Zahl unserer Lösungen wird somit

$$z(N, q_1 q_2 \ldots) = \prod_{n=1}^{\infty} \frac{(Q_n + q_n) \ldots (Q_n + 1)}{q_n!} = \prod_n \binom{Q_n + q_n}{q_n}, \tag{45}$$

wo die Q_n durch (44) definiert sind.

8. Wir werden nun nachweisen, daß wir die richtige Anzahl Lösungen erhalten haben.

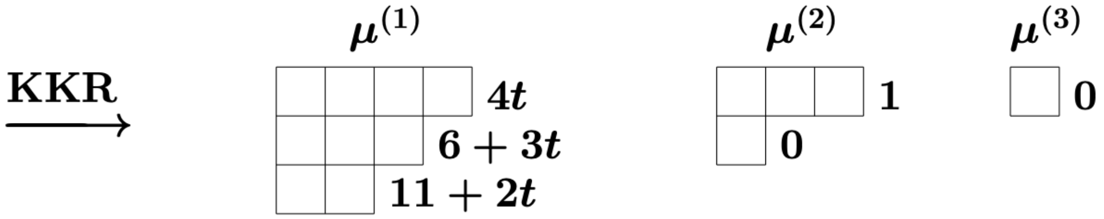
Example of KKR algorithm



Top left rigged configuration $\xmapsto{\textbf{KKR}}$ **00121021**

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000
$t = 1$: 000000001111000022103200000000000000000000000000
$t = 2$: 000000000000111100022132000000000000000000000000
$t = 3$: 000000000000000011110021322000000000000000000000
$t = 4$: 000000000000000000001110211322000000000000000000
$t = 5$: 000000000000000000000001100211322100000000000000
$t = 6$: 000000000000000000000000011000211032210000000000
$t = 7$: 000000000000000000000000000110000211003221000000



- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

$$\boxed{\text{Rigged configuration} = \text{action angle variable of BBS!}}$$

In particular, $\mu^{(1)}$ = list of amplitude of solitons.

(Remark: meaning of $\mu^{(2)}, \mu^{(3)}, \ldots$ is implicit, hence *not democratic.*)

Lascoux-Schützenberger charge in KKR theory

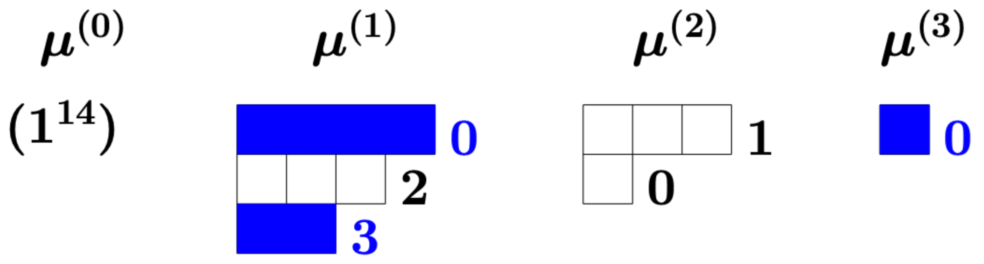
$$\color{red}{c(\mu, r)} = \frac{1}{2}\sum_{a,b=1}^{n-1} C_{ab}\min(\mu^{(a)}, \mu^{(b)}) - \min(\mu^{(0)}, \mu^{(1)}) + \sum_{a=1}^{n-1} |r^{(a)}|$$

$$\left(\begin{array}{c} \min(\lambda, \mu) = \sum_{ij}\min(\lambda_i, \mu_j), \quad |r| = \sum_i r_i \\ (C_{ab}) = \text{Cartan matrix of } sl_n \end{array}\right)$$

UD tau function

$$\tau_{\color{blue}{k},i} = -\min_{(\nu,s)}\{c(\nu, s) + |\nu^{(i)}|\} \quad (k \geq 1,\ 1 \leq i \leq n)$$

$\min\limits_{(\nu,s)}$: over all subsets $\color{blue}{(\nu, s)} \subseteq (\mu, r)$ with $\nu^{(0)} = (\mu_1^{(0)}, \ldots, \mu_{\color{blue}{k}}^{(0)})$.



Th. (K-Sakamoto-Yamada '07)

$\tau_{k,i}$ admits the following characterizations:

(1) UD of tau functions for KP hierarchy

(2) "corner transfer matrix" of box-ball system ■

(1) $$\tau_{k,i} = \lim_{\epsilon \to 0} \epsilon \log \langle i| \exp\Big(\sum_j c_j \psi(p_j)\psi^*(q_j)\Big)|i\rangle$$

