Generalized hydrodynamics in box-ball system

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(Joint with Grégoire Misguich and Vincent Pasquier)

Nordita, 12 March 2020

Q: Are there 1D deterministic cellular automata with the following features?

Commuting time evolutions, Family of conserved quantities,
Solitons obeying factorized scattering,
Equation of motion = discrete soliton equation,
Yang-Baxter equation, Bethe ansatz structure.

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provide exact quasi-particle picture to Bethe's formula for #{string solutions},
allow interpretation of corner transfer matrices as tau functions,
identify $2\pi i/\log(\text{Bethe eigenvalue})$ = Poincaré cycle, etc.

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Plan of the talk

- Basic features and integrability of BBS (Review)
- Recent progress on statistical aspects of randomized BBS in out of equilibrium

  1. Limit shape problem of soliton distributions
     by thermodynamic Bethe ansatz (TBA)

  2. Density plateaux emerging from domain wall initial conditions
     by generalized hydrodynamics (GHD)

n-color Box-ball system (BBS)

$n = 3$ example.

$$\cdots 00000000{\color{red}33211}0000000000000000000000000 \cdots$$
$$\cdots 0000000000000{\color{red}33211}00000000000000000000 \cdots$$
$$\cdots 000000000000000000{\color{red}33211}000000000000000 \cdots$$

0 =empty box, $\quad 1, 2, 3 =$ balls with colors

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$\cdots 00000000000003321100000000000000000000 \cdots$
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- time evolution = (move 1) · (move 2) · (move 3)

(move i) · Pick the leftmost ball with color i and move it to the nearest right empty box.

· Do the same for the other color i balls.

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· Do the same for the other color i balls.

- soliton=consecutive balls $i_1 \ldots i_a$ with color $i_1 \geq \cdots \geq i_a \geq 1$.
- velocity=amplitude.

- **Collisions of 2 solitons**

$$\cdots 00033211000032200000000000000000000000 \cdots$$
$$\cdots 00000000332110032200000000000000000000 \cdots$$
$$\cdots 00000000000003321132200000000000000000 \cdots$$
$$\cdots 00000000000000000021133322000000000000 \cdots$$
$$\cdots 00000000000000000000021100333220000000 \cdots$$
$$\cdots 00000000000000000000000021100003332200 \cdots$$

- **Amplitudes are individually conserved.**

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$$\cdots 00033211000032200000000000000000000000\cdots$$
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- **Amplitudes are individually conserved.**

- **Two body scattering:**

  Exchange of internal labels (colors) like quarks in hadrons

  Phase shift

Collision of 3 solitons

$\cdots$ 00321003100002000000000000000 $\cdots$
$\cdots$ 00000321031000200000000000000 $\cdots$
$\cdots$ 00000000320311020000000000000 $\cdots$
$\cdots$ 00000000003200312100000000000 $\cdots$
$\cdots$ 00000000000032001032100000000 $\cdots$
$\cdots$ 00000000000000320100032100000 $\cdots$
$\cdots$ 00000000000000003021000032100 $\cdots$

$\cdots$ 00321000031002000000000000000 $\cdots$
$\cdots$ 00000321000310200000000000000 $\cdots$
$\cdots$ 00000000321003120000000000000 $\cdots$
$\cdots$ 00000000000321013200000000000 $\cdots$
$\cdots$ 00000000000000302132100000000 $\cdots$
$\cdots$ 00000000000000030021032100000 $\cdots$
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$\cdots$ 00000000000000003000210032100 $\cdots$

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

Double (classical and quantum) origin of integrability

(1) Ultra-Discretization (UD) of soliton equations

Double (classical and quantum) origin of integrability

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- Key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) + \exp(\frac{b}{\varepsilon}) \right) = \max(a, b)$$

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) \times \exp(\frac{b}{\varepsilon}) \right) = a + b$$

$$(+, \times) \longrightarrow (\max, +)$$

keeps distributive law:

$$AB + AC = A(B + C) \rightarrow \max(a + b, a + c) = a + \max(b, c)$$

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$$AB+AC=A(B+C)\to\max(a+b,a+c)=a+\max(b,c)$$

- UD of a discrete KdV equation gives an evolution equation of the $n=1$ BBS (1996).

(2) Solvable lattice model at “ Temperature 0 ”

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Time evolution pattern

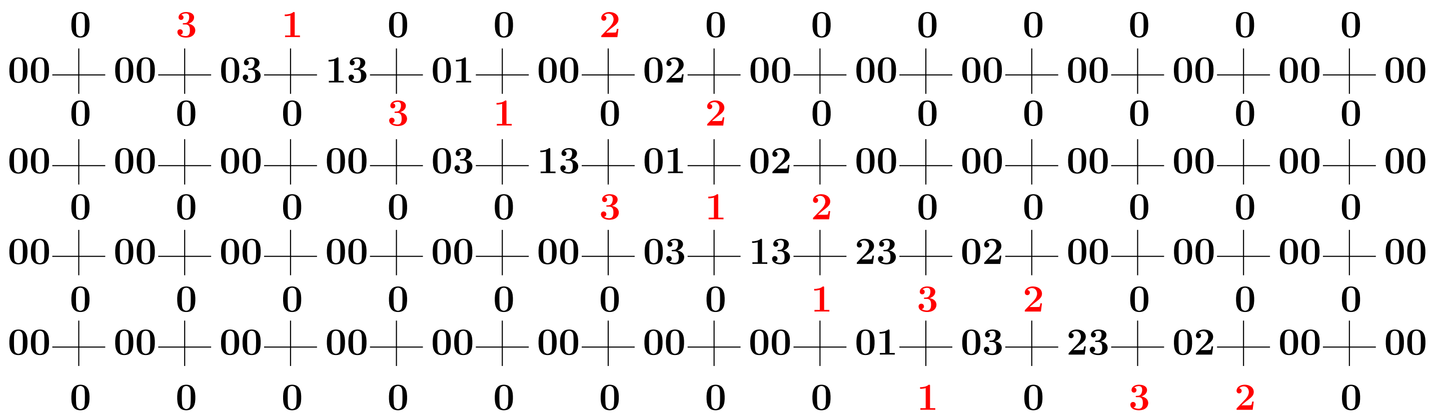
$$\begin{array}{c}
\cdots 0\mathbf{31}00\mathbf{2}0000000 \cdots \\
\cdots 000\mathbf{31}0\mathbf{2}000000 \cdots \\
\cdots 00000\mathbf{312}00000 \cdots \\
\cdots 0000000\mathbf{132}000 \cdots \\
\cdots 00000000\mathbf{1}0\mathbf{32}0 \cdots
\end{array}$$

(2) Solvable lattice model at “ Temperature 0 ”

Time evolution pattern

$$\cdots 0310020000000 \cdots$$
$$\cdots 0003102000000 \cdots$$
$$\cdots 0000031200000 \cdots$$
$$\cdots 0000000132000 \cdots$$
$$\cdots 0000000010320 \cdots$$

emerges from a configuration of a 2D lattice model in statistical mechanics



by forgetting the hidden variables on the horizontal edges.

- n-color box-ball system

  = 2D solvable vertex model associated with quantum group

  $$U_q(\widehat{sl}_{n+1}) \text{ at } q = 0 \quad (q \sim \text{temperature})$$

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  $U_q(\widehat{sl}_{n+1})$ at $q = 0$ ($q \sim$ temperature)

- Row transfer matrix at $q = 0$

  = deterministic map (defined by the unique configuration surviving at $q = 0$)

  = time evolution of box-ball system (forming a commuting family $T_1, T_2, \ldots T_\infty$)

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  = deterministic map (defined by the unique configuration surviving at $q = 0$)

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- Proper formulation uses *crystal base theory* (theory of quantum group at $q = 0$).

Some outcomes from such insight

- ∃ Integrable cellular automata with quantum group symmetry.

Example: $\widehat{\mathrm{so}}_{10}$-automaton

$$\cdots 000\bar{2}\bar{4}211\,0000\,\bar{1}\bar{1}\bar{4}\,00000000000000000000000000000000 \cdots$$
$$\cdots 00000000\,\bar{2}\bar{4}211\,00\,\bar{1}\bar{1}\bar{4}\,00000000000000000000000000000 \cdots$$
$$\cdots 0000000000000\,\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4}\,00000000000000000000000000 \cdots$$
$$\cdots 000000000000000000\,\bar{2}\bar{4}2\bar{0}0\bar{4}\,00000000000000000000000 \cdots$$
$$\cdots 00000000000000000000000\,3\bar{0}\bar{0}\bar{3}\bar{4}\bar{4}\,000000000000000000 \cdots$$
$$\cdots 00000000000000000000000000\,3111\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}\,0000000000000 \cdots$$
$$\cdots 00000000000000000000000000000\,31100\,\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}\,00000000 \cdots$$
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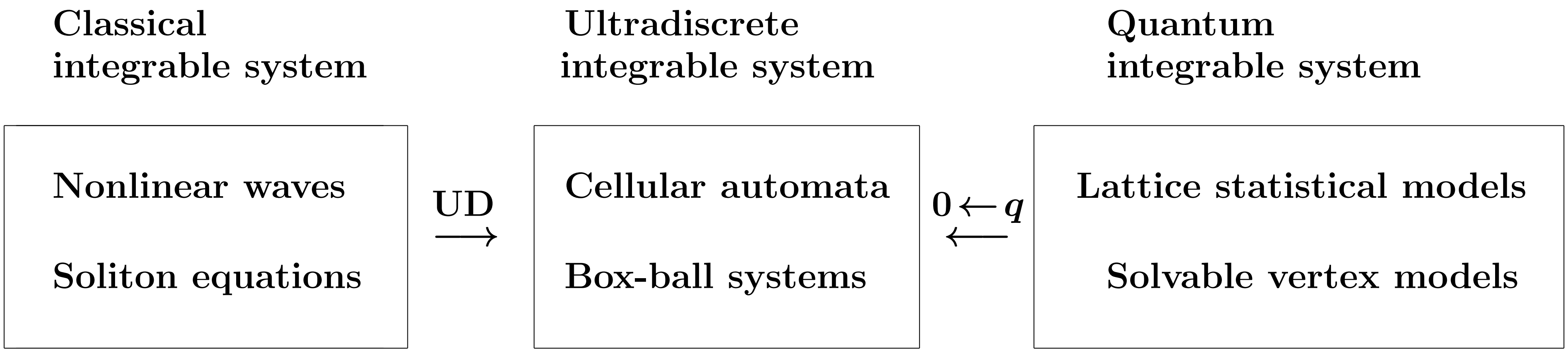
$$\cdots 000000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4}00000000000000000000000 \cdots$$
$$\cdots 000000000000000000000003\bar{0}\bar{0}\bar{3}\bar{4}\bar{4}000000000000000000 \cdots$$
$$\cdots 00000000000000000000000000311\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}0000000000000 \cdots$$
$$\cdots 0000000000000000000000000000031100\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}00000000 \cdots$$
$$\cdots 000000000000000000000000000000003110000\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}000 \cdots$$

- Particles and antiparticles undergo pair-creations/annihilations.

