Ramdomized box-ball systems:
density plateaux, current correlations and large deviations

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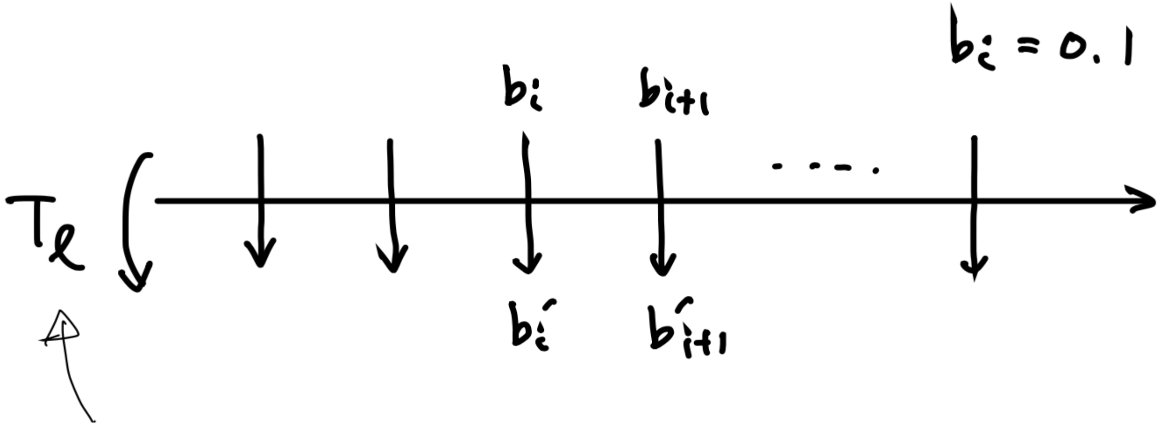
"*Box-Ball Systems from Integrable Systems and Probabilistic Perspectives*"
CRM Workshop, 23 September 2022

Based on

K-Lyu-Okado,
Randomized BBSs, limit shape of rigged configurations and TBA (2018)
K-Lyu,
Large deviations and one-sided scaling limit of multicolor BBS (2020)
K-Misguich-Pasquier,
GHD in BBS (2020)
GHD in complete BBS for $U_q(sl_n)$ (2021)
Current fluctuations, Drude weights and large deviations in a BBS (2022)

Box-ball system (single color)



Time evolution by capacity ℓ carrier .

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...00000000001110011100100000000000000000000000000...

...00000000000001100011011100000000000000000000000...

...00000000000000011000100011110000000000000000000...

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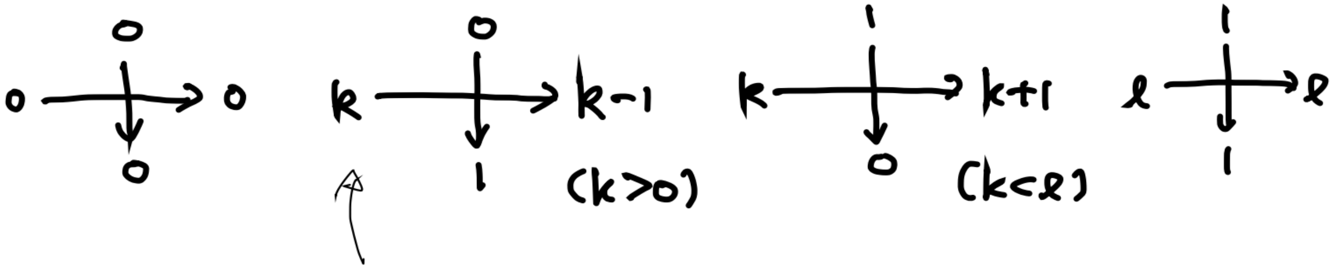
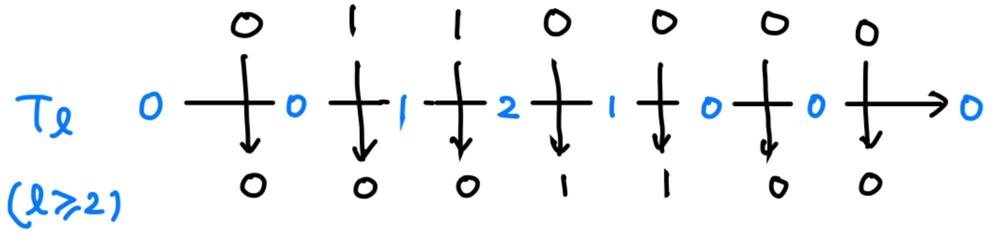


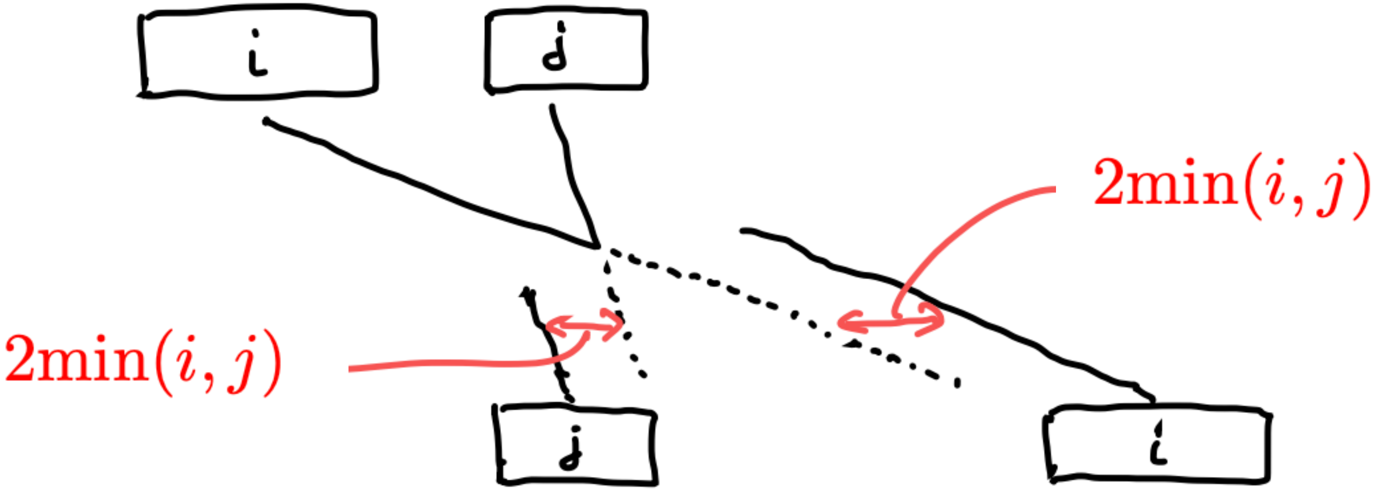
load of the carrier



$\cdots$ Combinatorial R of $\widehat{sl}_2$

T_ℓ = row transfer matrix with spin $\ell/2$ auxiliary space at $q = 0$

- $T_j T_\ell = T_\ell T_j$ $\cdots$ commuting dynamics
- $\cdots 00\overbrace{11\cdots 1}^{k}000\cdots$ ... k-soliton
- bare speed of k-soliton under T_ℓ is $\min(\ell, k)$.
- phase shift of i-soliton & j-soliton is $2\min(i,j)$.



Conserved quantities

$$E_1, E_2, E_3, \cdots E_\infty$$

$$E_i = \sum_j \min(i,j) m_j$$

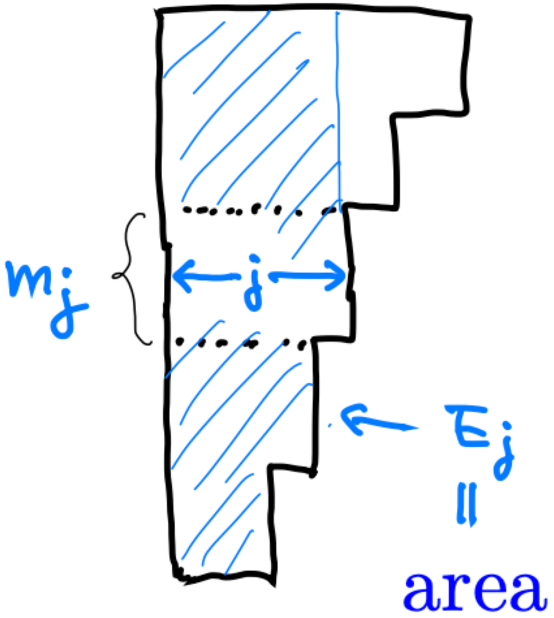
$\uparrow$

$\sharp(j\text{-soliton})$

$$E_1 = \sum_j m_j = \sharp(\text{soliton})$$

$$E_\infty = \sum_j j m_j = \sharp(\text{ball})$$

$$E_i(T_j(b)) = E_i(b) \qquad \forall i,j, \quad b = \text{BBS state}$$



$$\{E_j\} \longleftrightarrow \{m_j\}$$

energies

soliton contents

Generalized Gibbs ensemble (GGE)

$$Z_L = \sum_{\text{config}} \mathrm{e}^{-\beta_1 E_1 - \beta_2 E_2 - \cdots - \beta_\infty E_\infty}$$

system size

$$\underset{(L \to \infty)}{\simeq} \sum_{\{m_j\}} \mathrm{e}^{-\sum_i \beta_i E_i} \overbrace{\prod_i \binom{p_i + m_i}{m_i}}^{\text{Fermionic formula}}$$

sum over soliton contents

$$m_i = \sharp(i - \text{soliton})$$

$$p_i = \sharp(i - \text{hole}) = L - 2E_i \quad \text{(also called ``vacancy'')}$$

$$\mathcal{F} = -\lim_{L \to \infty} \frac{1}{L} \log Z_L \quad \cdots \text{free energy per site}$$

Randomized BBS in this talk

Randomness in the ensemble of initial states

Identical and independent (iid) distribution of balls with fugacity z

empty		ball	ball density
1	:	z	$\frac{z}{1+z}$

$\forall \beta_i = 0$ except β_∞ $\quad$ $z = \mathrm{e}^{-\beta_\infty}$

$0 < z < 1$ assumed throughout

Some results are also available for multi-temperature GGE.