$(c_j, p_j, q_j) \longleftrightarrow$ **strings in rigged configuration**

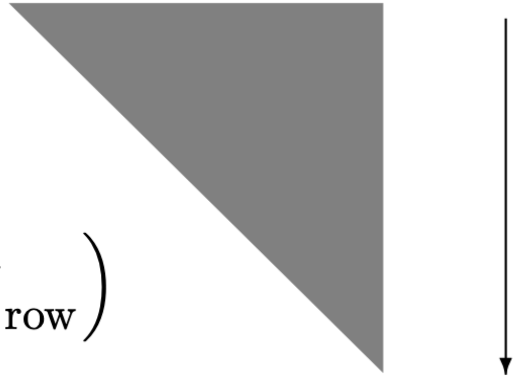
Corollary of (1). (UD Hirota bilinear eq.)

$$\bar{\tau}_{k,i-1} + \tau_{k-1,i} = \max(\bar{\tau}_{k,i} + \tau_{k-1,i-1}, \bar{\tau}_{k-1,i-1} + \tau_{k,i} - \mu_k^{(0)})$$

$\bar{\tau}_{k,i} = \tau_{k,i}$ **associated with** $T_\infty((\mu, r))$

(2) Suppose $b_1 b_2 \cdots b_L \xleftrightarrow{\text{KKR}} (\mu, r) \longrightarrow \{\tau_{k,i}\}$. Then,

$b_1\ b_2\ \cdots\ b_k$



$\tau_{k,i} = \sharp$ of balls in

$\left(\begin{array}{c}\text{only } 1, 2, ..., i-1 \\ \text{counted in the top row}\end{array}\right)$

box-ball system
time evolution T_∞

$\cdots$ "analogue" of Baxter's corner transfer matrix

UD Hirota bilinear = eq. of motion of box-ball system.

Corollary of (1)&(2).

- piecewise linear formula for KKR map: $(\mu, r) \overset{\mathrm{KKR}}{\longmapsto} b_1 \ldots b_L$

  semistandard tab. $b_k = (\overbrace{1\ldots 1}^{x_{k,1}} \ldots \overbrace{n \ldots n}^{x_{k,n}})$ is specified by

$$x_{k,i} = \tau_{k,i} - \tau_{k-1,i} - \tau_{k,i-1} + \tau_{k-1,i-1}.$$

- general N soliton solution of n-color BBS
  with arbitrary inhomogeneous box capacities

	Bethe ansatz	Corner transfer matrix
main combinatorial object	rigged configuration	charge (energy) in affine crystal
role in box-ball system	action-angle variable	tau function
dynamics	linear	bilinear

Part 2 Generalizations from various perspectives

$\hat{\mathfrak{g}}_n$-BBS (Hatayama-K-Takagi '00, HK-Okado-T-Yamada '02)

- formulated by KR crystals of $\hat{\mathfrak{g}}_n = A^{(2)}_{2n-1}, A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, D^{(2)}_{n+1}$
- Commuting time evolutions $T_1, T_2, \ldots T_\infty =: T$ and associated energies $E_1, E_2, \ldots$
- Particle/anti-particle system with pair creation/annihilation via a bound state
- Solitons exhibiting factorized scattering with S-matrix = combinatorial R of $\hat{\mathfrak{g}}_{n-1}$

$$\begin{array}{l}
\cdots 111\bar{3}\bar{5}3221111\bar{2}\bar{2}\bar{5}11111111111111111111111111111111 \cdots \\
\cdots 11111111\bar{3}\bar{5}32211\bar{2}\bar{2}\bar{5}11111111111111111111111111111 \cdots \\
\cdots 1111111111111\bar{3}\bar{5}322\bar{2}\bar{2}\bar{5}11111111111111111111111111 \cdots \\
\cdots 111111111111111111\bar{3}\bar{5}3\bar{1}\bar{1}\bar{5}11111111111111111111111 \cdots \\
\cdots 111111111111111111111114\bar{1}\bar{1}\bar{4}5\bar{5}111111111111111111 \cdots \\
\cdots 11111111111111111111111111422\bar{2}\bar{2}\bar{4}5\bar{5}1111111111111 \cdots \\
\cdots 1111111111111111111111111111142211\bar{2}\bar{2}\bar{4}5\bar{5}11111111 \cdots \\
\cdots 111111111111111111111111111111114221111\bar{2}\bar{2}\bar{4}5\bar{5}111 \cdots
\end{array}$$

A collision of two solitons in $D^{(1)}_5 = \widehat{so}_{10}$ BBS during several time steps. $3, \bar{3} \longrightarrow 4, \bar{4}$ is happening.

$$\boxed{\underline{\text{Th.}} \quad T = \text{translation in extended affine Weyl group} = (\text{Dynkin auto}) \prod (\text{simple reflection})}$$

Signature rule for the simple reflection S_i in Weyl group

(1) Assign i-sgn

(2) Eliminate $+-$ recursively

$$\longrightarrow \quad \overbrace{------}^{\alpha}\ \overbrace{+++}^{\beta}$$

(3) Change it into

$$\overbrace{---}^{\beta}\ \overbrace{+++++}^{\alpha}$$

$\widehat{sl}_3 \quad B_1$

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[ll, bend right, "0"'] \end{tikzcd}$$