- n-color BBS $= \widehat{sl}_{n+1}$-automaton $= \widehat{\mathrm{so}}_{2n+2}$-automaton in antipaticle-free sector.

- Soliton & scattering most naturally captured in quantum group framework.


Classical
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Ultradiscrete
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Quantum
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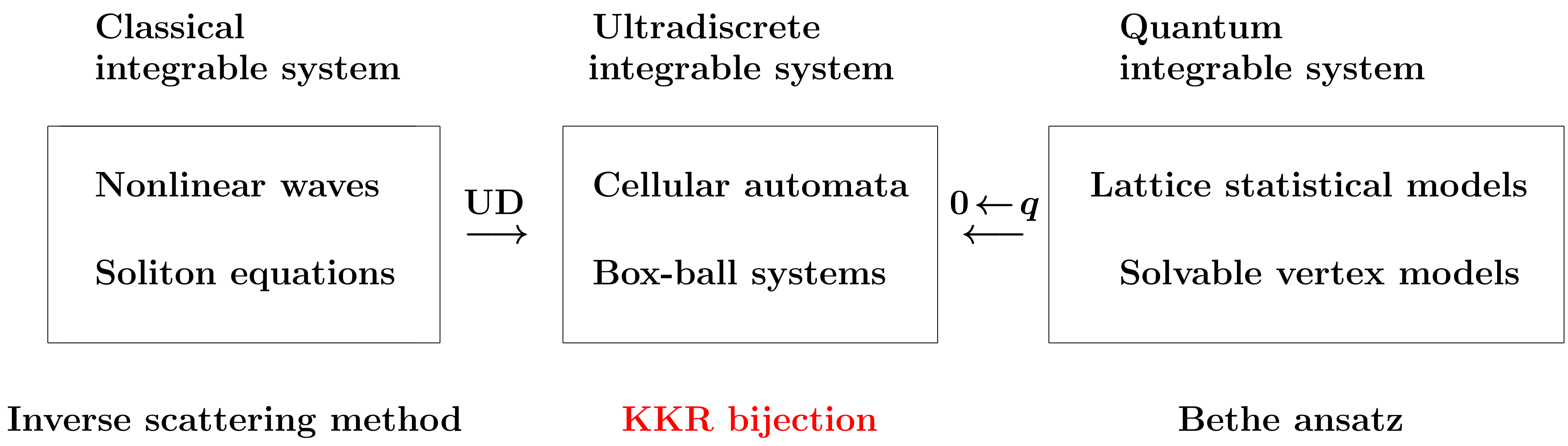
Nonlinear waves
Soliton equations
UD
Cellular automata
Box-ball systems
0 ← q
Lattice statistical models
Solvable vertex models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Inverse scattering method
KKR bijection
Bethe ansatz

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Nonlinear waves Soliton equations	$\xrightarrow{\text{UD}}$	Cellular automata Box-ball systems	$\xleftarrow{0 \leftarrow q}$	Lattice statistical models Solvable vertex models
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- Kerov-Kirillov-Reshetikhin (KKR) bijection (1986) asserts “formal completeness” of the hypothetical string solutions to the Bethe equation at combinatorial level.

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	UD $\longrightarrow$	Cellular automata Box-ball systems	$0 \leftarrow q$ $\longleftarrow$	Lattice statistical models Solvable vertex models
Inverse scattering method		KKR bijection		Bethe ansatz

- Kerov-Kirillov-Reshetikhin (KKR) bijection (1986) asserts “formal completeness” of the hypothetical string solutions to the Bethe equation at combinatorial level.

- Its remarkable connection to BBS was discovered in 2002.

- **Example. Spin $\frac{1}{2}$ periodic Heisenberg chain**

$$H = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)$$

For $L = 6$ sites in 3 down-spin sector, the Bethe equation reads

$$\left(\frac{u_1 + i}{u_1 - i}\right)^6 = \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)},$$
$$\left(\frac{u_2 + i}{u_2 - i}\right)^6 = \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)},$$
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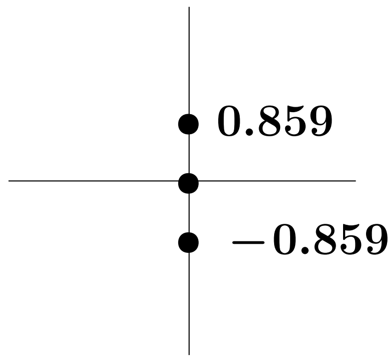
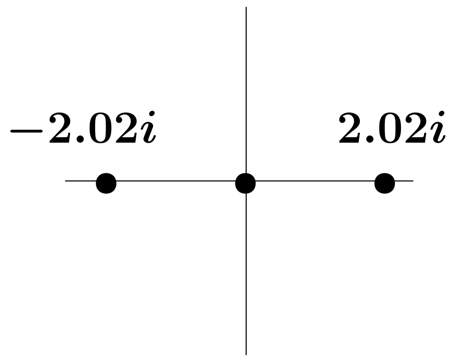
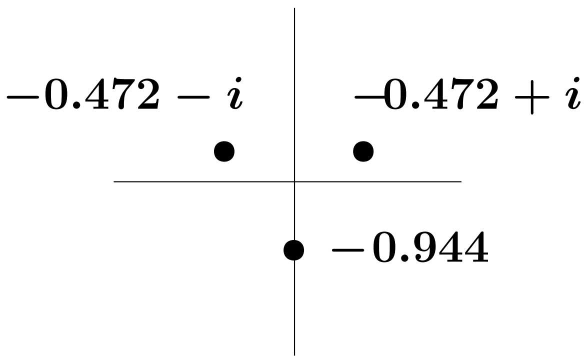
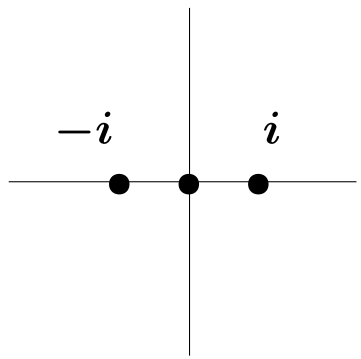
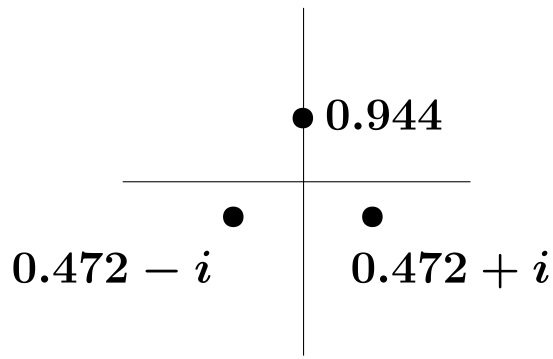
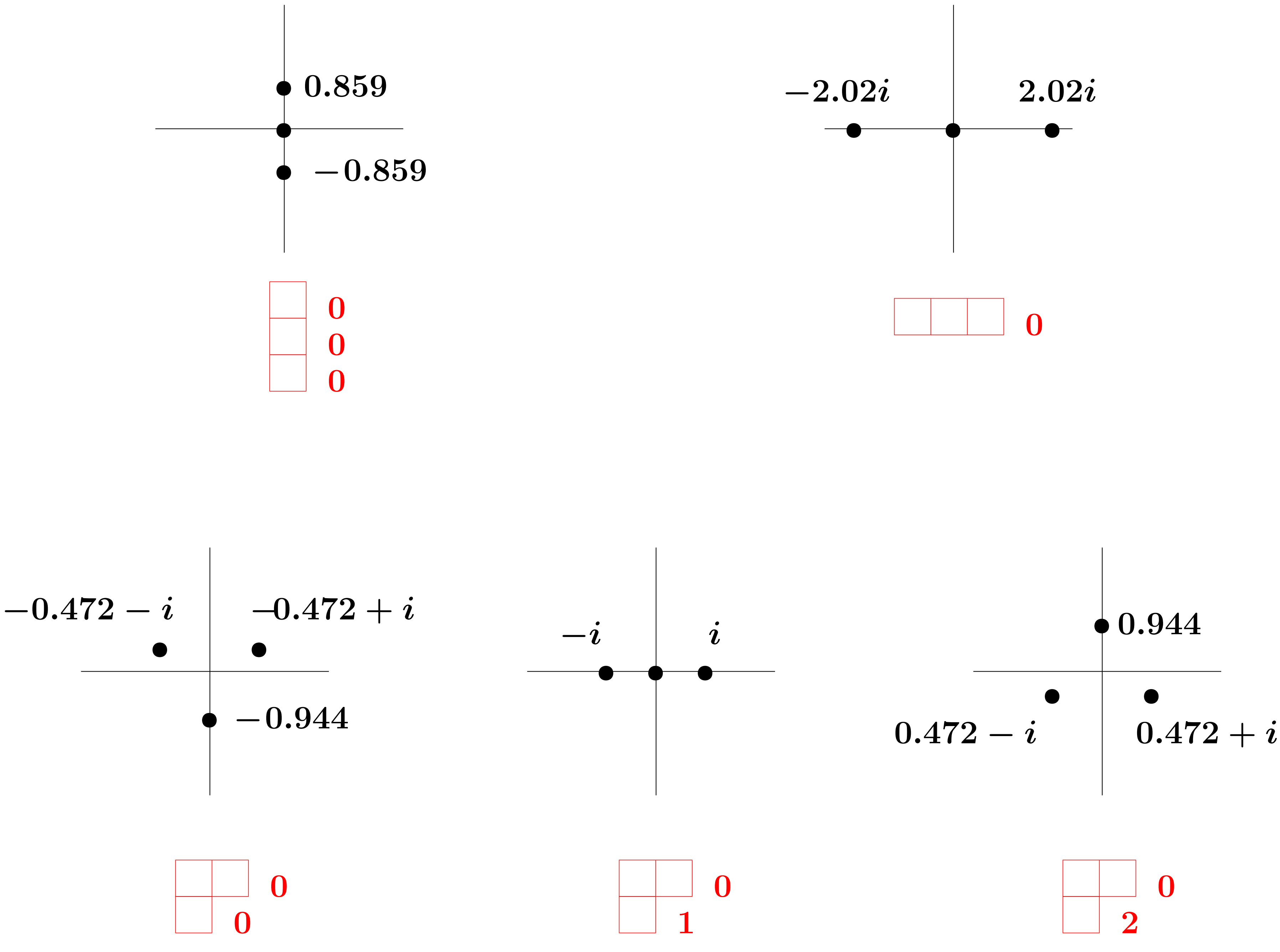