Thermodynamic Bethe ansatz (TBA)

Assume $m_i \simeq L\rho_i, \quad p_i \simeq L\sigma_i$ as $L \to \infty$

$$p_i = L - 2E_i = L - 2\sum_j \min(i,j)m_j$$

$$\boxed{\overset{\text{hole}}{\sigma_i} = 1 - 2\sum_j \min(i,j)\overset{\text{string}}{\rho_j}} \quad \ldots \quad \text{“Bethe eq.”}$$

Equilibrium free energy per site by Stirling formula

$$\mathcal{F} = \left[\beta_\infty \sum_i i\rho_i - \sum_i \{(\rho_i + \sigma_i)\log(\rho_i + \sigma_i) - \rho_i \log \rho_i - \sigma_i \log \sigma_i\}\right]_{\min}$$

Equilibrium condition (TBA eq.)

$$\frac{\partial \mathcal{F}}{\partial \rho_i} = 0 \iff \begin{cases} \underline{y_i^{-2} = (1+y_{i-1}^{-1})(1+y_{i+1}^{-1})} & i \geq 1 \\ & (y_0^{-1} = 0) \\ \lim\limits_{i \to \infty} \dfrac{1+y_i^{-1}}{1+y_{i+1}^{-1}} = \mathrm{e}^{-\beta_\infty} = z & \end{cases}$$

for $\boxed{y_i = \dfrac{\rho_i}{\sigma_i}}$

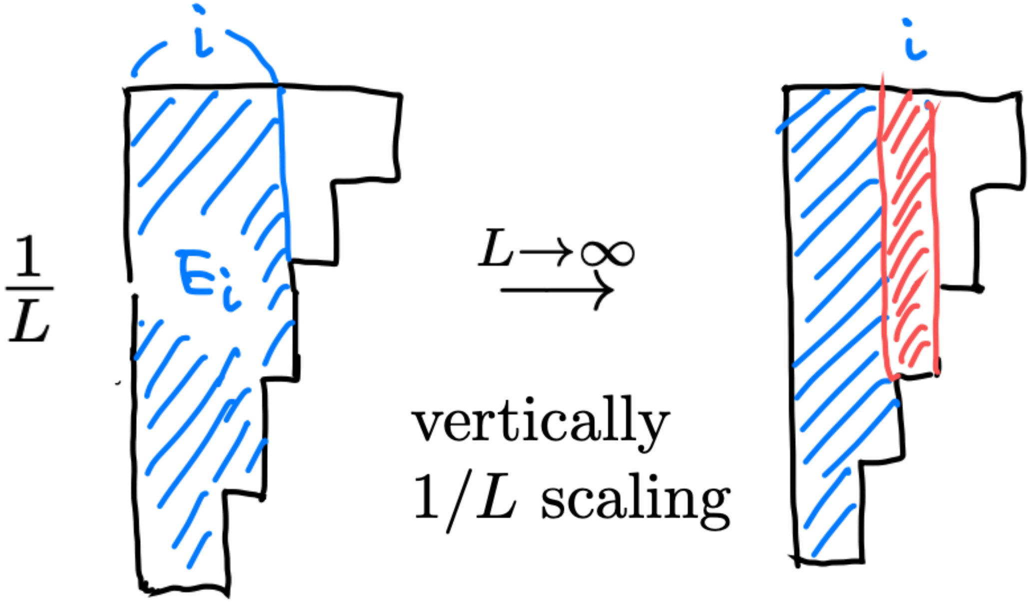
$\underline{\qquad\qquad}$ $\cdots$ algebraic form of TBA eq. called “Y-system”

It has been formulated for all the affine Lie algebras.
Many results from cluster algebra theory available.

Solution $$y_i = \frac{z^i(1-z)^2}{(1-z^i)(1-z^{i+2})}, \qquad \sigma_i = \frac{(1-z)(1+z^{i+1})}{(1+z)(1-z^{i+1})}$$

Recall $$p_i = L - 2E_i \xrightarrow{L\to\infty} \sigma_i = 1 - 2e_i$$

density of E_i



$\xrightarrow{L\to\infty}$

vertically
$1/L$ scaling

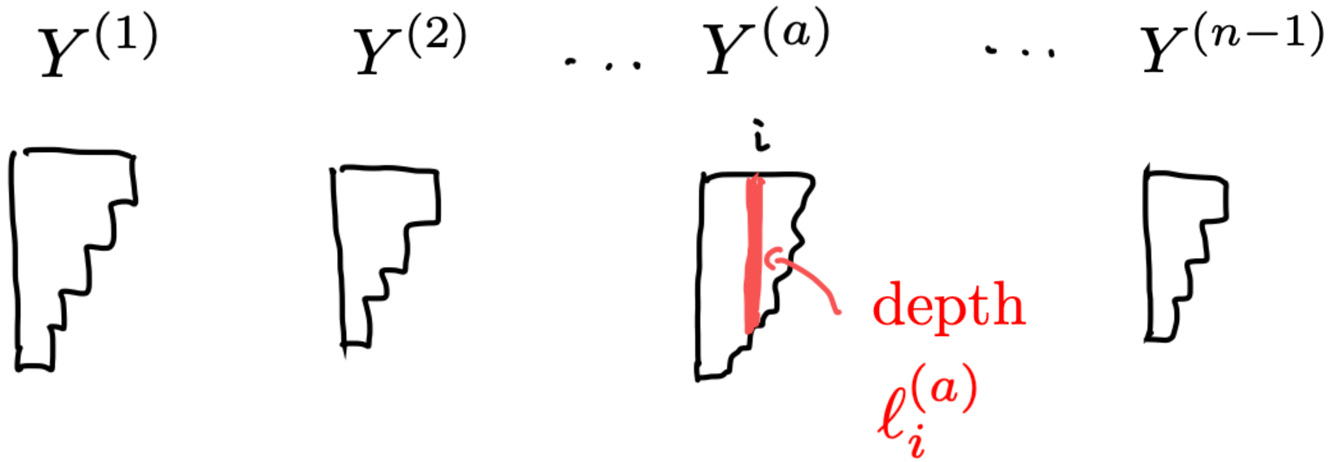
depth of ▨

$$= e_i - e_{i-1}$$
$$= \frac{1}{2}(\sigma_{i-1} - \sigma_i)$$
$$= \frac{z^i(1-z)^2}{(1+z)(1-z^i)(1-z^{i+1})}$$

⋯ Equilibrium limit shape of conserved Young diagram

$\widehat{sl}_n$ case: $(n-1)$-color BBS (Remark)

There are $(n-1)$-tuple of conserved Young diagrams



$$\lim_{L\to\infty}\frac{1}{L}\ell_i^{(a)} = \frac{S_{i-1}^{(a-1)}S_i^{(a+1)}}{S_i^{(a)}S_{i-1}^{(a)}S_1^{(1)}},$$

$$S_i^{(a)} = S_{\overset{i}{\boxed{\quad\quad}}\,a}(z_1,\dots,z_n)$$: Schur function for $a\times i$ rectangle

z_i = fugacity of color-i ball

$\widehat{sl}_3$ example $\quad \mathrm{prob}(1) = \frac{7}{18}, \ \mathrm{prob}(2) = \frac{6}{18}, \ \mathrm{prob}(3) = \frac{5}{18}$

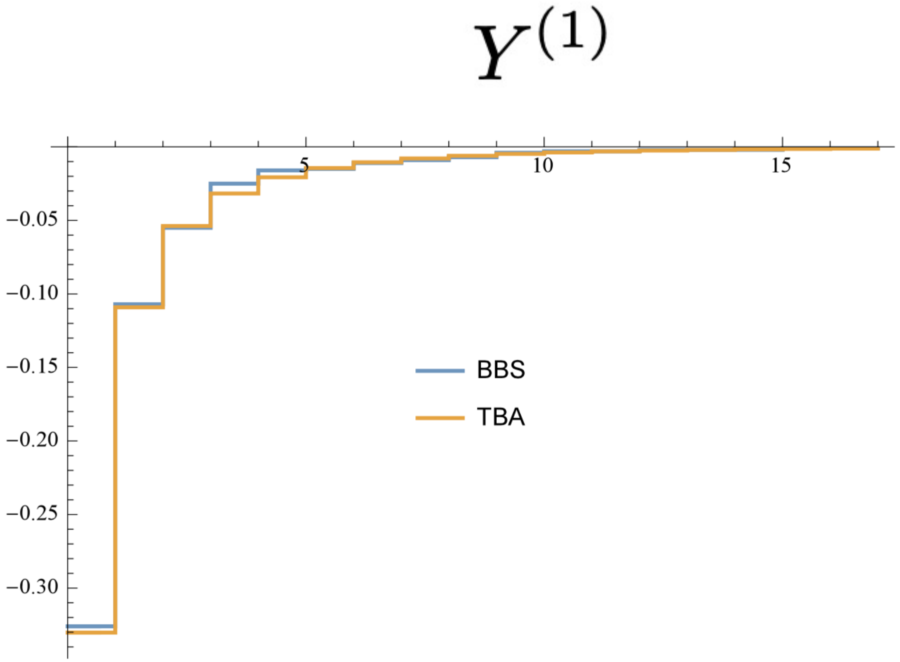

$Y^{(1)}$

Fig. 1. Vertically $1/L$ scaled Young diagram μ_1. $L = 1000$, $(p_1, p_2, p_3) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$. (For interpretatio
in the figure(s), the reader is referred to the web version of this article.)