	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
1-sgn	$+$	$-$	
2-sgn		$+$	$-$
0-sgn	$-$		$+$

	$\boxed{3}\otimes$	$\boxed{3}\otimes$	$\boxed{2}\otimes$	$\boxed{1}\otimes$	$\boxed{2}\otimes$	$\boxed{3}\otimes$	$\boxed{1}\otimes$	$\boxed{3}\otimes$	$\boxed{3}\otimes$	$\boxed{1}\otimes$	$\boxed{2}$
2-sgn	$-$	$-$	$(+$		$(+$	$-)$		$-)$	$-$		$+$
S_2											
2-sgn	$-$	$+$	$(+$		$(+$	$-)$		$-)$	$+$		$+$
	$\boxed{3}\otimes$	$\boxed{2}\otimes$	$\boxed{2}\otimes$	$\boxed{1}\otimes$	$\boxed{2}\otimes$	$\boxed{3}\otimes$	$\boxed{1}\otimes$	$\boxed{3}\otimes$	$\boxed{2}\otimes$	$\boxed{1}\otimes$	$\boxed{2}$

$=$ Pitman's transformation up to gauge

$A_n^{(1)}$ case: S_i corresponds, up to gauge, to moving balls with a specific color.

Example. $A_2^{(1)} = \widehat{sl}_3$ $\qquad T = \sigma S_2 S_0$

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"'] & \boxed{2} \arrow[r, "2"'] & \boxed{3} \arrow[ll, bend right, "0"'] \end{tikzcd}$$

	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$
1-sgn	$+$	$-$	
2-sgn		$+$	$-$
0-sgn	$-$		$+$

$$\begin{array}{ll}
 & 1\ 3\ 3\ 2\ 1\ 2\ 3\ 1\ 1\ 1\ 1 \\
S_0 \downarrow & -(+(+\ \ -)\ \ \ (+\ -)-)-- \qquad \leftarrow \text{0-sgn} \\
 & 3\ 3\ 3\ 2\ 1\ 2\ 3\ 1\ 1\ 3\ 3 \\
S_2 \downarrow & -\ -\ -(+\ (+\ -)\ \ \ \ \ -)- \qquad \leftarrow \text{2-sgn} \\
 & 2\ 2\ 2\ 2\ 1\ 2\ 3\ 1\ 1\ 3\ 2 \\
\sigma \downarrow & 1\ 1\ 1\ 1\ 3\ 1\ 2\ 3\ 3\ 2\ 1
\end{array}$$

$$\sigma : \boxed{a} \longmapsto \boxed{\text{a-1}} \quad \text{mod } 3$$

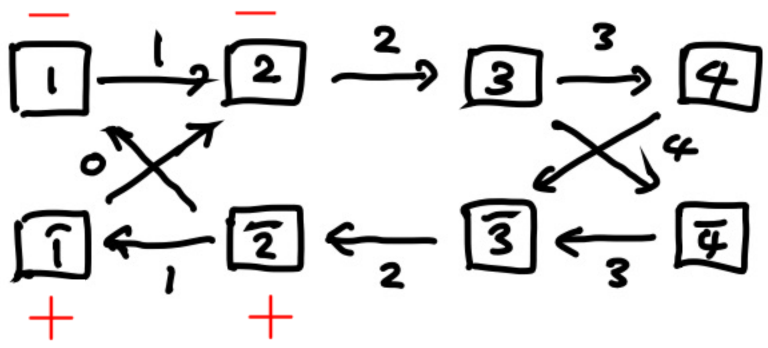
$$T = \underbrace{(\sigma S_2 s_0)}_{\text{move } 2}\underbrace{(s_0 S_0)}_{\text{move } 3}$$

s_i = simple reflection *within* each component.

$(s_i^2 = 1)$

Example. $D_4^{(1)} = \widehat{so}_8$

0-sgn



empty box	1	2 3 4	particle
bound state	$\bar{1}$	$\bar{2}$ $\bar{3}$ $\bar{4}$	anti-particle

$$T = K_2 K_3 K_4 K_{\bar{4}} K_{\bar{3}} K_{\bar{2}} \quad \left(\longleftarrow\ T = \sigma S_0 S_2 S_3 S_4 S_2 S_0, \quad \sigma : 1 \leftrightarrow \bar{1},\ 4 \leftrightarrow \bar{4}\right)$$