0.859
−0.859


−2.02i
2.02i


−0.472 − i
−0.472 + i
−0.944


−i
i


0.944
0.472 − i
0.472 + i


0.859
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2.02i
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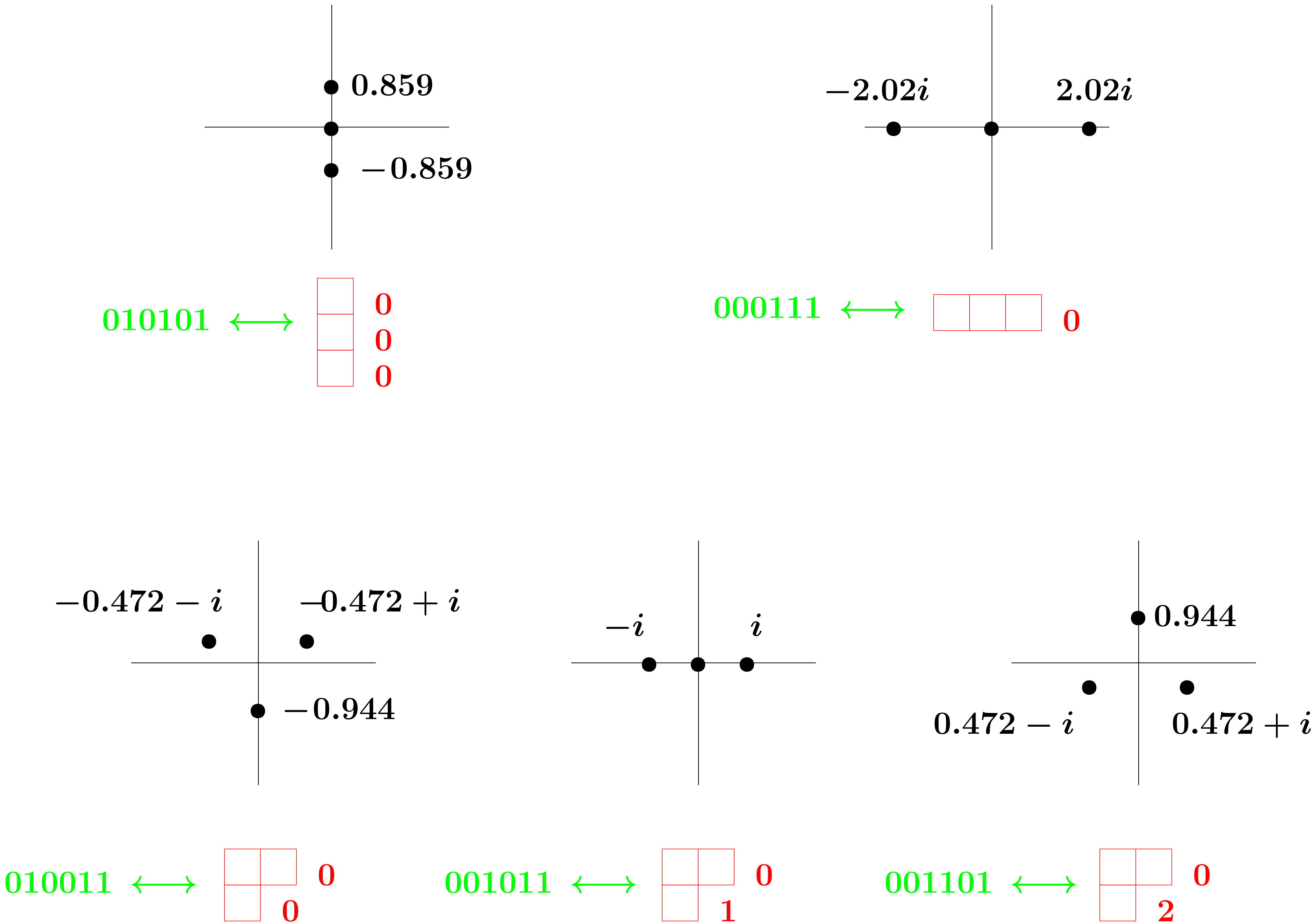
2


0.859
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010101 ⟷
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0
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010011 ⟷
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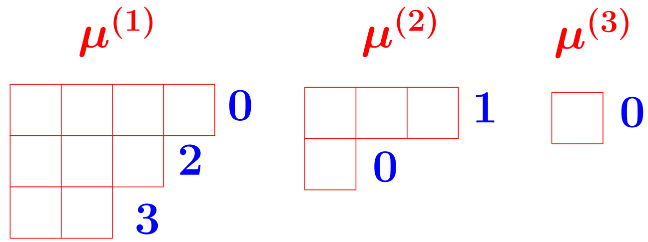
KKR bijection for sl_{n+1}

$$\{\text{highest states}\} \overset{1:1}{\longleftrightarrow} \{\text{rigged configurations}\}$$

$n = 3$ example

$$000011102113220000 \longleftrightarrow$$



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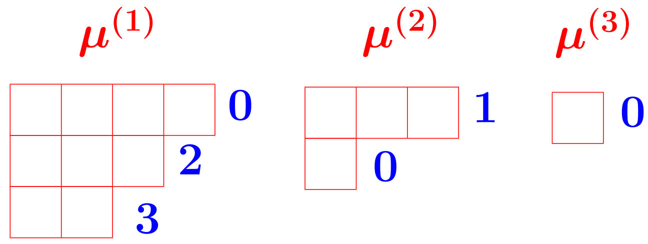

“Bethe vectors”	“Bethe roots”
Solitons	Strings (bound states of magnons)

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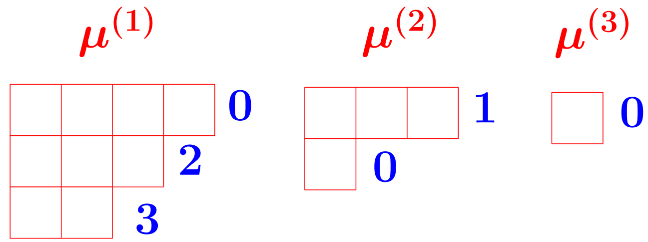

“Bethe vectors” — “Bethe roots”

Solitons — Strings (bound states of magnons)

- highest states $= i_1 i_2 \ldots i_L$ $(0 \le i_k \le n)$ satisfying the highest condition:

$$\#_0\{i_1,\ldots,i_k\} \ge \#_1\{i_1,\ldots,i_k\} \ge \cdots \ge \#_n\{i_1,\ldots,i_k\} \quad (\forall k)$$

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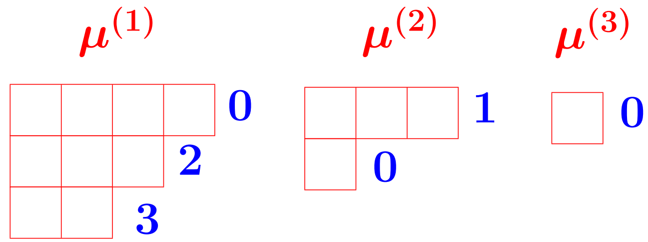

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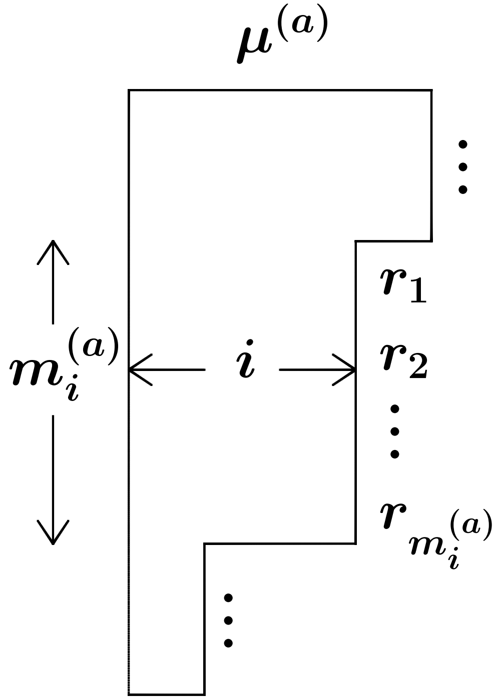
- rigged configuration: $((\mu^{(1)}, r^{(1)}), \ldots, (\mu^{(n)}, r^{(n)}))$

