Tetrahedron equation, 3D reflection equation and generalized quantum groups

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"New development in Teihumüler space theory; MCM2017"
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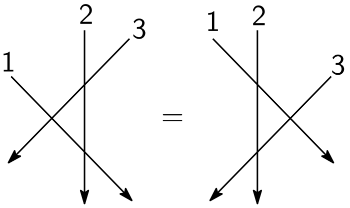
Key to integrability in 2D

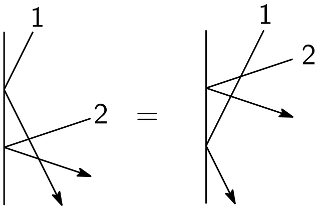
Yang-Baxter equation

$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

Reflection equation

$R_{21}K_2R_{12}K_1 = K_1R_{21}K_2R_{12}$





R : 2 particle scattering

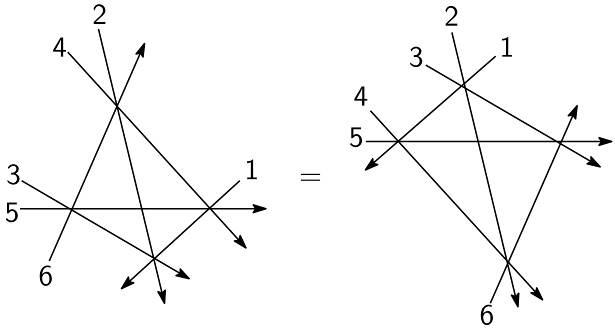
K : Reflection at boundary

What about 3D?

Tetrahedron equation (A.B. Zamolodchikov, 1980)

$$R : F \otimes F \otimes F \to F \otimes F \otimes F \qquad \text{(3D } R\text{)}$$

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$$



$$R = \begin{cases} \text{3 string scattering amplitude in (2+1)D} \\ \text{local Boltzmann weight of the vertex in 3D} \end{cases}$$

Status of finding solutions and relevant maths

Yang-Baxter eq. (2D)

- Infinitely many solutions constructed systematically by representation theory of the Drinfeld-Jimbo quantum affine algebra $U_q(\hat{g})$ ($\hat{g}$ = affine Kac-Moody algebra).

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- One such approach is by quantized algebra of functions $A_q(g)$ (g = finite dimensional simple Lie algebra).
- $A_q(g)$ is the quantum group corresponding to the dual of $U_q(g)$. Studied by Drinfeld (87), Vaksman-Soibelman (89,91), Reshetikhin-Takhtajan-Faddeev (90), Noumi-Yamada-Mimachi (92), Kashiwara (93), Yakimov (2010), Geiss-Leclerc-Schröer (2011-) etc.

- **Simplest example:**

$$SL_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid [t_{ij}, t_{kl}] = 0,\ t_{11}t_{22} - t_{12}t_{21} = 1 \right\}.$$

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$A_q(sl_2)$ is generated by $t_{11}, t_{12}, t_{21}, t_{22}$ with the relations

$$t_{11}t_{21} = qt_{21}t_{11},\ t_{12}t_{22} = qt_{22}t_{12},\ t_{11}t_{12} = qt_{12}t_{11},\ t_{21}t_{22} = qt_{22}t_{21},$$
$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1.$$

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Hopf algebra with coproduct $\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$.

- **Fock representation** $\boldsymbol{\pi_1 : \mathbf{A_q(sl_2)} \to \mathrm{End}(\mathbf{F_q})}$

$F_q = \oplus_{m \geq 0} \mathbb{C}|m\rangle$: q-boson Fock space

$$\pi_1 : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \mathbf{a}^- & \mathbf{k} \\ -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}$$
$$\mathbf{k}|m\rangle = q^m|m\rangle,\ \mathbf{a}^+|m\rangle = |m+1\rangle,\ \mathbf{a}^-|m\rangle = (1 - q^{2m})|m-1\rangle.$$

Theorem (Classification of irreducible representations. Soibelman 1991)

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2. *The irreducible rep. corresponding to the reduced expression* $s_{i_1} \cdots s_{i_r} \in W(g)$ *is realized as the tensor product* $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$.