Algorithm for K_j

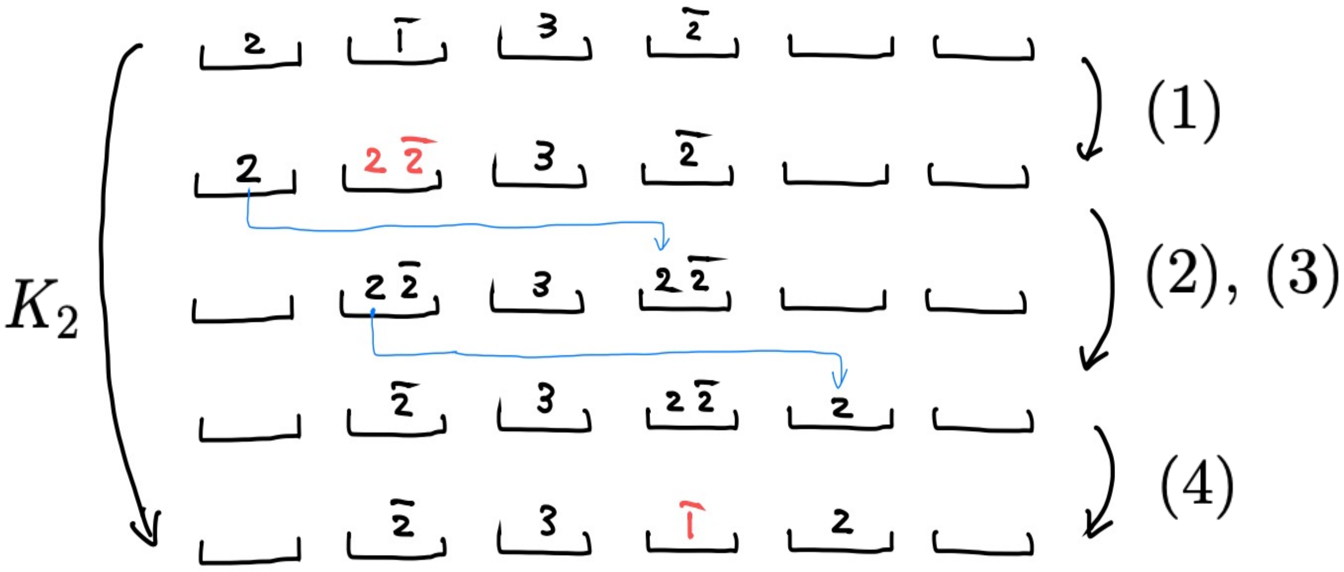
(1) Replace $\bar{1}$ by a pair $j, \bar{j}$ within a box.

(2) Move the leftmost j (if any) to the nearest right box which is empty or containing just $\bar{j}$.

(3) Repeat (2) until all j is moved once.

(4) Replace a pair $j, \bar{j}$ by $\bar{1}$ within a box.

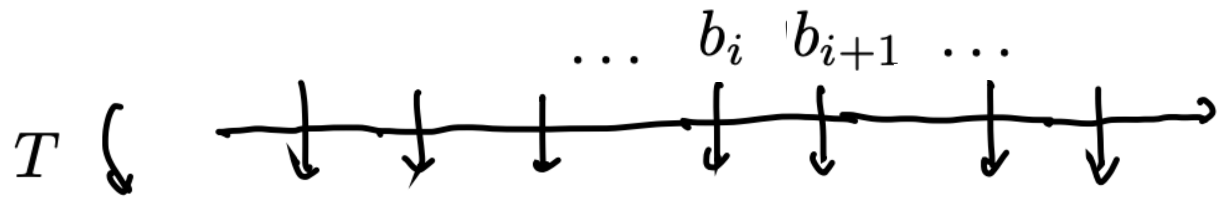
Propagation of particle 2 exhibiting pair creation/annihilation with anti-particle $\overline{2}$ via a bound state $\overline{1}$.



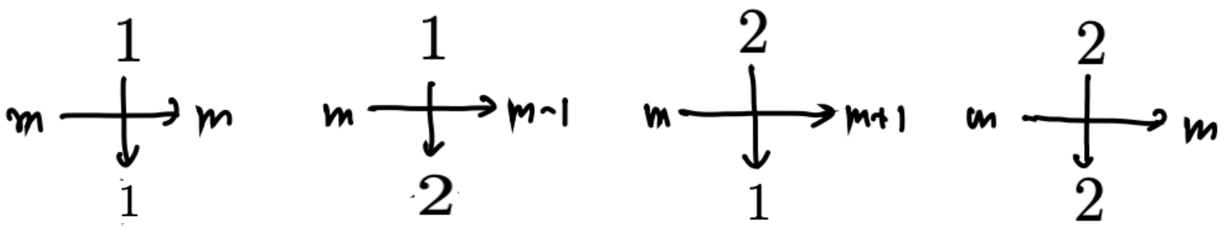
Usual (type A) BBS corresponds to the special sector free from anti-particles.