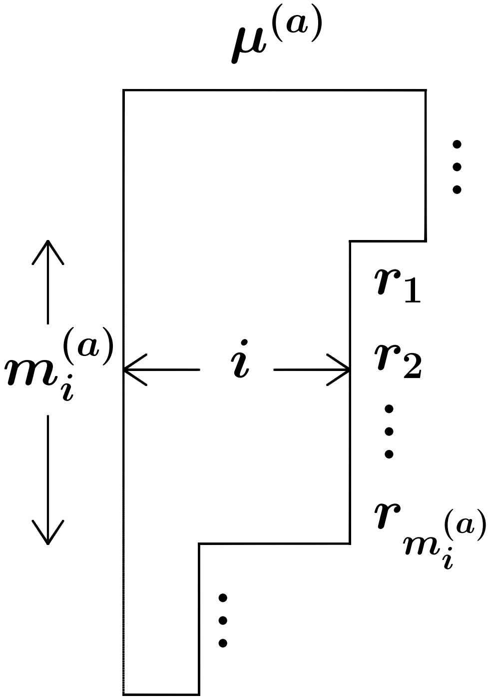
$$\left.\begin{array}{l} \mu^{(1)}, \ldots, \mu^{(n)} : \text{configuration} = n\text{-tuple of Young diagrams} \\ r^{(1)}, \ldots, r^{(n)} : \text{rigging} = \text{integers assigned to each row} \end{array}\right\} + \text{selection rule (next page)}$$



$$m_i^{(a)} = \#(\text{length } i \text{ rows in } \mu^{(a)}), \;\; \sum_{i\geq 1} i m_i^{(a)} = |\mu^{(a)}|$$

$$0 \leq r_1 \leq \cdots \leq r_{m_i^{(a)}} \leq h_i^{(a)}$$

$\cdots$ **“Fermionic” selection rule**



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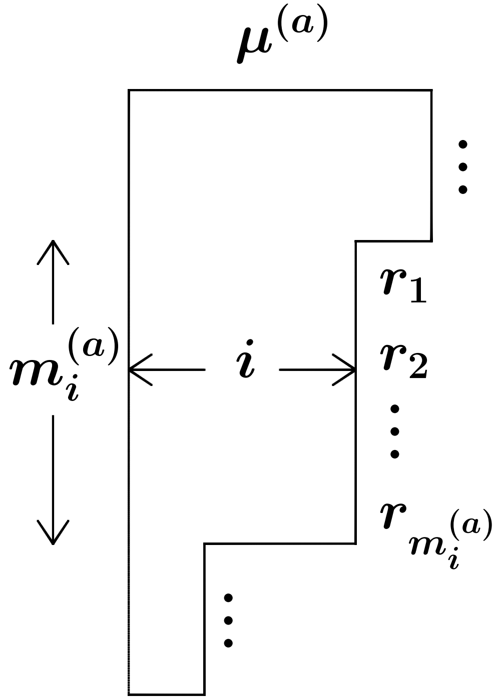
$$h_i^{(a)} = L\delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j\ge 1} \min(i,j) m_j^{(b)}$$

$\cdots$ vacancy for *holes*

$$C_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1}$$

(C_{ab}) $\cdots$ Cartan matrix of sl_{n+1}



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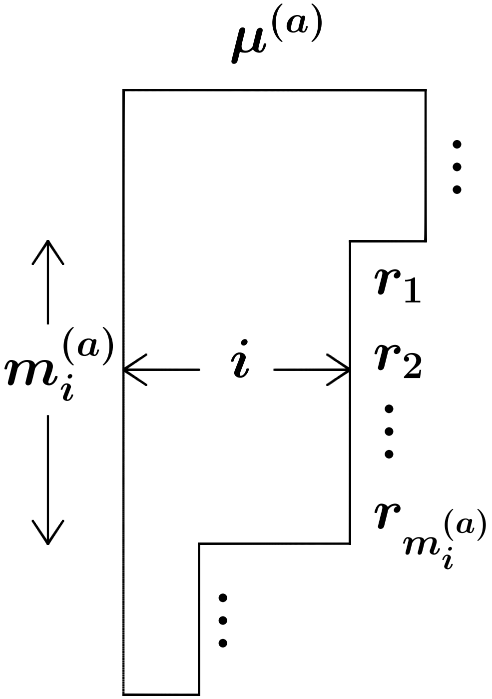
$$\# \text{ of rigging choices for a fixed configuration} = \prod_{a=1}^{n} \prod_{i\geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}$$



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This is an sl_{n+1} generalization of Bethe's formula for # of string solutions (1931).

hat also eine Möglichkeit weniger, die des letzten Komplexes von n Wellen, λ_{q_n}, kann schließlich nur noch

$$Q'_n - (q_n - 1) = Q_n + 1$$

verschiedene Werte annehmen, wo

$$Q_n(N, q_1 q_2 \ldots) = N - 2 \sum_{p<n} p\, q_p - 2 \sum_{p \geqq n} n\, q_p. \tag{44}$$

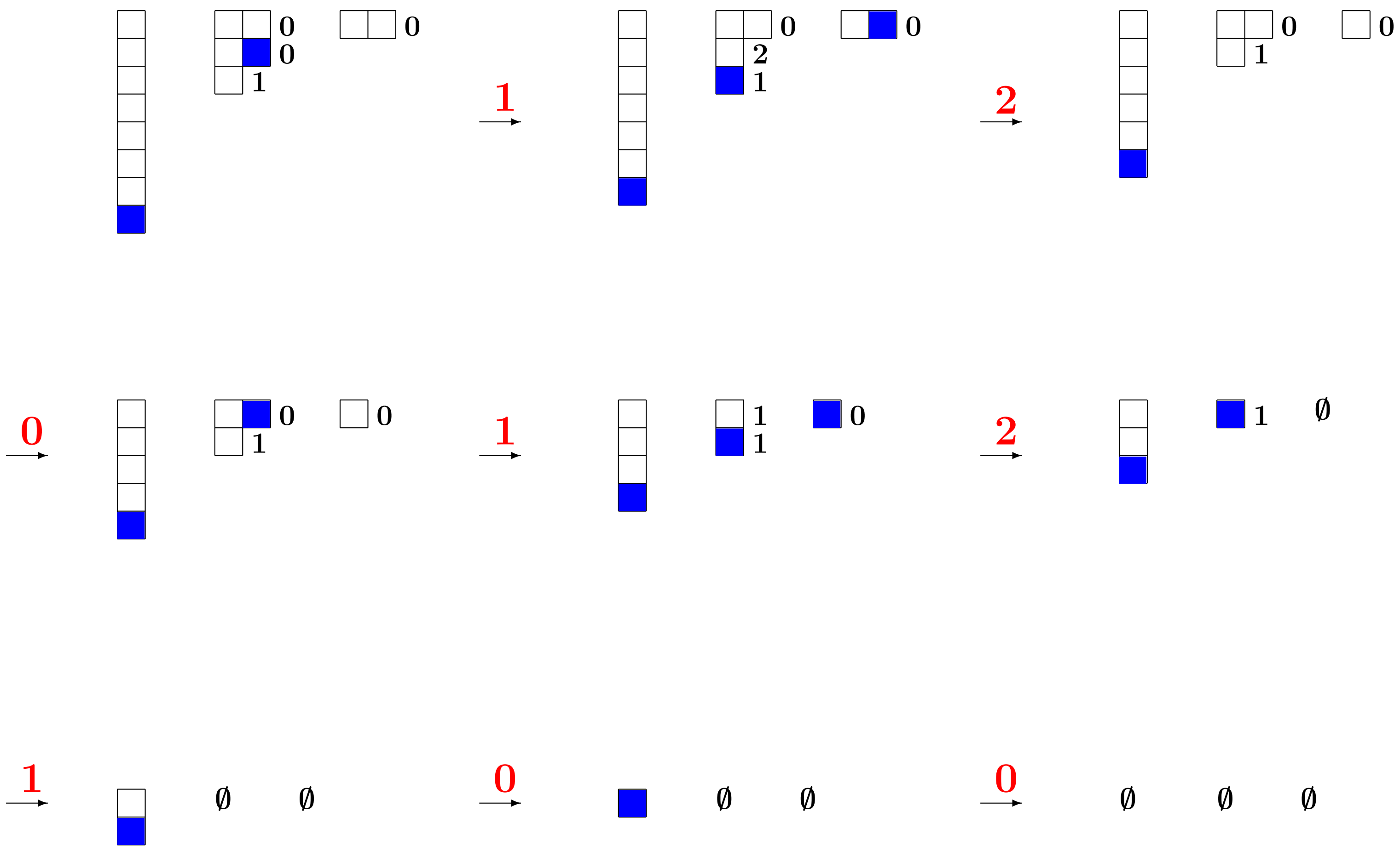
Schließlich ist zu berücksichtigen, daß Vertauschung der λ der verschiedenen Wellenkomplexe mit gleicher Anzahl n von Wellen nicht zu neuen Lösungen führt. Die gesamte Zahl unserer Lösungen wird somit

$$z(N, q_1 q_2 \ldots) = \prod_{n=1}^{\infty} \frac{(Q_n + q_n) \ldots (Q_n + 1)}{q_n!} = \prod_n \binom{Q_n + q_n}{q_n}, \tag{45}$$

wo die Q_n durch (44) definiert sind.