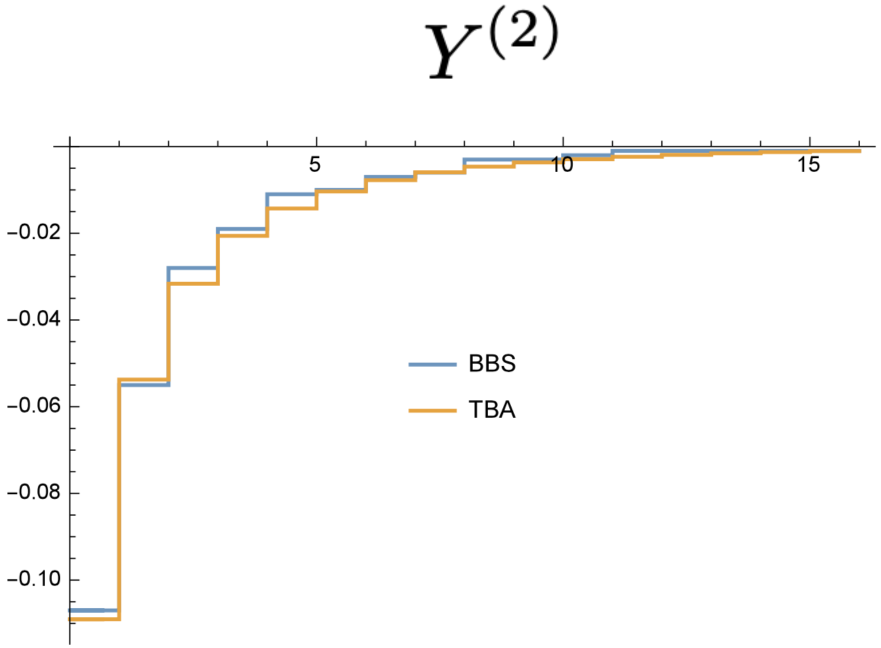

$Y^{(2)}$

Fig. 2. Vertically $1/L$ scaled Young diagram μ_2. $L = 1000$, $(p_1, p_2, p_3) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$.

Generalized hydrodynamics (GHD)

Assume $\rho_i, \sigma_i, y_i = \frac{\rho_i}{\sigma_i}$ are (r,t)-dependent

$$\left.\begin{array}{l}\bullet\ \text{Speed eq.}\\[1ex] \bullet\ \text{Continuity eq.}\end{array}\right\}\longrightarrow \text{separated eq. on } y_i \quad (\cdots\ \text{normal mode}).$$

Speed eq. for T_ℓ-dynamics $\quad v_i = v_i^{(\ell)}$

$$\boxed{v_i = \overset{\text{bare speed}}{\min(i,\ell)} + \sum_j \overbrace{2\min(i,j)}^{\text{phase shift}}\,\overbrace{(v_i - v_j)\rho_j}^{\sharp\text{collisions}}}$$

Bethe. eq

$$\sigma_i v_i + 2\sum_j \min(i,j)\underbrace{\rho_j v_j}_{\substack{\shortparallel \\ y_j\sigma_j v_j}} = \min(i,\ell)$$

Effective speed

speed eq. in matrix form $$(I + \underset{\underset{(2\min(i,j))_{i,j\geq 1}}{\nearrow}}{M}\underset{\underset{\mathrm{diag}(y_1, y_2, \ldots)}{\uparrow}}{\mathbf{y}})(\sigma * \underset{\underset{\begin{pmatrix} \sigma_1 v_1 \\ \sigma_2 v_2 \\ \vdots \end{pmatrix}}{\nwarrow}}{v}) = \underset{\underset{\begin{pmatrix} \min(1,\ell) \\ \min(2,\ell) \\ \vdots \end{pmatrix}}{\nwarrow}}{\kappa_\ell}$$

Bethe eq. $$(I + M\mathbf{y})\sigma = \kappa_1$$

$$v = \begin{pmatrix} v_1(\mathbf{y}) \\ v_2(\mathbf{y}) \\ \vdots \end{pmatrix} = \frac{(I + M\mathbf{y})^{-1}\kappa_\ell}{(I + M\mathbf{y})^{-1}\kappa_1}$$

$$\mathrm{e}^{\frac{\beta_1}{2}} = \frac{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}}, \quad \mathrm{e}^{-\beta_\infty} = z$$

two temperature GGE(a, z) case

$$\Downarrow$$

$$v_i = v_i^{(\ell)} = \sum_{k=1}^{\min(i,\ell)} \frac{\sigma_\ell}{\sigma_{k-1}\sigma_k} = \frac{1 + az^\ell}{1 - az^\ell}\left(\frac{1+a}{1-a}n - \frac{2a(1+z)(1-z^n)}{(1-a)(1-z)(1+az^n)}\right)\Bigg|_{a=z} \quad (n = \min(i, \ell))$$

$\cdots$ Result for the homogeneous system.

Continuity eq. for T_ℓ-dynamics

$$\underset{\underset{G\kappa_1}{\Uparrow}}{\partial_t \sigma_i} + \partial_r(\underset{\underset{G\kappa_\ell}{\Uparrow}}{\sigma_i} v_i) = 0 \qquad (t = t_\ell,\ v_i = v_i^{(\ell)})$$

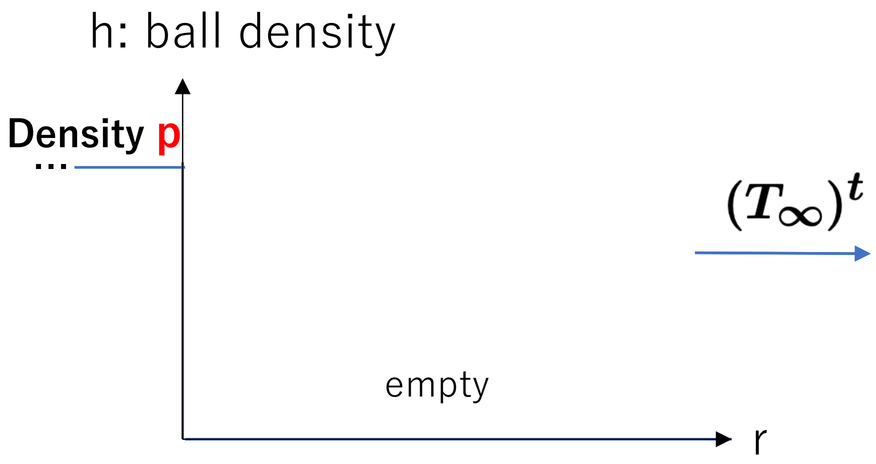
$$G = (I + M\mathbf{y})^{-1}$$

$$\partial_\alpha G = -GM(\partial_\alpha \mathbf{y})G \quad \text{leads to}$$

$$-GM(\partial_t \mathbf{y})\underset{\underset{\sigma}{\Uparrow}}{\underline{G\kappa_1}} - GM(\partial_r \mathbf{y})\underset{\underset{\sigma * v}{\Uparrow}}{\underline{G\kappa_\ell}} = 0$$

$$\boxed{\partial_t y_i + v_i \partial_r y_i = 0} \quad \cdots \quad \text{separated eq.}$$

Density Plateaux emerging from domain wall initial condition



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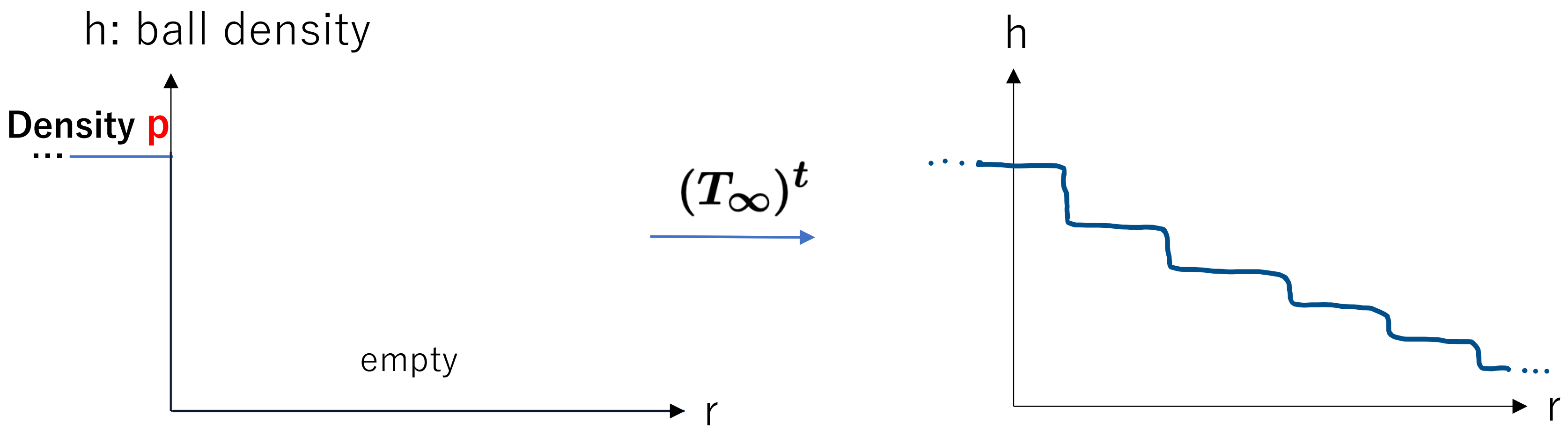

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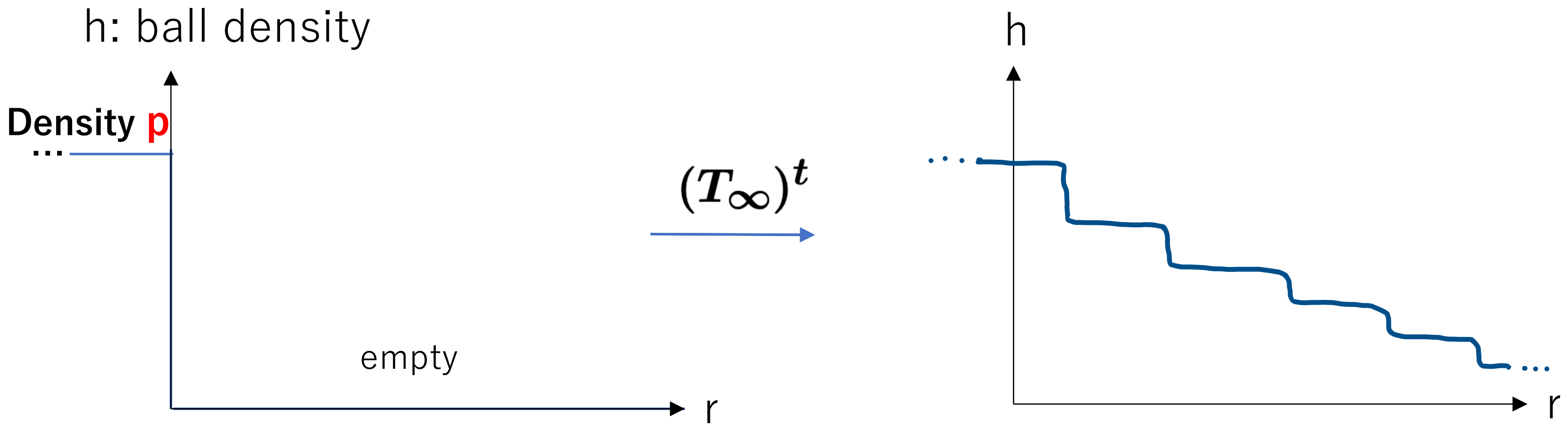

Plateaux broaden linearly in time t. The plot against $\zeta = r/t$ collapses into a single curve.