Quantized BBS

(Inoue-K-Okado '04)



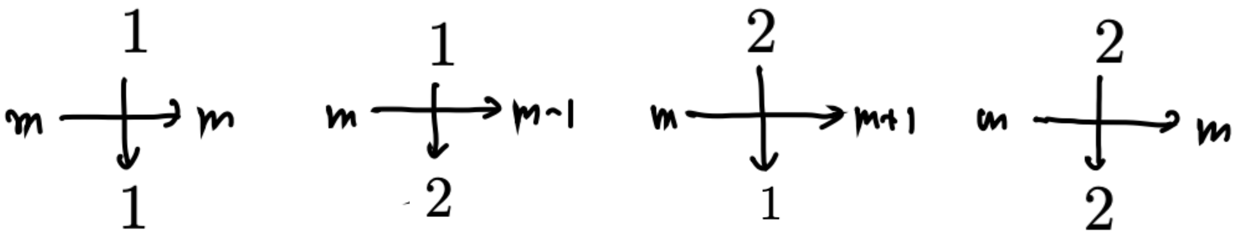
Time evolution is induced by carrier



1=empty box
2= box containing a ball

amplitude δ_{m0} $1-\delta_{m0}$ 1 0

quantization



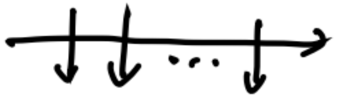
q^m $1-q^{2m}$ 1 $-q^{m+1}$

= Soibelman's representation of quantized coordinate ring $A_q(sl_2)$ up to gauge

<u>Th.</u>

(1) $\displaystyle \lim_{\ell\to\infty} \Bigg($ Transfer matrix of $U_q(\widehat{sl}_2)$ vertex model in an asymptotic domain

spin 1/2



spin $\ell/2$ $\Bigg)$

= Quantized T in the previous page

$$T(|b\rangle) = \sum_c A_{b,c}|c\rangle \quad (\,|b\rangle = |\ldots b_{i-1}b_ib_{i+1}\ldots\rangle,\ \ |c\rangle = |\ldots c_{i-1}c_ic_{i+1}\ldots\rangle)$$

(2) The amplitude satisfies $\displaystyle\sum_d A_{bd}A_{cd} = \delta_{bc}$ (orthogonality), $\displaystyle\sum_c A_{bc} = 1$ (‘linear norm’ preserving).

Example $\quad |b\rangle = |\cdots \circ\circ\bullet\bullet\circ\circ\circ\cdots\rangle$

$(A_{bd})^2$

$$\begin{array}{ll}
\begin{array}{l}\bullet\ \bullet\\ \bullet\ \bullet\end{array} & (-q)^4 \\[2ex]
\begin{array}{l}\bullet\ \bullet\\ \bullet\ \circ\underbrace{\cdots}_{k}\bullet\end{array} & (-q)^2\sum_{k\geq 0}(q^k)^2(1-q^2)^2 = q^2(1-q^2) \\[2ex]
\begin{array}{l}\bullet\ \bullet\\ \circ\ \bullet\underbrace{\cdots}_{k}\bullet\end{array} & (-q^2)^2\sum_{k\geq 0}(q^k)^2(1-q^2)^2 = q^4(1-q^2) \\[2ex]
\begin{array}{l}\bullet\ \bullet\\ \circ\ \circ\underbrace{\cdots}_{k_1}\bullet\underbrace{\cdots}_{k_2}\bullet\end{array} & \sum_{k_1,k_2\geq 0}(q^{2k_1+k_2})^2(1-q^2)^2(1-q^4)^2 = (1-q^2)(1-q^4)
\end{array}\Bigg\}$$