8. Wir werden nun nachweisen, daß wir die richtige Anzahl Lösungen erhalten haben.

Example of KKR algorithm



Top left rigged configuration $\xrightarrow{\text{KKR}}$ **00121021**

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000

$t = 1$: 000000001111000022103200000000000000000000000000

$t = 2$: 000000000000111100022132000000000000000000000000

$t = 3$: 000000000000000011110021322000000000000000000000

$t = 4$: 000000000000000000001110211322000000000000000000

$t = 5$: 000000000000000000000001100211322100000000000000

$t = 6$: 000000000000000000000000011000211032210000000000

$t = 7$: 000000000000000000000000000110000211003221000000

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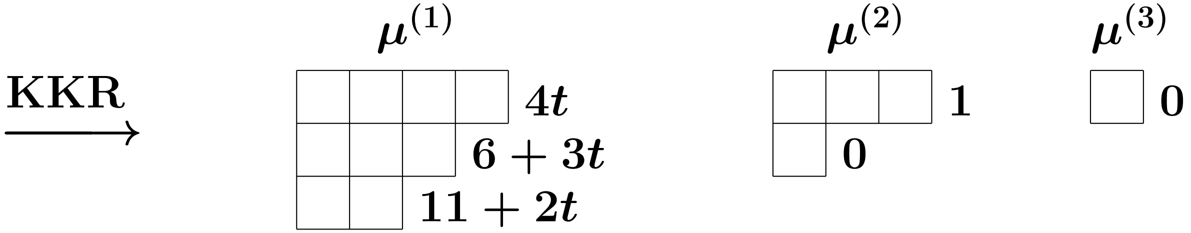
```



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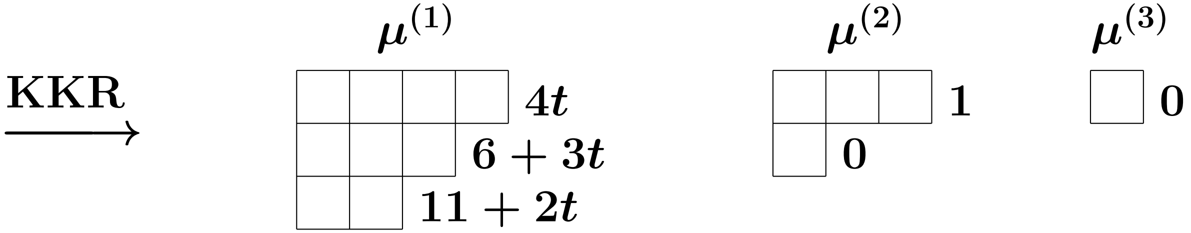

- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
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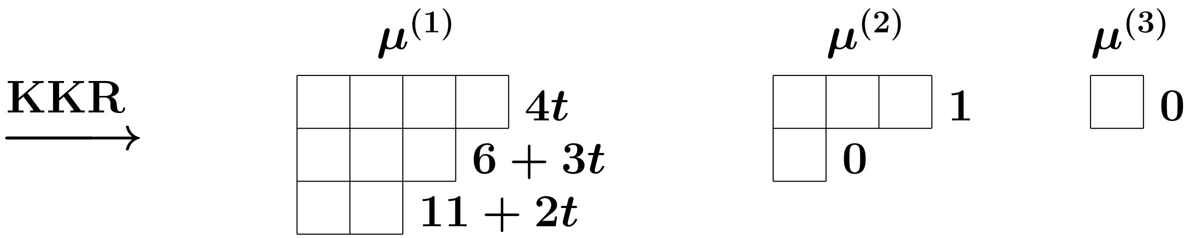

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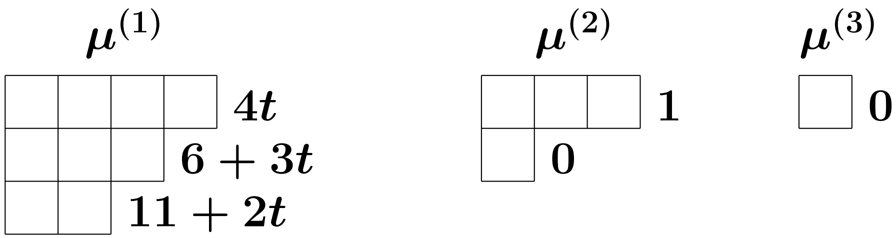
$\xrightarrow{\text{KKR}}$



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Rigged configuration = action angle variable of BBS!

In particular, $\mu^{(1)}$ = list of amplitude of solitons.

$(\mu^{(1)}, \ldots, \mu^{(n)})$ will be called a soliton content (= generalized amplitude).

Quiz: What are the soliton contents or at least their amplitude?

110111100111000111000000000000000000000000000000

210221001220012211000000000000000000000000000000

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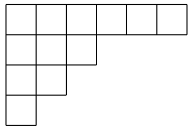
110111100111000111000000000000000000000000000000

001000011000111000111111000000000000000000000000

000100000110000111000000111111000000000000000000

000010000001100000111000000000111111000000000000

$\longrightarrow$ $\boldsymbol{\mu}^{(1)}$



210221001220012211000000000000000000000000000000

002100220112201000221100000000000000000000000000

000021002001120221000022110000000000000000000000

000000210200012100221000002211000000000000000000

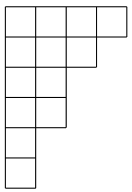
000000002120001021000221000000221100000000000000

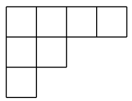
000000000012200100210000221000000022110000000000

000000000001022010002100000221000000002211000000

000000000000100202100021000000221000000000221100

$\longrightarrow$ $\boldsymbol{\mu}^{(1)}$ $\boldsymbol{\mu}^{(2)}$





Randomized box-ball system

BBS state		Soliton content