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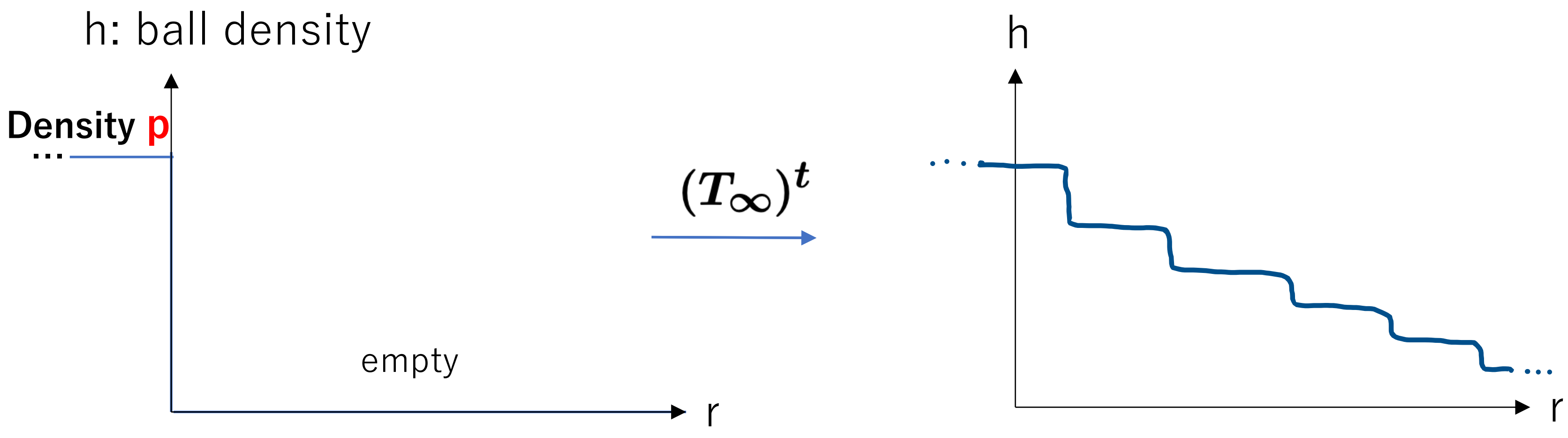

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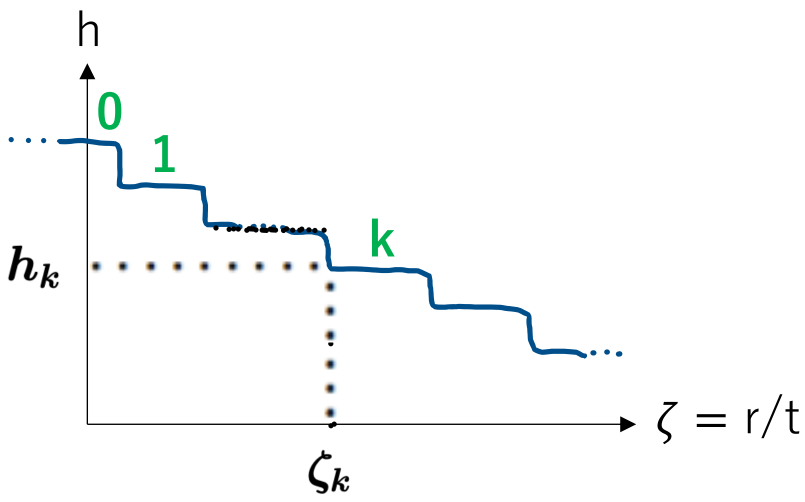

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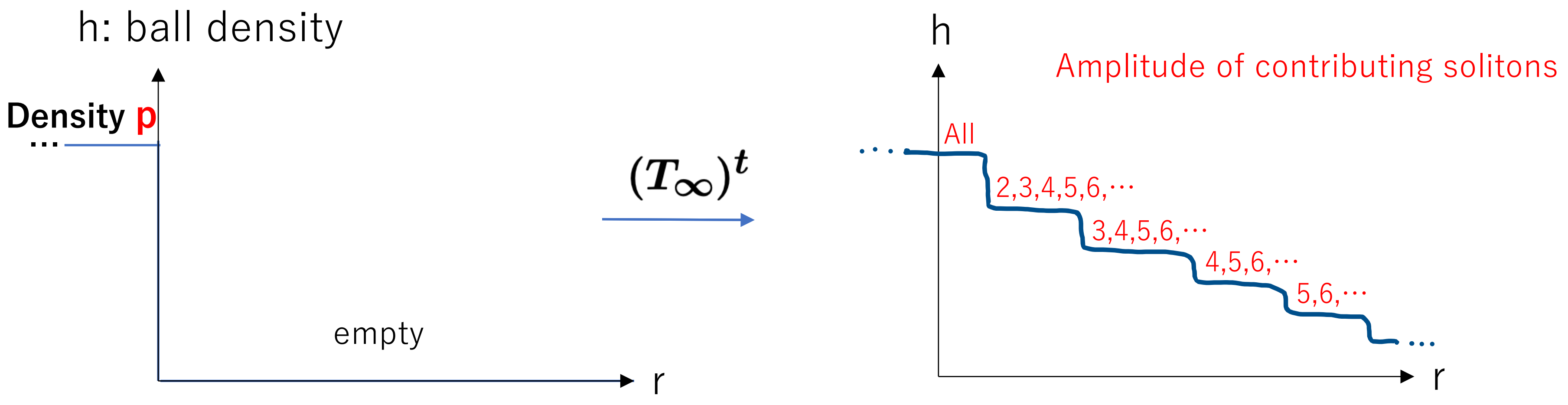

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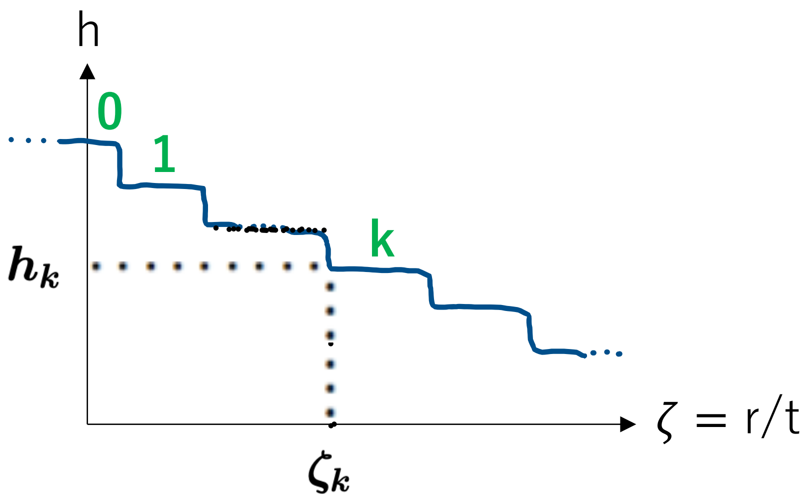

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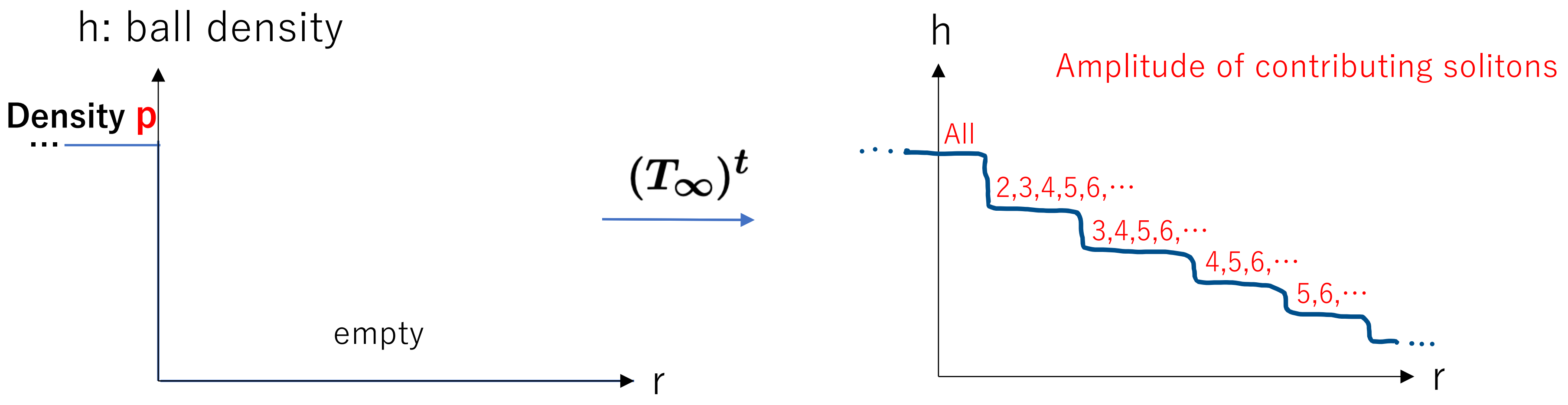