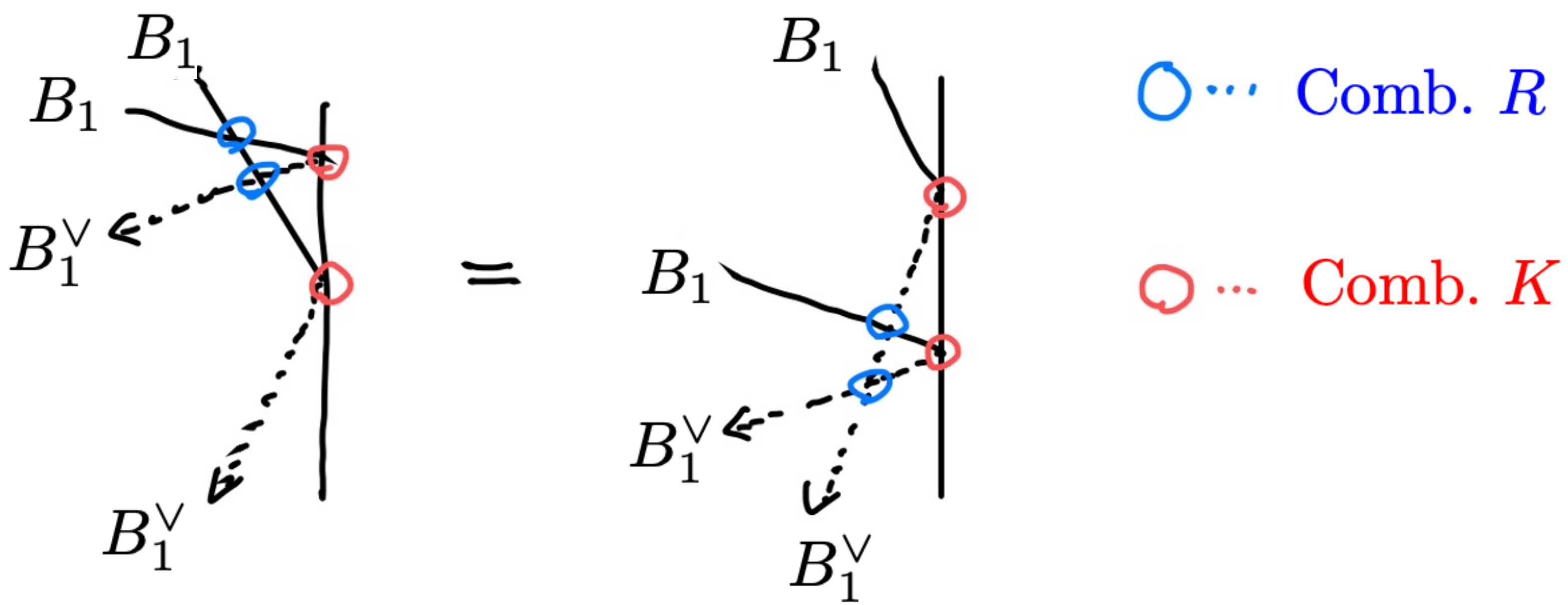
Sum $= \sum_d (A_{bd})^2 = 1$

BBS with reflecting end (K-Okado-Yamada '05)

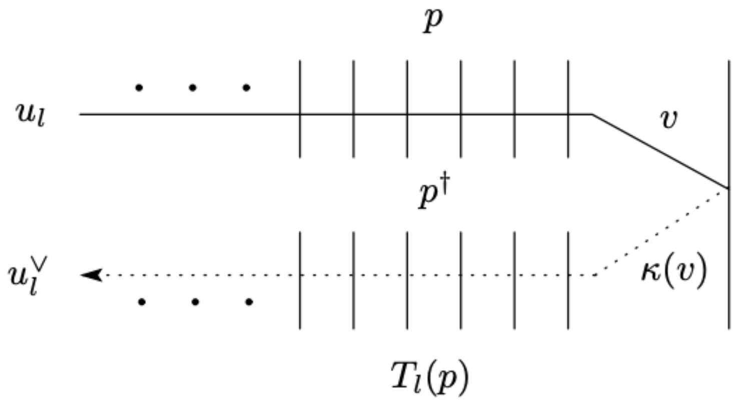
Idea

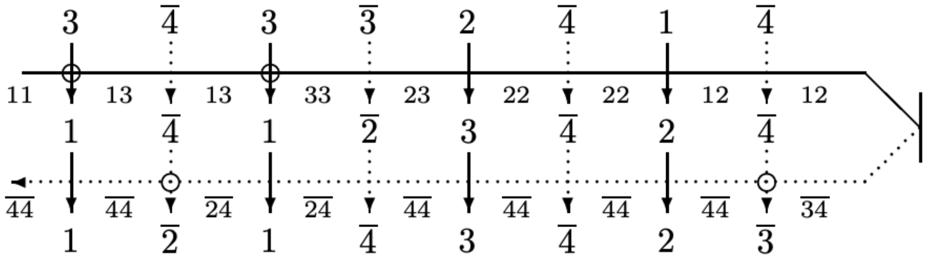
(1) Take the local states from $B_1 \otimes B_1^\vee$ (B_1: vector rep., $B_1^\vee$: anti-vector rep.)

(2) Time evolution = double row transfer matrix at $q = 0$
involving **combinatorial** K, which satisfies
the combinatorial reflection equation $KRKR = RKRK$.



Double row transfer matrix in crystal setting





Reflection of a soliton

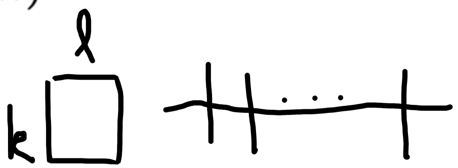
$$
\begin{array}{ll}
0: & ..\ ..\ 5\bar{6}\ 4\bar{6}\ 4\bar{6}\ 2\bar{6}\ 2\bar{6}\ ..\ ..\ .. \\
1: & ..\ ..\ ..\ ..\ ..\ 5\bar{6}\ 4\bar{6}\ 4\bar{6}\ 2\bar{6}\ 2\bar{6} \\
2: & ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{3}\ 5\bar{3}\ 4\bar{5} \\
3: & ..\ ..\ ..\ ..\ 1\bar{3}\ 1\bar{3}\ 1\bar{4}\ 1\bar{5}\ 1\bar{5}\ .. \\
4: & ..\ 1\bar{3}\ 1\bar{3}\ 1\bar{4}\ 1\bar{5}\ 1\bar{5}\ ..\ ..\ ..\ ..
\end{array}
$$