$i_1 i_2 \ldots i_L 00000 \ldots$	$\overset{\text{KKR}}{\longmapsto}$	$(\mu^{(1)}, \ldots, \mu^{(n)})$

Randomize $i_1 i_2 \ldots i_L$ by introducing the i.i.d. measure on the set of states:

$$\{0, 1, \ldots, n\} \to (0, 1); \qquad i \mapsto p_i \qquad (p_0 + \cdots + p_n = 1).$$

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Determine the scaling form of the most probable $(\mu^{(1)}, \ldots, \mu^{(n)})$ when $L \to \infty$.

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This can be done by TBA minimizing the free energy F associated with

$$\mathrm{Prob}(\mu^{(1)}, \ldots, \mu^{(n)}) \simeq Z_L^{-1} \mathrm{e}^{-\beta_1 |\mu^{(1)}| - \cdots - \beta_n |\mu^{(n)}|} \prod_{a=1}^{n} \prod_{i \geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}},$$

$$\mathrm{e}^{\beta_a} := p_{a-1}/p_a, \qquad Z_L = \text{normalization const (partition function)}.$$

Introduce the scaled string and hole densities $\rho_i^{(a)}, \sigma_i^{(a)}$ by

$$m_i^{(a)} \simeq L\rho_i^{(a)}, \quad h_i^{(a)} \simeq L\sigma_i^{(a)}, \quad \sigma_i^{(a)} = \delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i, j) \rho_j^{(b)},$$

Assume $p_0 \geq \cdots \geq p_n$ in the rest.

The condition $\frac{\delta F}{\delta \rho_i^{(a)}} = 0$ leads to the TBA equation

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \log(1 + (Y_j^{(b)})^{-1})$$

in terms of $Y_i^{(a)} = \dfrac{\sigma_i^{(a)}}{\rho_i^{(a)}}$ with the boundary condition $\displaystyle\lim_{i\to\infty} \frac{1 + Y_{i+1}^{(a)}}{1 + Y_i^{(a)}} = \mathrm{e}^{\beta_a}$.

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This is equivalent to the (constant) Y-system

$$\left(Y_i^{(a)}\right)^2 = \frac{(1 + Y_{i-1}^{(a)})(1 + Y_{i+1}^{(a)})}{\left(1 + (Y_i^{(a-1)})^{-1}\right)\left(1 + (Y_i^{(a+1)})^{-1}\right)}$$

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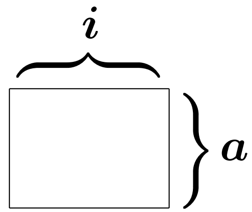
Solution (Rare case for which an exact formula can be given)

$$Y_i^{(a)} = \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{Q_i^{(a-1)} Q_i^{(a+1)}},$$

$$Q_i^{(a)} = Q_i^{(a)}(p_0, \ldots, p_n) = \frac{\det(p_k^{\lambda_j + n - j})_{j,k=0}^{n}}{\det(p_k^{n-j})_{j,k=0}^{n}} \qquad \Big((\lambda_0, \ldots, \lambda_n) = (\overbrace{i \ldots i}^{a}\ \overbrace{0, \ldots, 0}^{n+1-a}) \Big)$$

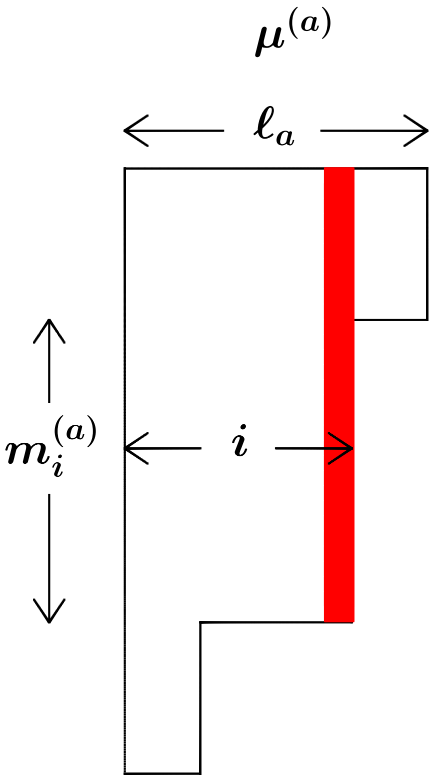
= Schur function for $a \times i$ rectangular Young diagram



Result. The limit shape of soliton content $(\mu^{(1)}, \dots, \mu^{(n)})$ is given by