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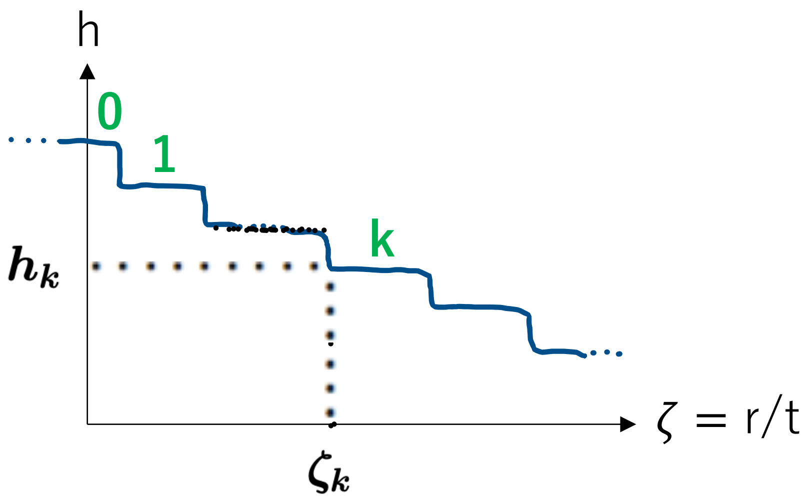


Ballistic approximation

→ h_k , ζ_k

$$\left(p = \frac{z}{1+z}\right)$$

Diffusive correction

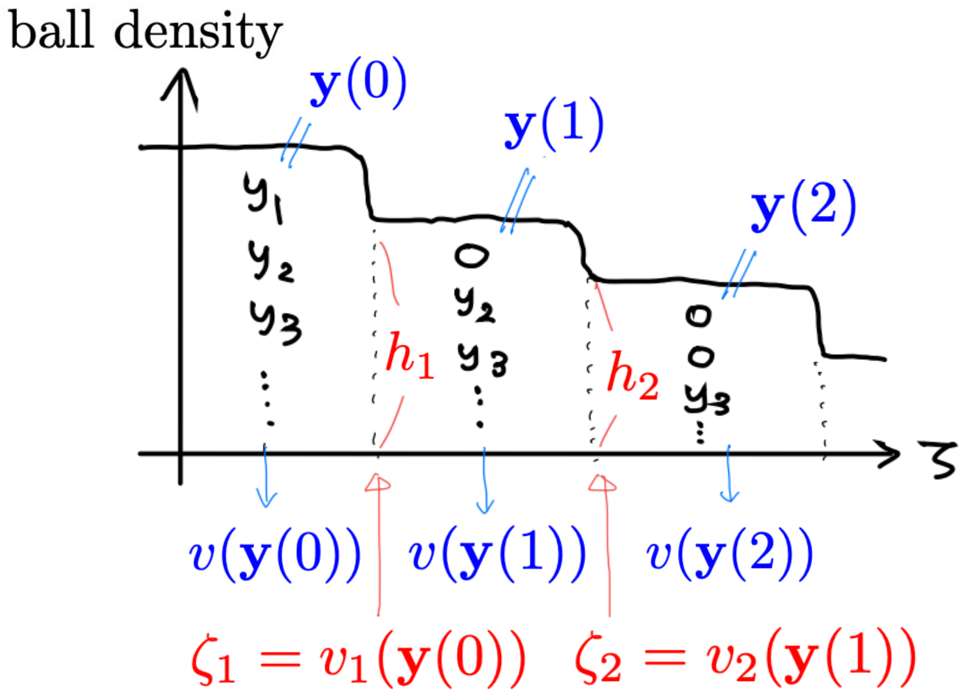
→ broadening of plateau edges

Ballistic approximation

jump discontinuity at v_i

$$y_i = y_i(r,t) \longrightarrow y_i(\zeta = r/t)$$

$$\text{separated eq.} \longrightarrow (\zeta - v_i)\partial_\zeta y_i = 0 \longrightarrow y_i(\zeta) = \begin{cases} y_i(-\infty) & \zeta < v_i \\ y_i(+\infty) & \zeta > v_i \end{cases}$$



$$h_1 = \sum_k k\rho_k(\mathbf{y}(1)) \qquad h_2 = \sum_k k\rho_k(\mathbf{y}(2))$$

$$h_k = \frac{z^{k+1}(1 - z^{k+2} + k(1-z))}{1 - z^{2k+3} + (2k+1)(1-z)z^{k+1}}$$

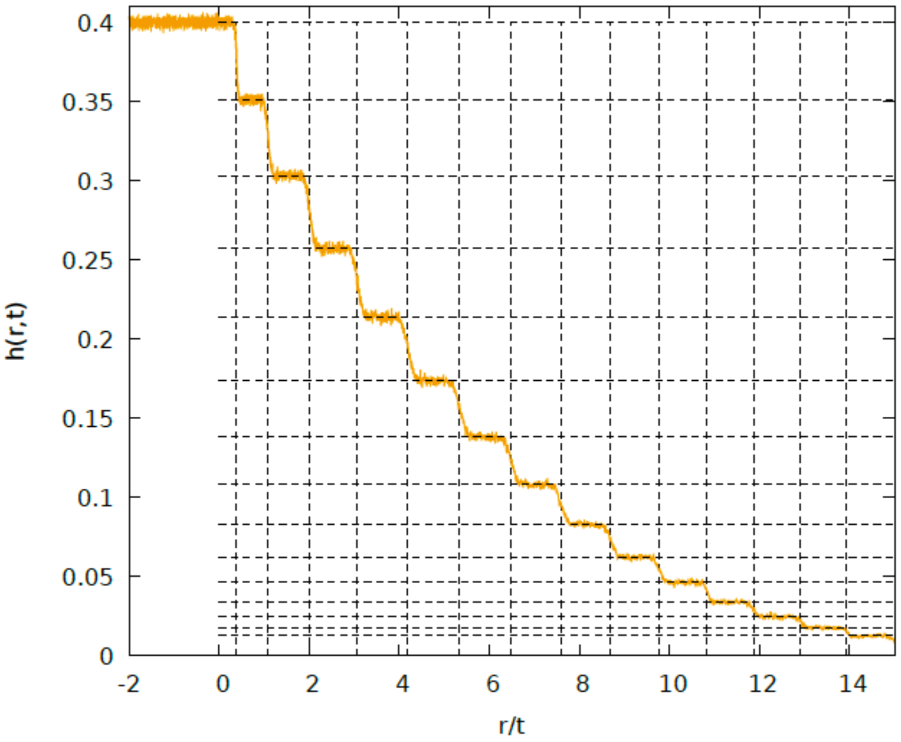
$$\zeta_k = \frac{k(1 - z^{k+1})(1 + z^{\ell+1})}{(1 + z^{k+1})(1 - z^{\ell+1})}$$

(for T_ℓ-dynamics)

Simulation with $N_{\text{samples}} = 50000$

(Plots of ball density vs $\zeta = r/t$. Dotted lines are GHD predictions)

p=0.4, q=0.666⋯, t=500.

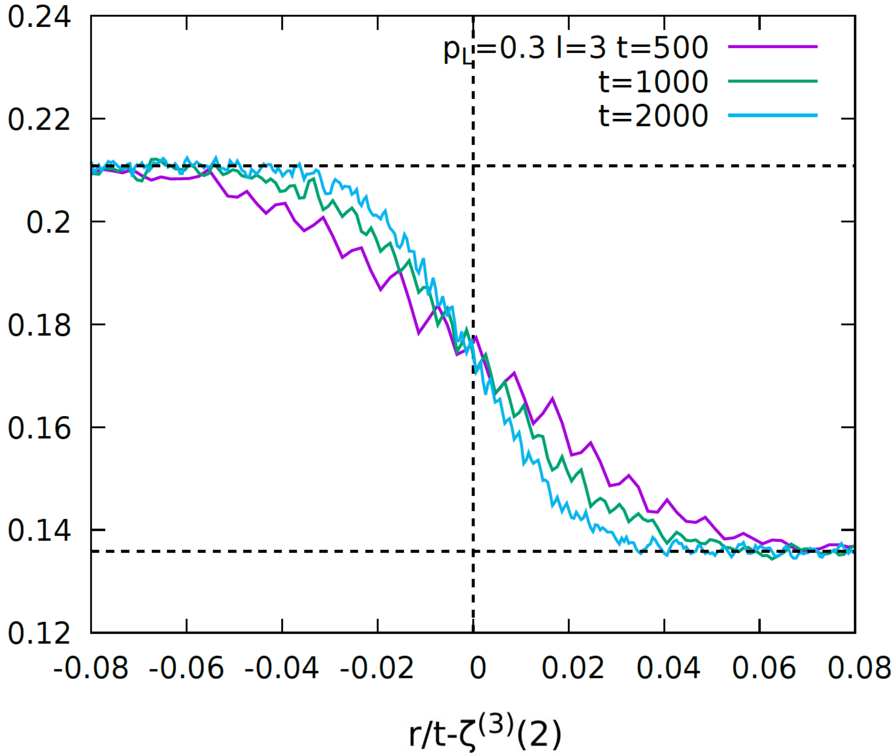


Actual plateau edges exhibit broadening, which may be viewed as a finite t effect.