Solitons at the boundary satisfy the reflection equation

$$
\begin{array}{l}
111\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ .. \\
111\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ .. \\
113\ 1\bar{3}\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
113\ 2\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
133\ 4\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
113\ 2\bar{4}\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{1} \\
111\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3} \\
111\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 4\bar{2}\ 3\bar{3}\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ .. \\
111\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 3\bar{1} \\
113\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3} \\
133\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ .. \\
111\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{1}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ .. \\
111\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{1}\ 3\bar{2} \\
123\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ .. \\
111\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ 1\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ 3\bar{4}\ 2\bar{2}\ 1\bar{2}\ 1\bar{3}\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
111\ ..\ 1\bar{2}\ 1\bar{2}\ 4\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ ..\ ..\ .. \\
133\ 1\bar{3}\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ .. \\
113\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ .. \\
111\ ..\ ..\ 3\bar{4}\ 3\bar{4}\ 2\bar{4}\ ..\ ..\ ..\ ..\ 3\bar{4}\ 2\bar{4}\ ..\ ..
\end{array}
$$

Complete BBS for $\widehat{sl}_n \ (n \geq 3)$

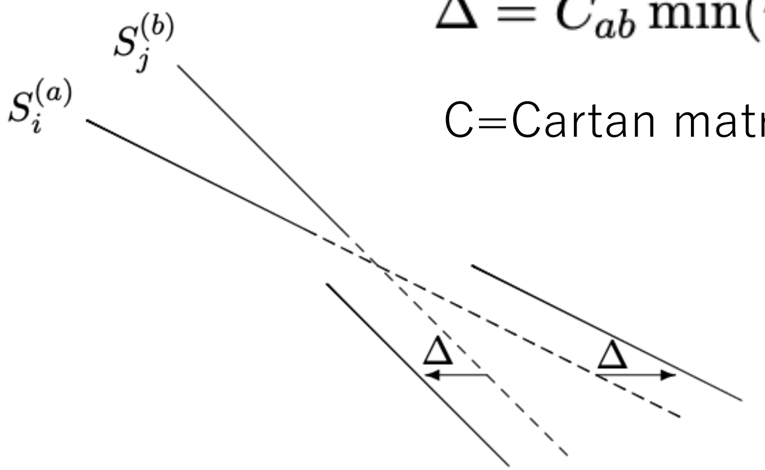
$T^{(k)}_\ell$



complete set of fundamental representations are built in

	Traditional $(n-1)$-color BBS (Takahashi '93)	Complete BBS (K-Misguich-Pasquier '21)
Vacuum	$\boxed{1} \otimes \boxed{1} \otimes \boxed{1} \otimes \cdots$	$\boxed{1} \otimes \begin{array}{\|c\|}\hline 1\\2\\\hline\end{array} \otimes \begin{array}{\|c\|}\hline 1\\2\\3\\\hline\end{array} \otimes \boxed{1} \otimes \begin{array}{\|c\|}\hline 1\\2\\\hline\end{array} \otimes \begin{array}{\|c\|}\hline 1\\2\\3\\\hline\end{array} \otimes \cdots$ $(\leftarrow n = 4\,\text{case})$
Dynamics	$T^{(1)}_\ell$ only	$T^{(k)}_\ell \ \ \forall k, \ell$
Solitons	$\widehat{sl}_{n-1}$-crystal	$(k, \ell) \in \{1, \ldots, n-1\} \times \mathbb{Z}_{\geq 1}$
S-matrix	combinatorial R of $\widehat{sl}_{n-1}$	Diagonal $(\rightarrow$ GHD$)$
Rigged conf.	higher-color part has implicit meaning.	every part is directly connected to solitons. Complete democracy achieved! Y(^,^)Y

Solitons undergo the phase shift only



$$\Delta = C_{ab}\min(i,j),$$

C=Cartan matrix.

Successive collisions of $S_3^{(3)}$ with $S_1^{(1)}$ and $S_2^{(2)}$ under the time evolution $T_l^{(3)}$ with $l \geq 3$.

$$, \ , \substack{1\\12\\124} , \substack{1\\12\\124} , \substack{1\\12\\124} , \ , \ , \ , \ , \substack{1\\12\\223} , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \substack{1\\12\\124} , \substack{1\\12\\124} , \substack{1\\12\\124} , \ , \substack{1\\12\\223} , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \ , \ , \ , \substack{1\\12\\124} , \substack{1\\12\\224} , \substack{1\\12\\124} , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \ , \ , \ , \ , \substack{1\\12\\223} , \ , \substack{1\\12\\124} , \substack{1\\12\\144} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \ , \ , \ , \ , \substack{1\\12\\223} , \ , \ , \ , \substack{1\\13\\144} , \substack{1\\12\\124} , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \ , \ , \ , \ , \substack{1\\12\\223} , \ , \ , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\134} , \substack{1\\12\\124} , \substack{1\\12\\124} , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \ , \ , \ , \ , \substack{1\\12\\223} , \ , \ , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \substack{1\\12\\124} , \substack{1\\12\\124} , \substack{1\\12\\124} , \ ,$$