$$\eta_i^{(a)} := \lim_{L\to\infty} \frac{1}{L}(\text{Length of the } i\text{ th column of } \mu^{(a)}) = \frac{Q_{i-1}^{(a-1)} Q_i^{(a+1)}}{Q_i^{(a)} Q_{i-1}^{(a)} Q_1^{(1)}}$$

$$\text{width } \ell_a \text{ of } \mu^{(a)} \simeq \frac{\log L}{\log \frac{p_{a-1}}{p_a}} \quad (L \to \infty \text{ if } p_0 > \cdots > p_n)$$



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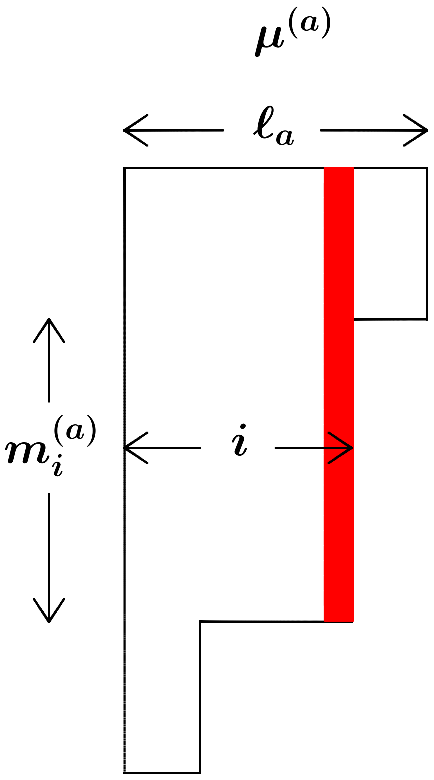
Special case $p_a = \frac{q^a}{1+q+\cdots+q^n} \quad (0 < q \leq 1).$



Scaled column length of $\mu^{(a)}$

$$\eta_i^{(a)} = \frac{q^{i+a-1}(1-q)(1-q^a)(1-q^{n+1-a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})}$$

Strings

$$\rho_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} m_i^{(a)} = \frac{q^{i+a-1}(1-q)^2(1-q^a)(1-q^{n+1-a})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

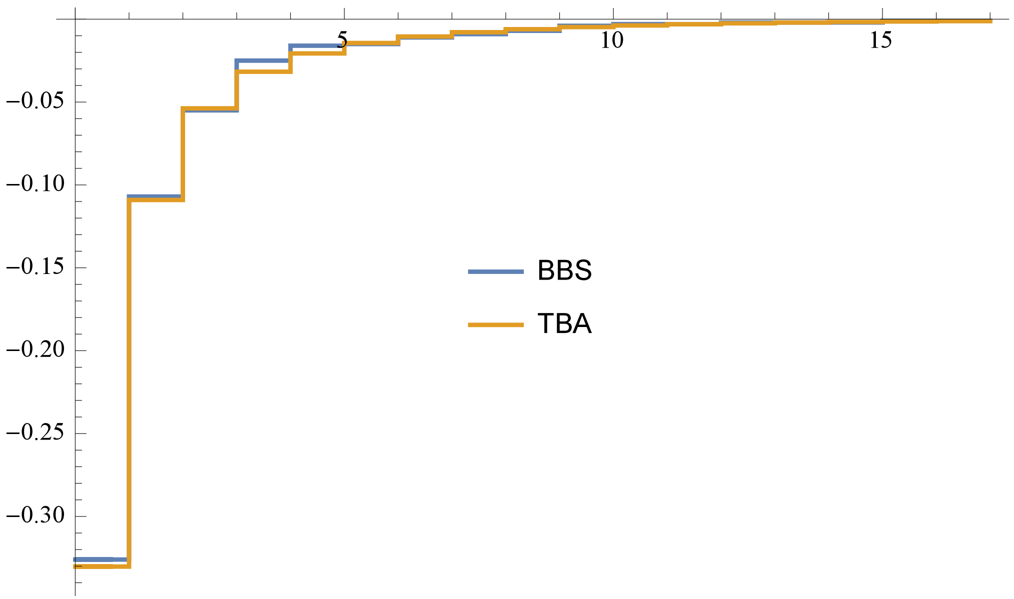
Holes

$$\sigma_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} h_i^{(a)} = \frac{q^{a-1}(1-q)^2(1-q^i)(1-q^{n+i+1})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

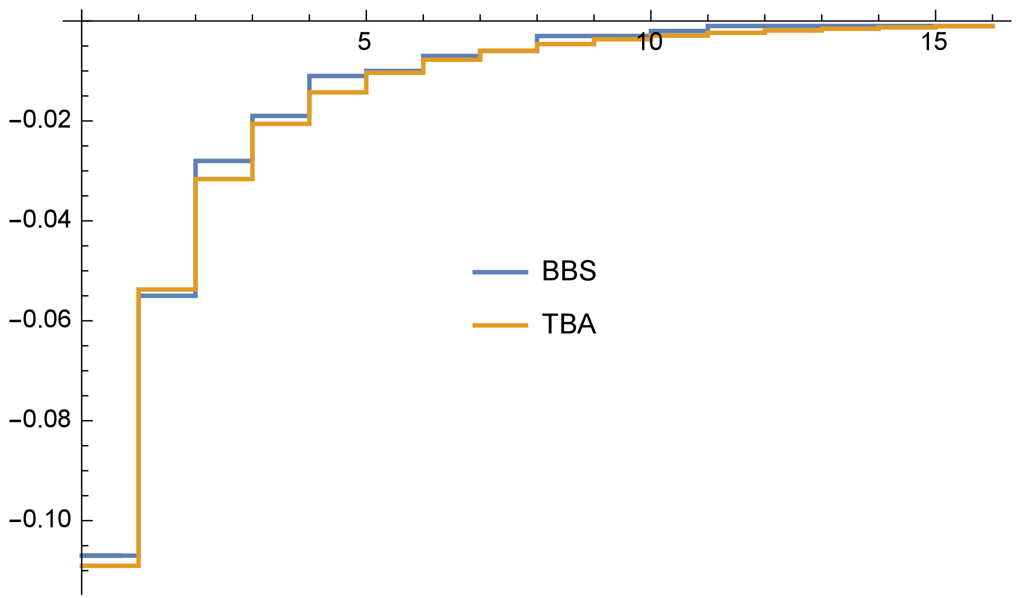
2-color BBS with $L = 1000$ sites with distribution $(p_0, p_1, p_2) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$.

Vertically L^{-1} scaled soliton contents.

$\mu^{(1)}$



$\mu^{(2)}$



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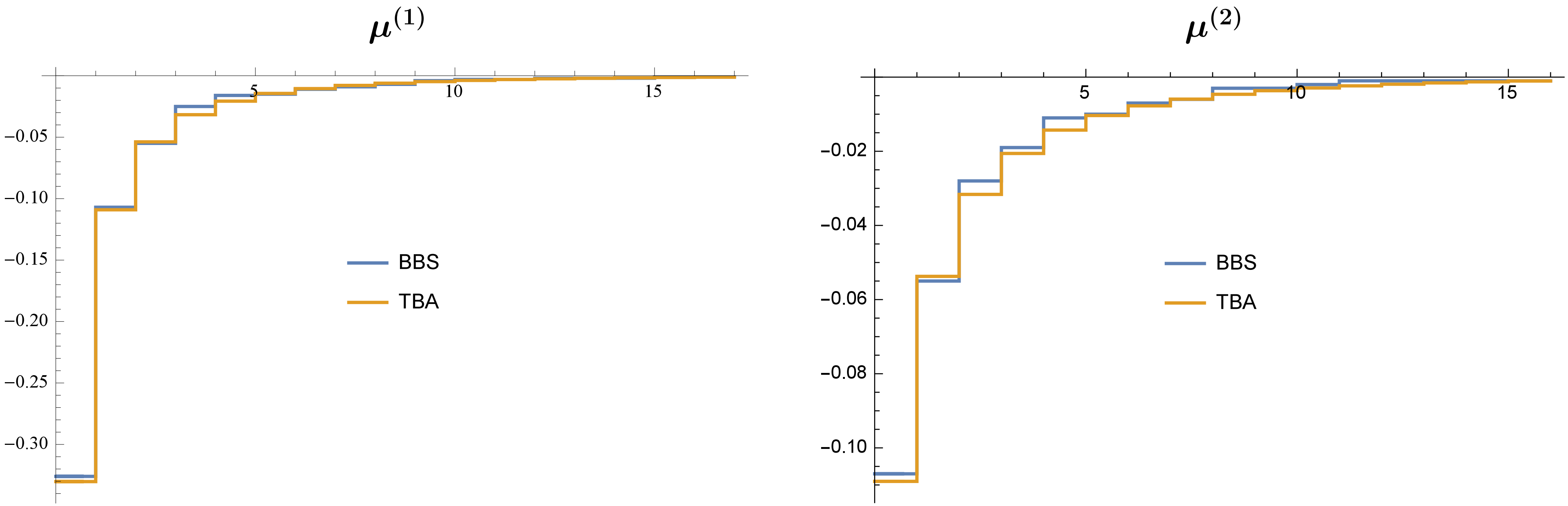

- **$0 < q < 1$: Subcritical**

  $\mu^{(a)}$ scales with depth $O(L)$, width $O(\log L)$

- **$q = 1$: Critical**

  $\mu^{(1)}$ scales with depth $O(L)$, width $O(\sqrt{L})$

Generalized hydrodynamics, GHD (from here 1-color BBS only)

[Castro-Alvaredo, Doyon, Yoshimura, Bertini, Collura, De Nardis, Fagotti, . . . 2016∼]

Densities: ρ_i (amplitude i-solitons), σ_i (hole)

Bethe equation: $\sigma_i = 1 - 2\sum_j \min(i,j)\rho_j$

Speed equation: $v_i = i + \sum_j 2\min(i,j)(v_i - v_j)\rho_j$ $\left(\begin{array}{c} v_i = \text{effective speed} \\ 2\min(i,j) = \text{phase shift} \end{array}\right)$

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Solution for homogeneous BBS:

$$v_i = i\frac{1+q}{1-q} - \frac{2q(1+q)(1-q^i)}{(1-q)^2(1+q^{i+1})}, \qquad (\rho_i, \sigma_i) = (\rho_i^{(1)}, \sigma_i^{(1)}) \text{ in previous pages}$$

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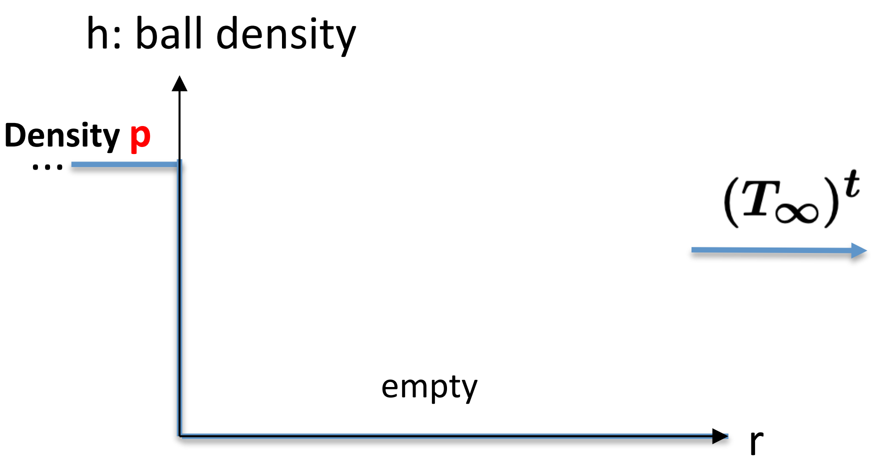
A most fruitful application of this is the Riemann problem:

> "Describe the profile of the fluid starting from the *homogeneous* initial state except a *single discontinuity* at some point."

Density Plateaux emerging from domain wall initial condition