Analytical description of the **diffusive broadening** of plateau edges

Position of plateau edge fluctuates over the scale

$$\frac{1}{\sqrt{(\textbf{Diffusion const})t}}$$



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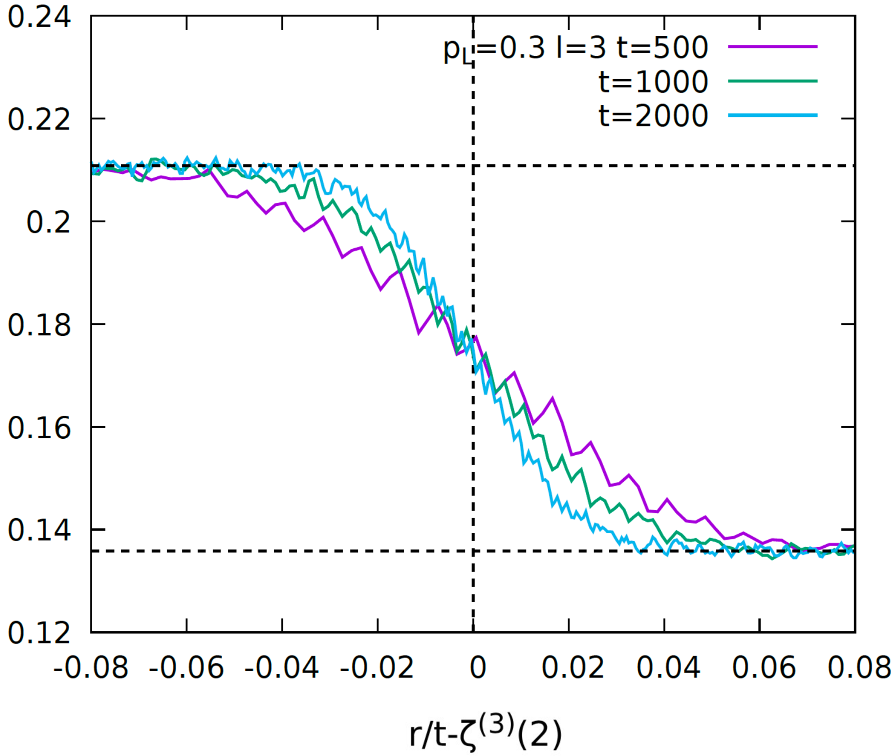

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \mathrm{e}^{-s^2} ds$$

$$\langle \boldsymbol{\rho_j(r,t)} \rangle = \frac{\mathbf{1}}{\mathbf{2}} (\boldsymbol{\rho_j(k-1) - \rho_j(k)}) \,\mathbf{erfc}\left(\frac{\boldsymbol{r - \zeta(k)t}}{\boldsymbol{\sqrt{2t}\,\Sigma_k^{(l)}}} \right) + \boldsymbol{\rho_j(k)}$$

averaged j-soliton density around the k th plateau edge $r = \zeta(k)t$ under the time evolution T_l.

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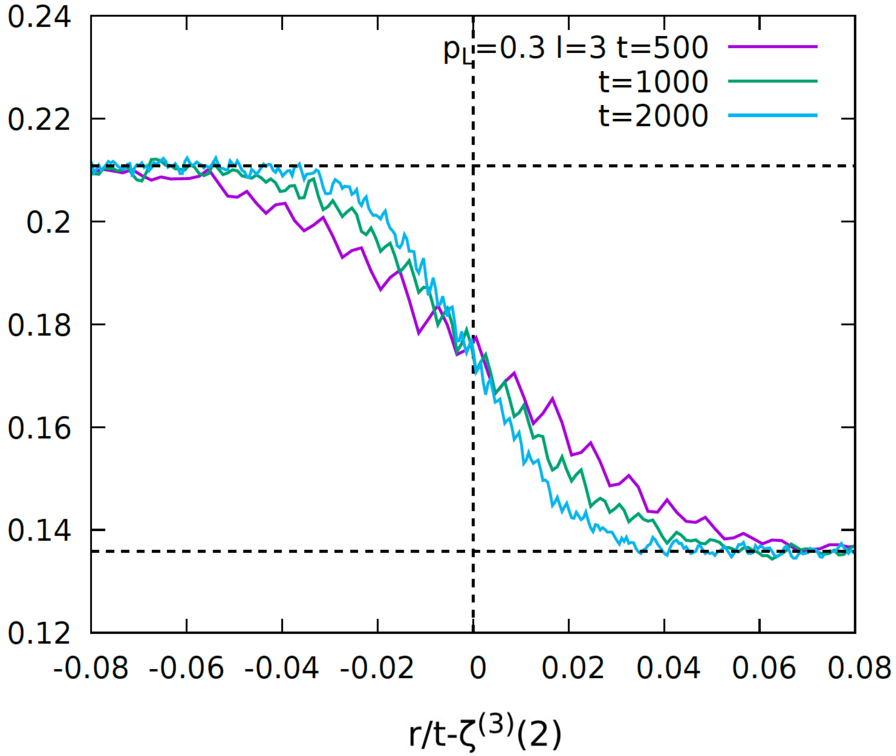

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$$2\left(\Sigma_k^{(l)}\right)^2 = \frac{8k^2 z^{k+1}(1-z^{k+1})(1-z^{l-k})(1+z^{l+k+2})}{(1+z^{k+1})^3(1-z^{l+1})^2}$$

← GHD (k<l)
(Bethe ansatz)

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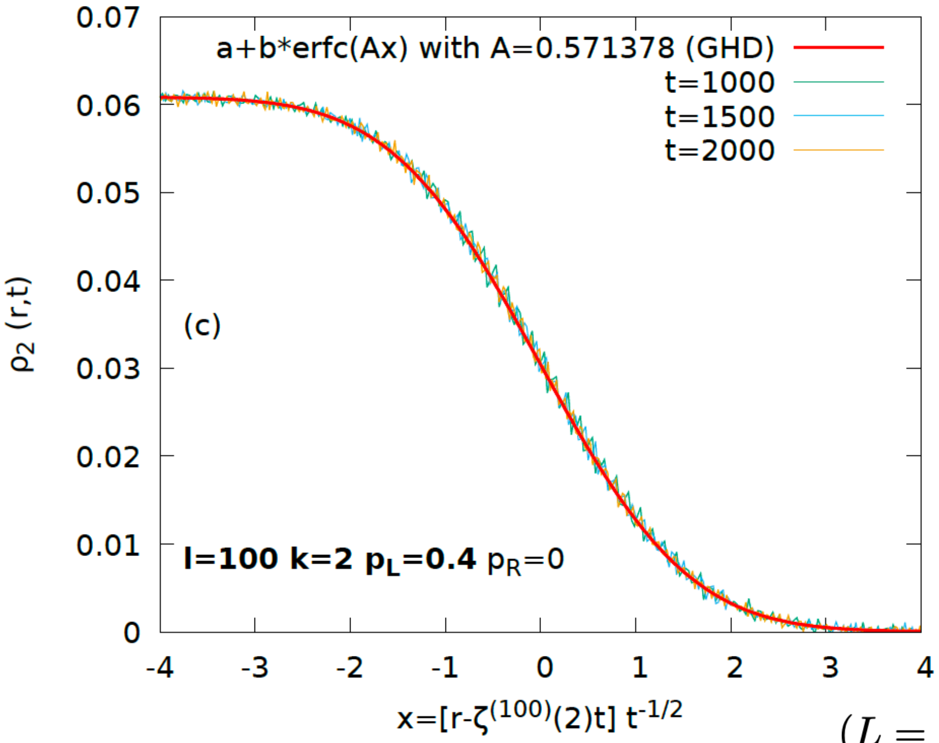

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$(L = 10^6,\ N_{\text{samples}} = 10^6)$

$$\langle \rho_j(r,t) \rangle = \frac{1}{2} (\rho_j(k-1) - \rho_j(k))\, \mathrm{erfc}\left(\frac{r - \zeta(k)t}{\sqrt{2t}\, \Sigma_k^{(l)}} \right) + \rho_j(k)$$

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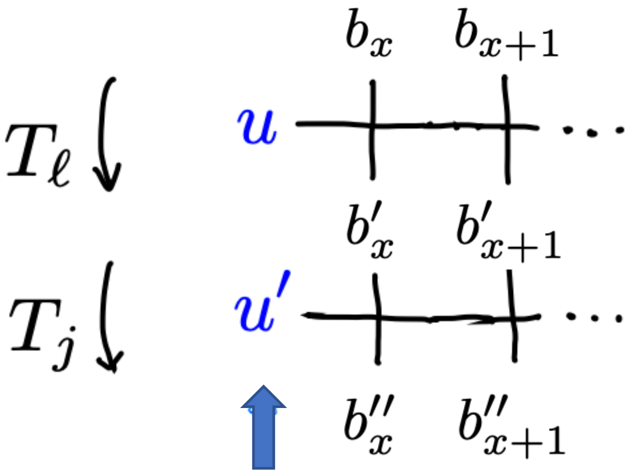
Generalized currents

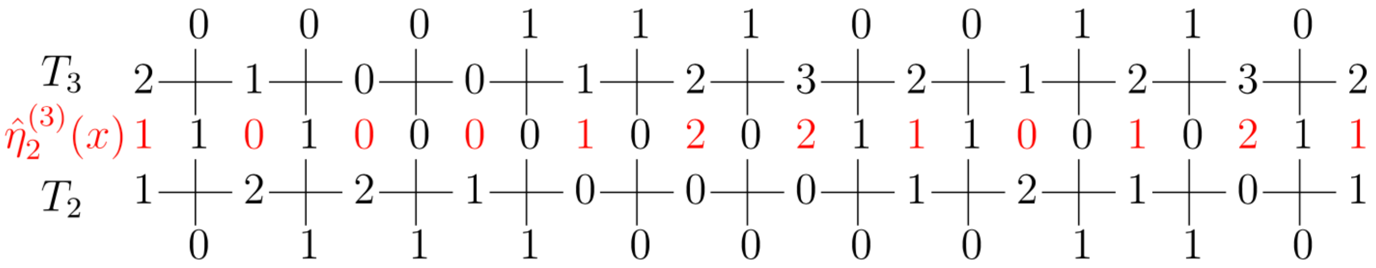
$\eta_j^{(\ell)}$:= current of energy E_j under the time evolution T_ℓ