Any state splits into solitons after enough time evolutions

$$, \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \substack{1\\12\\233} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \substack{1\\12\\323} , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \substack{1\\22\\133} , \ , \ , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \substack{1\\12\\233} , \ , \ , \ , \ , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \substack{1\\12\\223} , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \substack{1\\12\\133} , \ , \ , \ , \ ,$$

$$, \ , \ , \ , \ , \substack{1\\12\\223} , \ , \substack{1\\12\\133} , \ , \ , \ , \ , \ , \ , \ , \ , \ , \ , \substack{1\\12\\133} , \substack{1\\12\\133} , \substack{1\\12\\133} , \ ,$$

Duality with combinatorial Yang's system

$$B := \bigcup_{\rho \in \mathfrak{S}_L} B_{\ell_{\rho_1}} \otimes \cdots \otimes B_{\ell_{\rho_L}} \qquad \widehat{sl}_n\text{-crystal}$$

$(\ell_1, \dots, \ell_L; \text{fixed})$

$$\begin{tikzcd} B \arrow[r, "\tilde{W}(\widehat{sl}_L)"] \arrow[d, "\tilde{W}(\widehat{sl}_n)"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \\ B \arrow[r] & B \end{tikzcd}$$

$$\tilde{W}(\widehat{sl}_n) = \langle S_1, \dots, S_n, \sigma \rangle$$

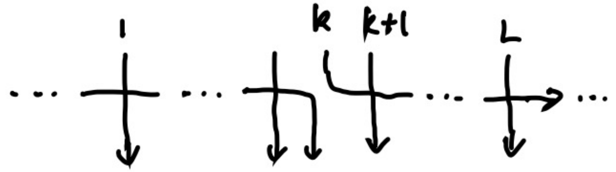
$$\tilde{W}(\widehat{sl}_L) = \langle R_1, \dots, R_L, P \rangle \qquad P = \text{cyclic shift}$$

R_k =comb. R acting on $k, (k+1)$th components

Translations in extended affine Weyl groups

$$\mathcal{T}_a = \sigma S_{a+1} \cdots S_{a+n} \qquad \mathcal{T}_a \mathcal{T}_b = \mathcal{T}_b \mathcal{T}_a$$

$$\mathcal{Y}_k = R_{k-1} \cdots R_1 P R_{L-1} \cdots R_k$$

$=$ 

$\mathcal{Y}_k \mathcal{Y}_\ell = \mathcal{Y}_\ell \mathcal{Y}_k$ (Yang '67)

dual picture $\updownarrow$

	dynamics	symmetry
periodic BBS	$\mathcal{T}_a$'s	$\mathcal{Y}_k$'s
combinatorial Yang's system	$\mathcal{Y}_k$'s	$\mathcal{T}_a$'s

Combinatorial Yang's dynamics on $B_4 \otimes B_3 \otimes B_2 \otimes B_1$

	$\mathcal{Y}_1^t(p)$	$\mathcal{Y}_3^t(p)$	$\mathcal{Y}_4^t(p)$
$t=0:$	$1134 \cdot 234 \cdot 24 \cdot 1$	$1134 \cdot 234 \cdot 24 \cdot 1$	$1134 \cdot 234 \cdot 24 \cdot 1$
$t=1:$	$1224 \cdot 113 \cdot 34 \cdot 4$	$1112 \cdot 334 \cdot 24 \cdot 4$	$1114 \cdot 233 \cdot 44 \cdot 2$
$t=2:$	$1144 \cdot 224 \cdot 13 \cdot 3$	$1244 \cdot 114 \cdot 33 \cdot 2$	$1112 \cdot 334 \cdot 24 \cdot 4$
$t=3:$	$1334 \cdot 114 \cdot 24 \cdot 2$	$1223 \cdot 444 \cdot 11 \cdot 3$	$1124 \cdot 134 \cdot 23 \cdot 4$
$t=4:$	$2244 \cdot 334 \cdot 11 \cdot 1$	$1233 \cdot 124 \cdot 44 \cdot 1$	$1244 \cdot 114 \cdot 33 \cdot 2$
$t=5:$	$1113 \cdot 224 \cdot 44 \cdot 3$	$1334 \cdot 112 \cdot 24 \cdot 4$	$1224 \cdot 144 \cdot 13 \cdot 3$

$$
\begin{tikzcd}
 & & 1112 \arrow[d] & & 334 \arrow[d] & & 24 \arrow[d] & & 4 \arrow[d] & & \\
\cdots \arrow[r] & 44 \arrow[r] & {} \arrow[r] & 11 \arrow[r] & {} \arrow[r] & 33 \arrow[r] & {} \arrow[r] & 24 \arrow[r] & {} \arrow[r] & 44 \arrow[r] & \cdots \\
 & & 1244 & & 114 & & 33 & & 2 & &
\end{tikzcd}
$$

For *periodic* BBS, stories are even more intriguing.

Review: Inoue-K-Takagi '12

A.K. "*Bethe Ansatz and Combinatorics*" (2011)
(monograph in Japanese, contents in English visible at
http://webpark1739.sakura.ne.jp/atsuo/Contents.pdf)