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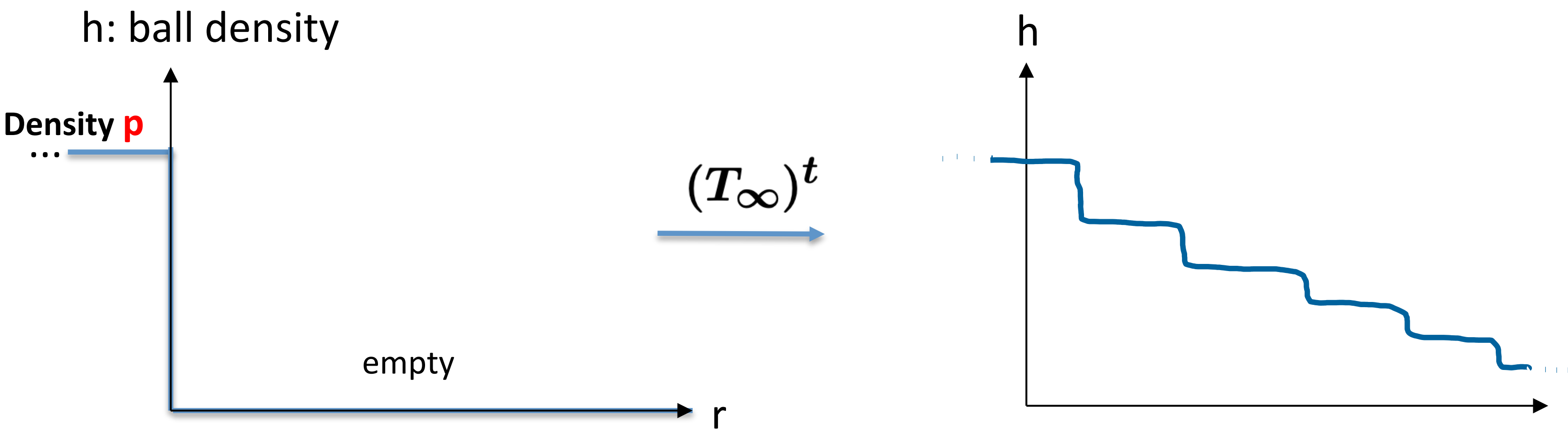

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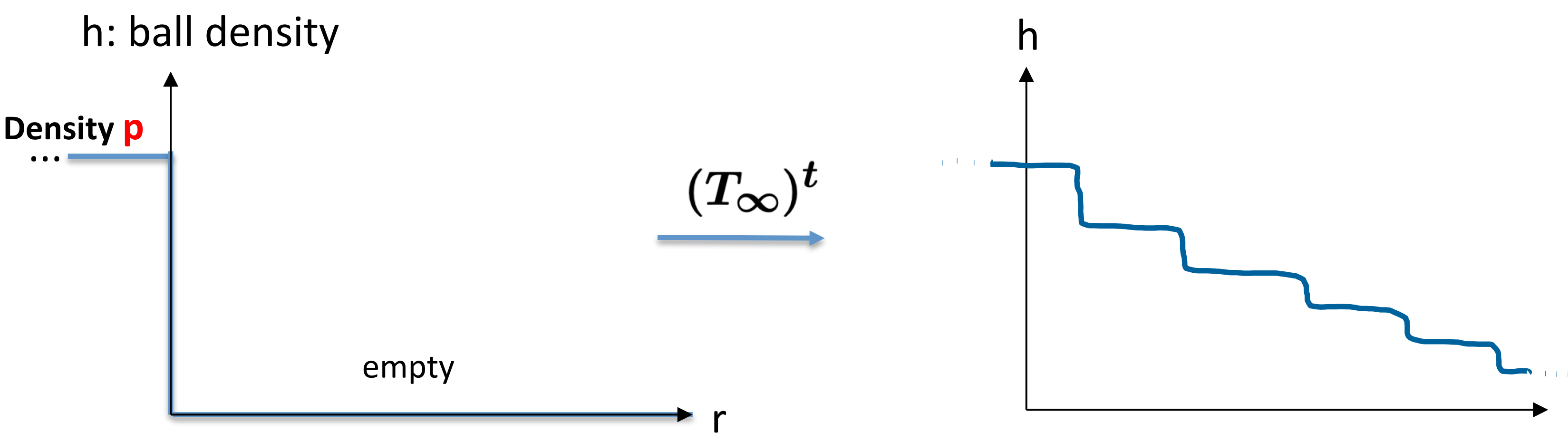

Plateaux broaden linearly in time t. The plot against $\zeta = r/t$ collapses into a single curve.

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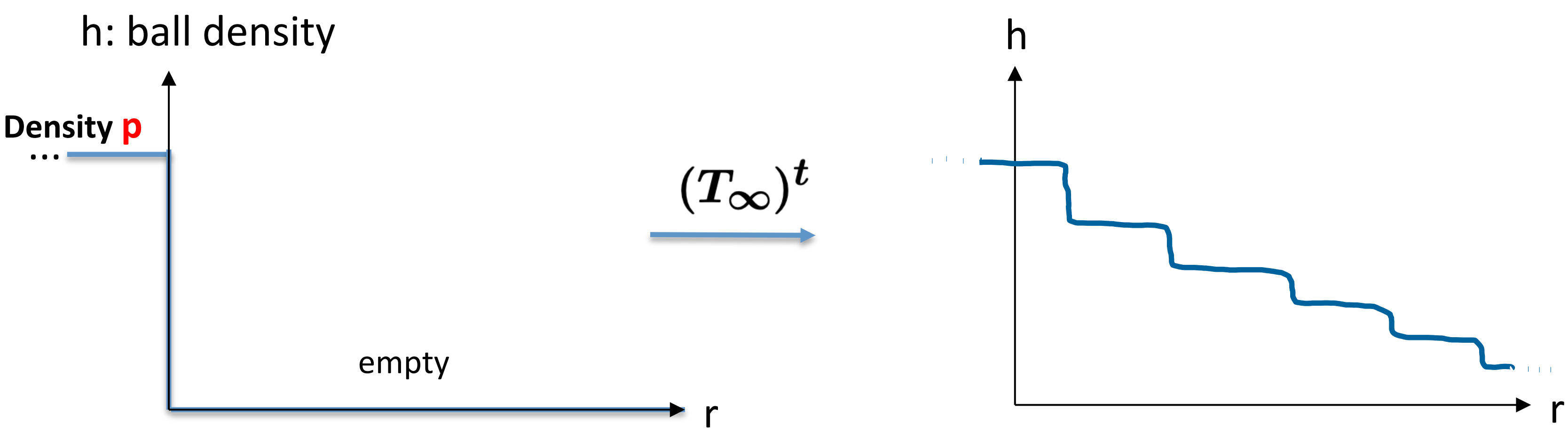

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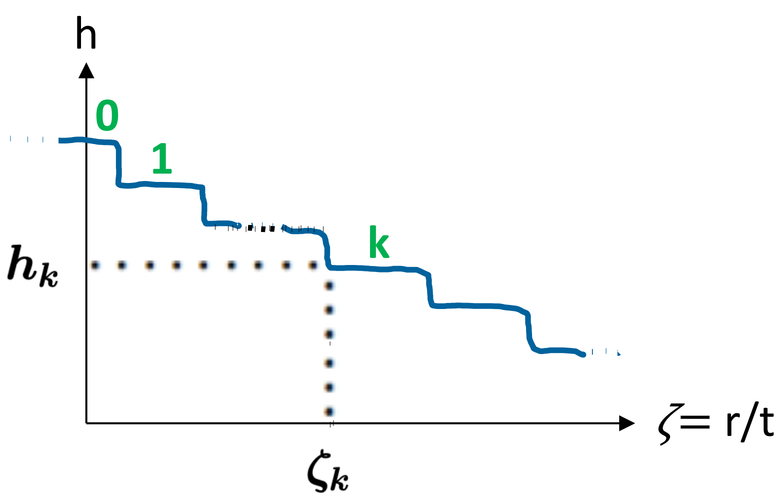

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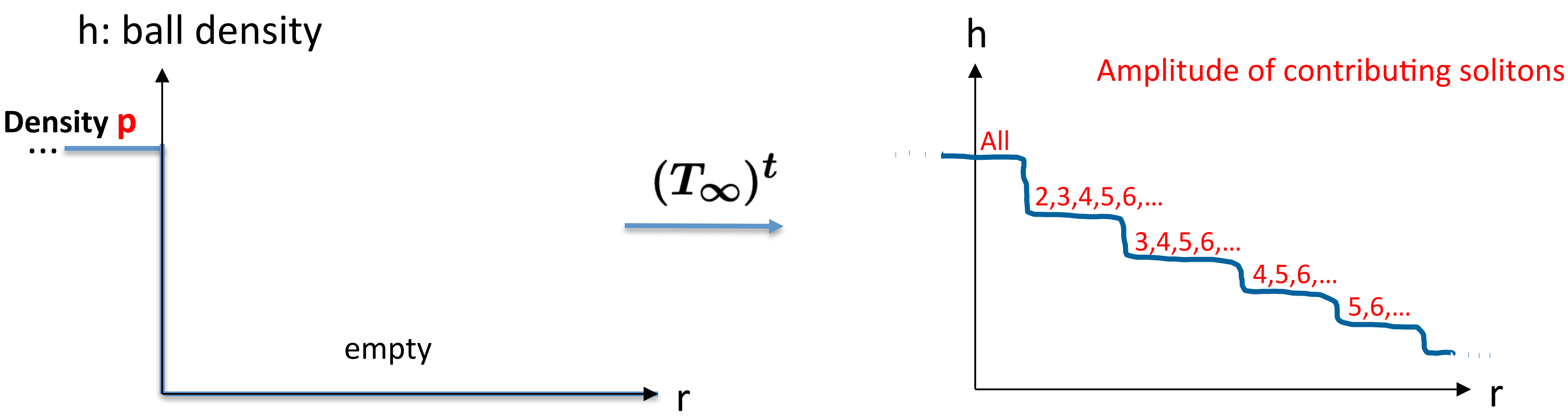

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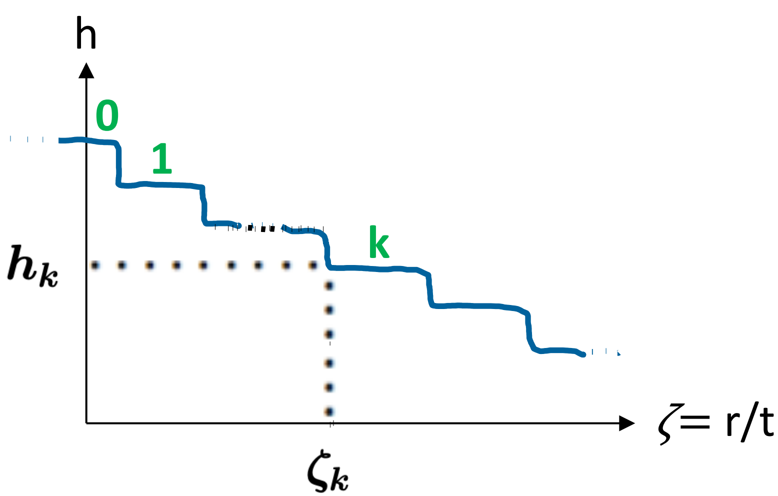

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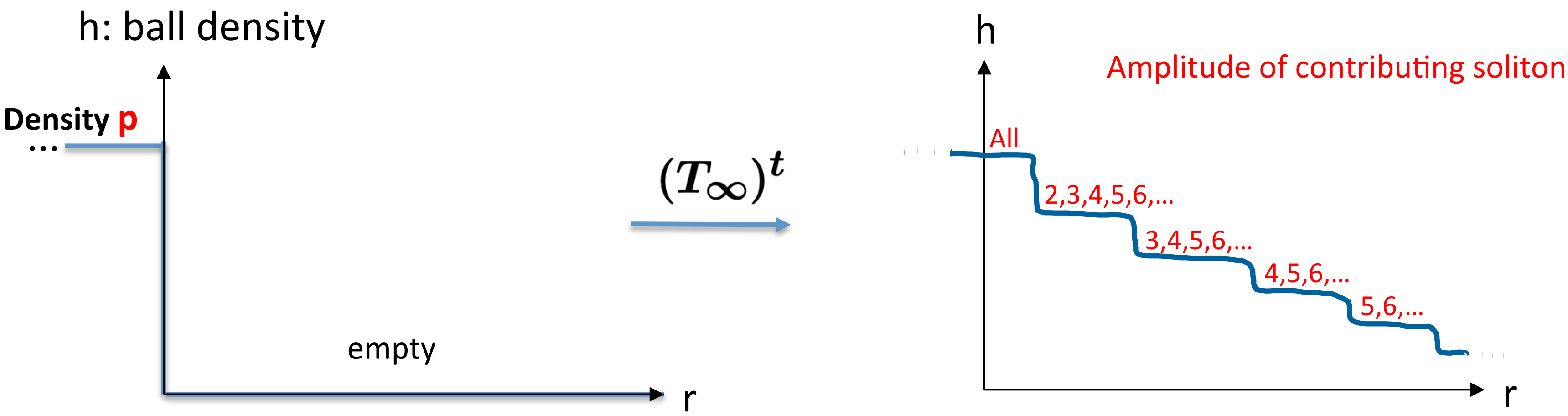

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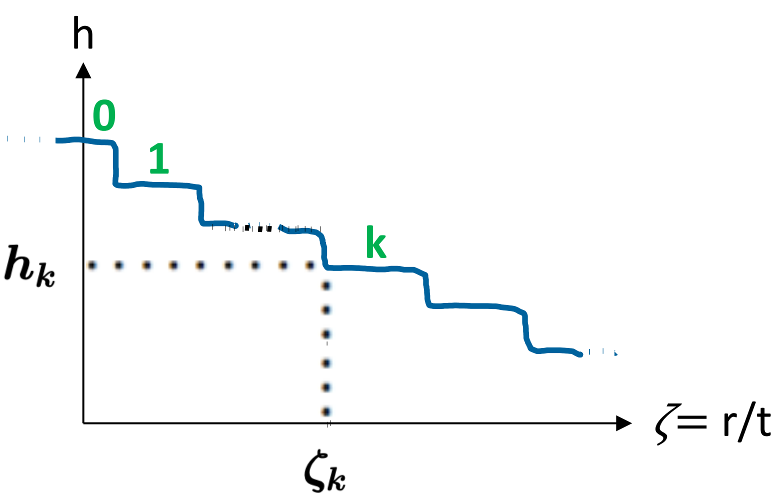


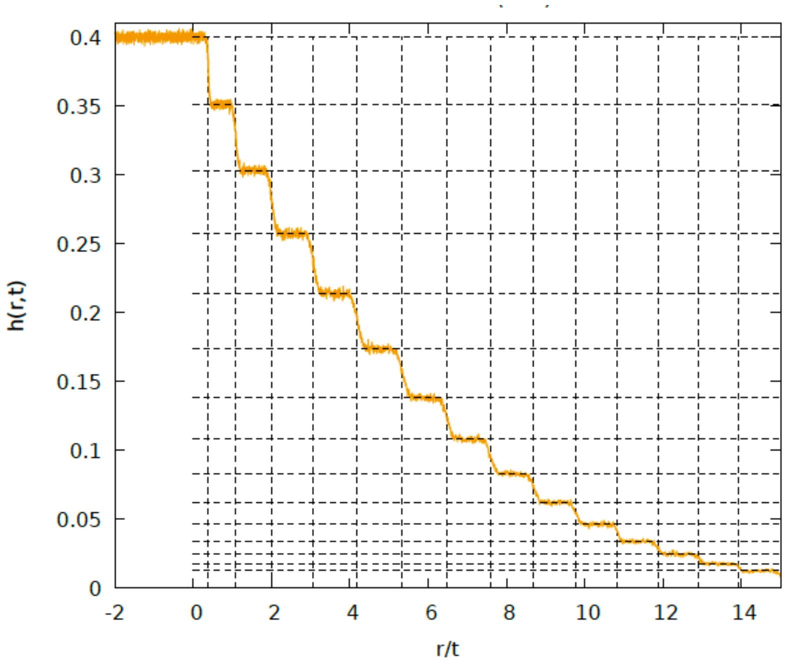
Generalized hydrodynamics (GHD) predicts

$$h_k = \frac{q^{k+1}(1-q^{k+2}+k(1-q))}{1-q^{2k+3}+(2k+1)(1-q)q^{k+1}}$$

$$\zeta_k = \frac{k(1-q^{k+1})}{1+q^{k+1}} \qquad \left(p = \frac{q}{1+q}\right)$$

Simulation with $N_{samples}$ = 50000

(Plots of ball density vs ζ = r/t. Dotted lines are GHD predictions)



p=0.4, q=0.666…, t=500.

Width of plateau edge $\propto \sqrt{t}$ for finite t.

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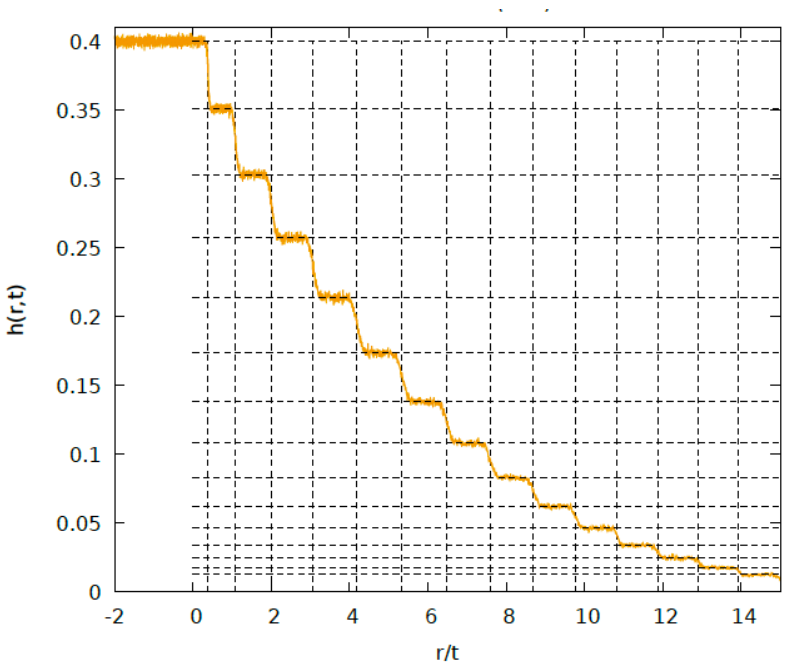
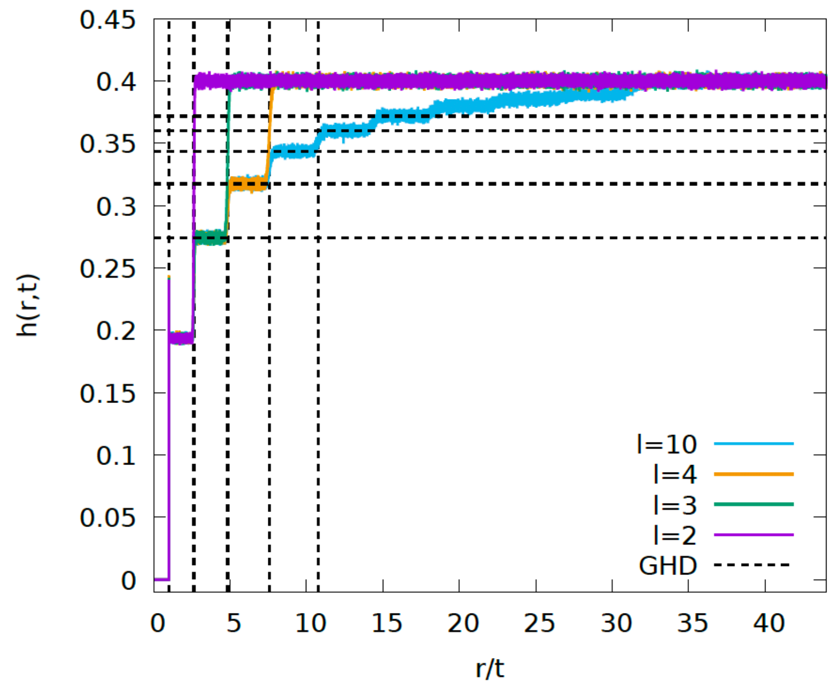
(Plots of ball density vs ζ= r/t. Dotted lines are GHD predictions)





p=0.4, q=0.666…, t=500.

Plateaux emerging from the *opposite domain wall* with empty left region by T_l (l=2,3,4,10)

Width of plateau edge $\propto \sqrt{t}$ for finite t.

Left edge of k th plateau $\zeta = \textbf{effective speed } v_k^{(k)}$

Height of k th plateau ($1 \le k < l$)

$$= \frac{q(1-q^{2k+2}-(k+1)(1-q^2)q^k)}{(1+q)\left(1-(2k+3)(1-q)q^{k+1}-q^{2k+3}\right)}$$

Summary

1. **BBS is a Yang-Baxter integrable cellular automaton with explicit action-angle variables originating in Bethe strings.**

2. **Limit shape of soliton content in randomized BBS is determined by TBA.**

3. **Density plateaux emerging from domain wall initial condition is analytically described by GHD.**

Reference

Review part:

R.Inoue, AK and T.Takagi
"Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry", JPA Topical Review (2012), arXiv:1109.5349.

Limit shape problem:

AK, H.Lyu and M.Okado
"Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz", NPB(2018), arXiv:1808.02626.

Generalized hydrodynamics of BBS:

AK, G.Misguich and V.Pasquier in preparation.