$\eta_j^{(\ell)}(x)$:= local $\eta_j^{(\ell)}$ at x

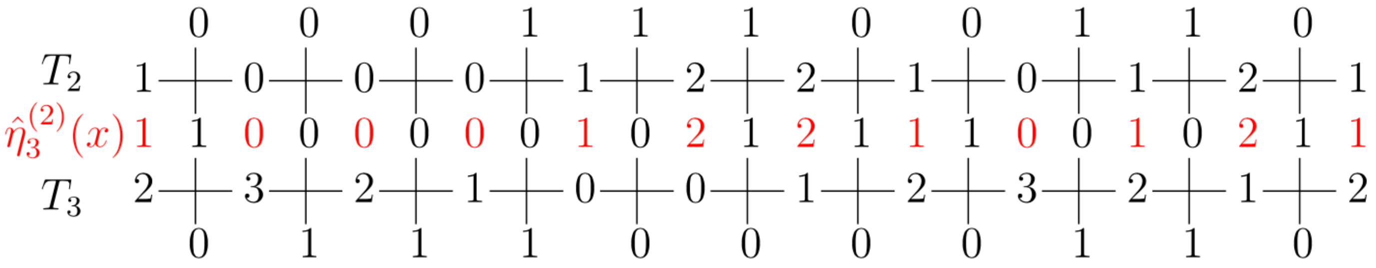


♯(ball) in the carriers

$$\boxed{\eta_j^{(\ell)} = \min(j-u', u)}$$



		0		0		0		1		1		1		0		0		1		1		0	
T_3	2		1		0		0		1		2		3		2		1		2		3		2
$\hat{\eta}_2^{(3)}(x)$	1	1	0	1	0	0	0	0	1	0	2	0	2	1	1	1	0	0	1	0	2	1	1
T_2	1		2		2		1		0		0		0		1		2		1		0		1
		0		1		1		1		0		0		0		0		1		1		0	



		0		0		0		1		1		1		0		0		1		1		0	
T_2	1		0		0		0		1		2		2		1		0		1		2		1
$\hat{\eta}_3^{(2)}(x)$	1	1	0	0	0	0	0	0	1	0	2	1	2	1	1	1	0	0	1	0	2	1	1
T_3	2		3		2		1		0		0		1		2		3		2		1		2
		0		1		1		1		0		0		0		0		1		1		0	

The symmetry $\eta_j^{(\ell)}(x) = \eta_\ell^{(j)}(x)$
holds as exemplified above.

Mean value in homogeneous system

$$\eta_j^{(\ell)} = \langle \eta_j^{(\ell)}(x) \rangle = \sum_k \min(j,k)\rho_k v_k^{(\ell)}$$

$$= \frac{z(1-z^{\min(j,\ell)})(1+z^{\max(j,\ell)})}{(1-z)(1-z^{j+1})(1-z^{\ell+1})} - \frac{\min(j,\ell)z(z^j+z^\ell)}{(1-z^{j+1})(1-z^{\ell+1})}$$

Special cases

$$\eta_k^{(1)} = \sum_k \min(j,k)\rho_k = \text{density of energy } E_j \quad (\because \color{red}{v_k^{(\ell=1)} = 1})$$

$$\eta_1^{(1)} = \sum_k \rho_k = \text{soliton density}, \quad \eta_\infty^{(1)} = \sum_k k\rho_k = \text{ball density}$$

$$\eta_\infty^{(\ell)} = \sum_k k\rho_k v_k^{(\ell)} = \text{ball current under } \ T_\ell$$

Time averaged correlations

$$C^{m,\ell,n}_{i,j} := \lim_{t_n \to \infty} \frac{1}{t_n} \int_0^{t_n} ds \sum_x \langle \eta^{(m)}_i (x,s) \eta^{(\ell)}_j (0,0) \rangle^c \overset{\substack{\text{numerically}\\ \text{confirmed}\\ \downarrow}}{=} \lim_{t_n \to \infty} \sum_x \langle \eta^{(m)}_i (x, \overset{\substack{T_n\text{-dynamics}\\ \downarrow}}{t_n}) \eta^{(\ell)}_j (0,0) \rangle^c$$

Conjecture: $C^{m,\ell,n}_{i,j}$ is n-independent for $n \geq \min(m,\ell) =: C^{m,\ell}_{i,j}$

Special cases

(1) $C^{1,1}_{\infty,\infty} = \lim \sum \langle \underset{\text{ball}}{\eta^{(1)}_\infty} \underset{\text{ball}}{\eta^{(1)}_\infty} \rangle^c = p(1-p)$ $\quad p$: ball density

(2) $C^{1,\ell}_{\infty,\infty} = \lim \sum_x \langle \underset{\text{ball}}{\eta^{(1)}_\infty} \underset{\text{current}}{\eta^{(\ell)}_\infty} \rangle^c = c_2$ $\quad$ 2nd cumulant of $\mathbf{N_t}$

$\mathbf{N_t} = \int_0^t \eta^{(\ell)}_\infty (0,s) ds = \sharp(\text{ball})$ passing through the origin during $[0,t]$ $\quad \langle \delta N_t^2 \rangle = \langle N_t^2 \rangle - \langle N_t \rangle^2 \simeq t c_2 \quad (t \to \infty)$

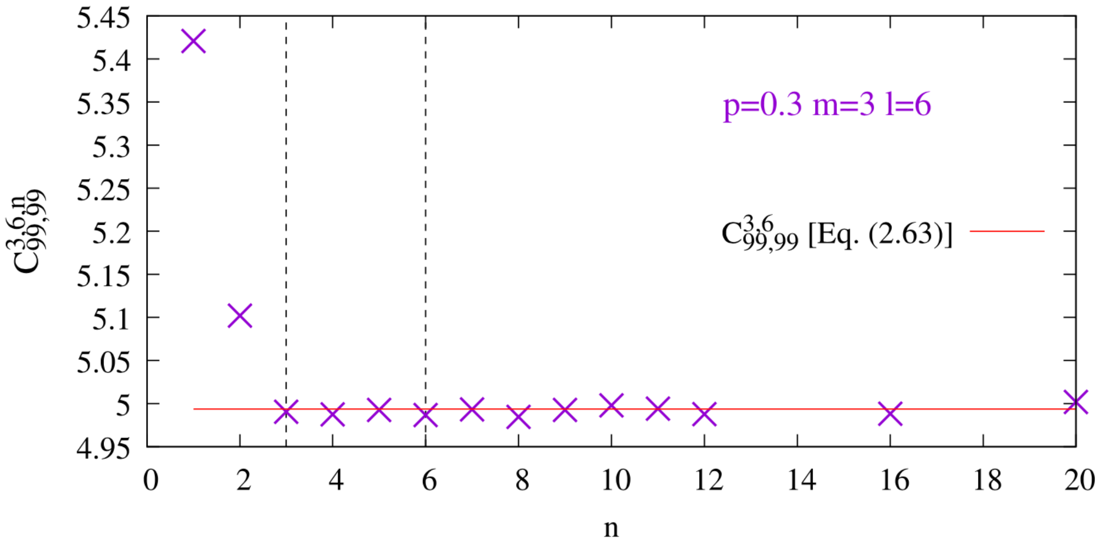
(3) $C^{\ell,\ell}_{\infty,\infty} = \lim \sum_x \langle \underset{\text{current}}{\eta^{(\ell)}_\infty} \underset{\text{current}}{\eta^{(\ell)}_\infty} \rangle^c = D$ $\quad$ Drude weight

TBA evaluation of $C_{i,j}^{m,\ell}$

$$\begin{cases} y_i = \rho_i/\sigma_i = \mathrm{e}^{-\varepsilon_i}, \qquad \langle \delta\varepsilon_i \delta\varepsilon_j \rangle = \delta_{ij}(\rho_i^{-1} + \sigma_i^{-1}) \quad \longleftarrow \left(\dfrac{\partial^2 \mathcal{F}}{\partial\varepsilon_i \partial\varepsilon_j} \right)^{-1} \text{ (free energy Hessian)} \\ \langle XY \rangle^c = \langle \delta X \delta Y \rangle = \displaystyle\sum_{j,k} \frac{\partial X}{\partial \varepsilon_j} \frac{\partial Y}{\partial \varepsilon_k} \langle \delta\varepsilon_j \delta\varepsilon_k \rangle = \sum_k \frac{\partial X}{\partial \varepsilon_k} \frac{\partial Y}{\partial \varepsilon_k} (\rho_k^{-1} + \sigma_k^{-1}) \\ \dfrac{\partial \eta_j^{(\ell)}}{\partial \varepsilon_k} = -\rho_k \sigma_k v_k^{(j)} v_k^{(\ell)} \end{cases}$$

$$\Longrightarrow \boxed{C_{i,j}^{m,\ell} = \textstyle\sum_k \rho_k \sigma_k (\rho_k + \sigma_k) v_k^{(i)} v_k^{(j)} v_k^{(\ell)} v_k^{(m)}}$$

$\cdots$ completely symmetric for i, j, ℓ, m.



Plot of $C_{99,99}^{3,6,n}$ vs n.

$n \geq \min(3,6) = 3$ case agrees with the above formula (red line).

$(p = 0.3, L = 3\times 10^4, t_n = 10^3, N_{\text{samples}} = 1.5\times 10^6)$

Scaled cumulant generating function

$(c_n = n\text{th cumulant})$

$$F(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log\langle e^{\lambda N_t}\rangle = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} c_n = \log\left(\frac{1-(ze^{\lambda})^{\ell+1}}{1-z^{\ell+1}} \frac{1-z}{1-ze^{\lambda}}\right) \quad \text{for } \; T_l - \text{dynamics}$$

Large deviation rate function

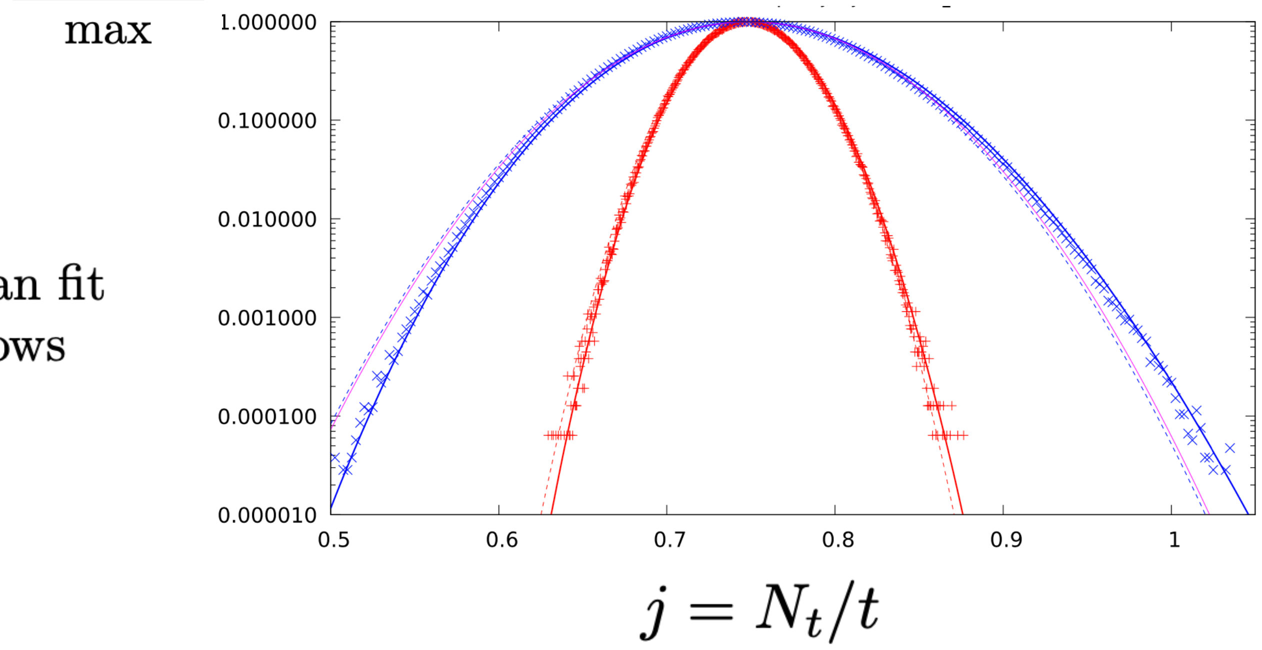
$$\mathrm{Prob}(N_t) \simeq \exp\Big(-tG(N_t/t)\Big)$$

$G(j) =$ Legendre transf. of $F(\lambda)$

Probability distribution of N_t for $t = 400$, $t = 2000$

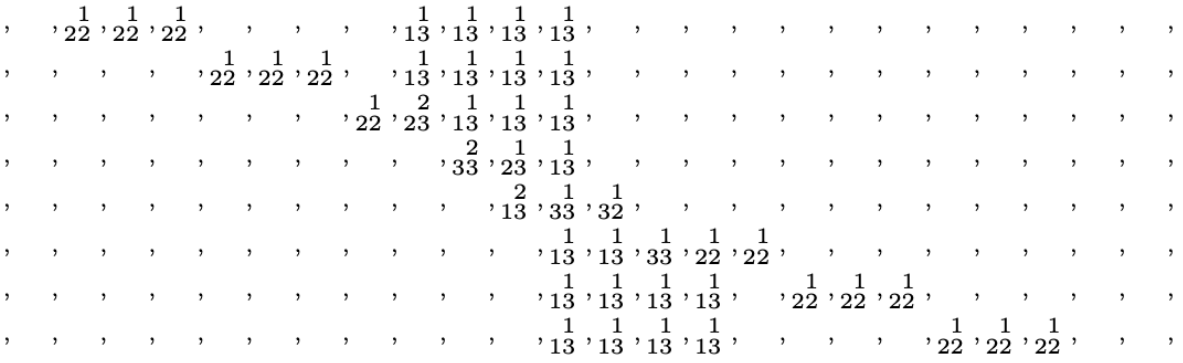
ball density $p = 0.3$, T_{10} dynamics, $\sim 10^6$ samples

$\langle j\rangle = 0.749, \; c_2 = 1.30.$

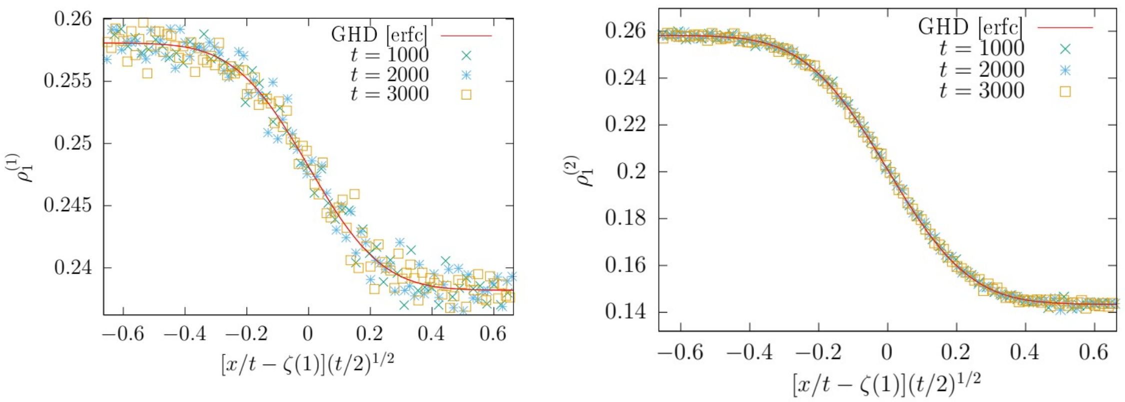


Numerical data deviates from the simple Gaussian fit $\exp(-(j-\langle j\rangle)^2/(2c_2))$ (magenta curve) and follows $\exp(-tG(j))$ for large deviations.

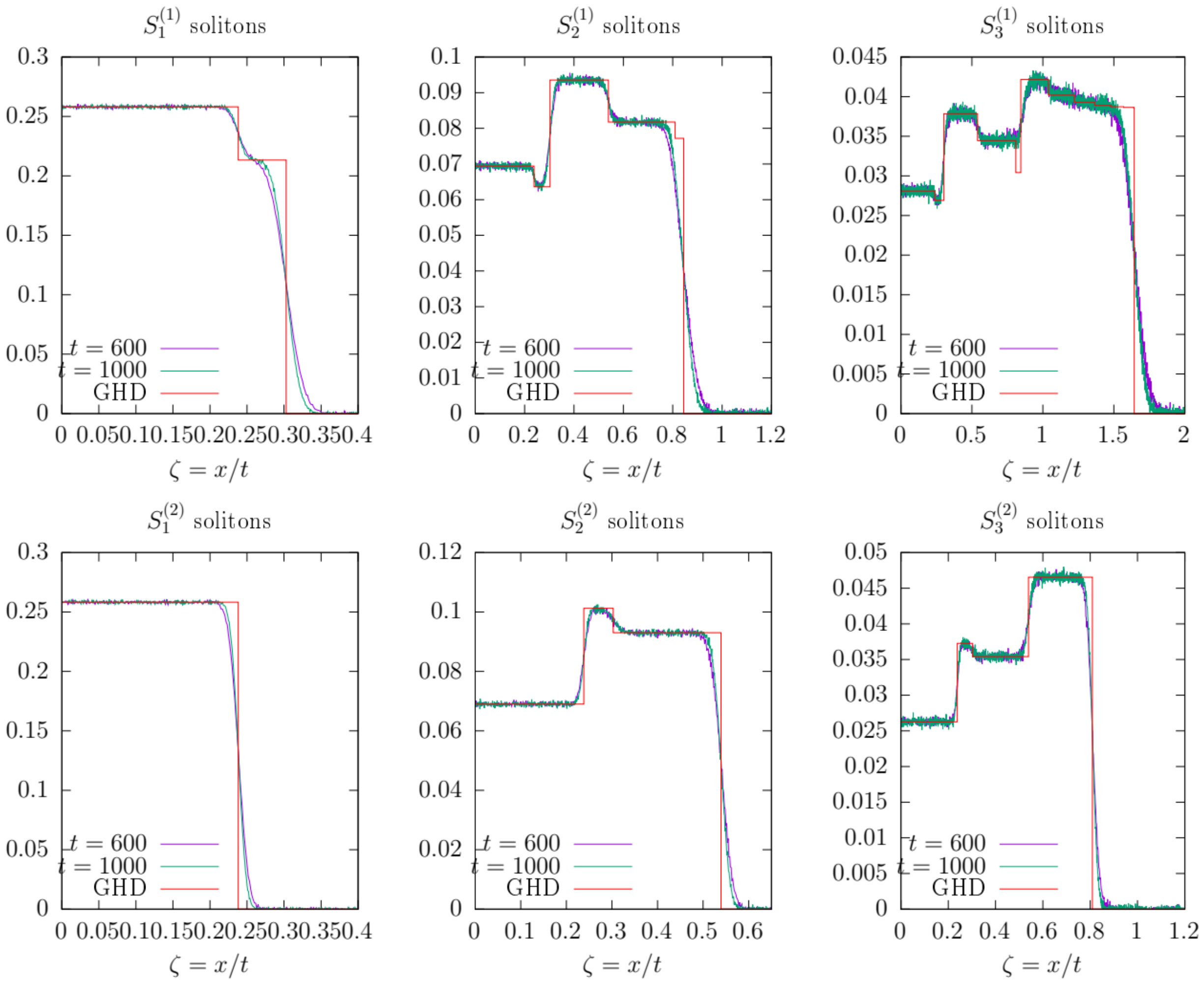
GHD for Complete BBS for $\widehat{sl}_3$



diagonal scattering of $S_3^{(1)}$ and $S_4^{(2)}$



Fit of diffusive broadening based on
$L \times t \times N_{\text{samples}} \simeq 10^6 \times 10^3 \times 10^5 \simeq 10^{14}$
applications of combinatorial R



density plateaux for solitons

GHD for the usual n-color BBS having non-diagonal S-matrix remains as a challenge.

$t=0$: 111122221111332114311111111111111111111111111111
$t=1$: 111111112222111332143111111111111111111111111111
$t=2$: 111111111111222211332431111111111111111111111111
$t=3$: 111111111111111122221132433111111111111111111111
$t=4$: 111111111111111111112221322433111111111111111111
$t=5$: 111111111111111111111112211322433211111111111111
$t=6$: 111111111111111111111111122111322143321111111111
$t=7$: 111111111111111111111111111221111322114332111111

How to set up the speed equation when the solitons change the “face” by collisions?

$$v_{2222}, v_{332}, v_{43} \longrightarrow v_{22}, v_{322}, v_{4332}$$