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Crucial Corollary

If $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}$ are 2 different reduced expressions, then

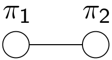
$$\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}.$$

$\Longrightarrow$ Exists the unique map Φ called intertwiner such that

$$(\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}) \circ \Phi = \Phi \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_r})$$

Example

$$A_q(sl_3) = \langle t_{ij} \rangle_{i,j=1}^{3}$$



Fock representations

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}}$$

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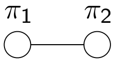

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$W(sl_3) = \langle s_1, s_2 \rangle.$ $\qquad s_2 s_1 s_2 = s_1 s_2 s_1$ (Coxeter relation)

$\Longrightarrow \quad \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_1 \otimes \pi_2 \otimes \pi_1$ as representations on $(F_q)^{\otimes 3}$

Exists the intertwiner $\Phi : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}$ such that

$$(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi = \Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1).$$

Explicit form

$$R := \Phi P_{13}, \quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x,$$

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R^{abc}_{ijk} |a\rangle \otimes |b\rangle \otimes |c\rangle.$$

$$R^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c} \sum_{\lambda,\mu\geq 0, \lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \times \begin{bmatrix} i, j, c+\mu \\ \mu, \lambda, i-\mu, j-\lambda, c \end{bmatrix}.$$

$$(q)_m = \prod_{j=1}^{m}(1-q^j), \quad \begin{bmatrix} i_1, \dots, i_r \\ j_1, \dots, j_s \end{bmatrix} = \frac{\prod_{m=1}^{r}(q^2)_{i_m}}{\prod_{m=1}^{s}(q^2)_{j_m}}$$

Example

$$
\begin{aligned}
R_{314}^{041} &= -q^2(1-q^4)(1-q^6)(1-q^8),\\
R_{314}^{132} &= (1-q^6)(1-q^8)(1-q^4-q^6-q^8-q^{10}),\\
R_{314}^{223} &= q^2(1+q^2)(1+q^4)(1-q^6)(1-q^6-q^{10}),\\
R_{314}^{314} &= q^6(1+q^2+q^4-q^8-q^{10}-q^{12}-q^{14}),\\
R_{314}^{405} &= q^{12}.
\end{aligned}
$$

As these examples indicate, all the matrix elements of R are polynomials in q with integer coefficients.

Example

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$$R^{132}_{314} = (1-q^6)(1-q^8)(1-q^4-q^6-q^8-q^{10}),$$
$$R^{223}_{314} = q^2(1+q^2)(1+q^4)(1-q^6)(1-q^6-q^{10}),$$
$$R^{314}_{314} = q^6(1+q^2+q^4-q^8-q^{10}-q^{12}-q^{14}),$$
$$R^{405}_{314} = q^{12}.$$

As these examples indicate, all the matrix elements of R are polynomials in q with integer coefficients.

Furthermore, for any $(i,j,k) \in \mathbb{Z}^3_{\geq 0}$, there is a unique (a,b,c) such that

$$R^{abc}_{ijk}|_{q=0} = 1.$$

This property will be utilized later to define the tropical 3D R.

Theorem (Kapranov-Voevodsky 1994)

R satisfies the tetrahedron eq. $R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$.

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Essence of proof. Consider $A_q(sl_4)$ and $W(sl_4) = \langle s_1, s_2, s_3 \rangle$.

$$s_2 s_1 s_2 = s_1 s_2 s_1, \quad s_3 s_2 s_3 = s_2 s_3 s_2, \quad s_1 s_3 = s_3 s_1,$$

$$s_1 s_2 s_3 s_1 s_2 s_1 = s_3 s_2 s_3 s_1 s_2 s_3 \quad \text{(longest element)}$$

The intertwiner for the last one is constructed in 2 different ways as

$$\begin{array}{ll}
123\underline{121} & \Phi_{456} \\
1\underline{232}12 & \Phi_{234} \\
\underline{13}2\underline{31}2 & P_{12}P_{45} \\
3\underline{121}32 & \Phi_{234} \\
321\underline{232} & \Phi_{456} \\
32\underline{13}23 & P_{34} \\
323123 &
\end{array}
\qquad\qquad
\begin{array}{ll}
12\underline{31}21 & P_{34} \\
\underline{121}321 & \Phi_{123} \\
21\underline{232}1 & \Phi_{345} \\
2\underline{13}2\underline{31} & P_{23}P_{56} \\
23\underline{121}3 & \Phi_{345} \\
\underline{232}123 & \Phi_{123} \\
323123 &
\end{array}$$

Equate the 2 sides, substitute $\Phi_{ijk} = R_{ijk}P_{ik}$ and cancel P_{ij}'s. □

Summary so far (type A case)

Weyl group elements $\longleftrightarrow$ "Multi-string states"

Cubic Coxeter relation $\longleftrightarrow$ 3D R

Reduced words for longest element $\longleftrightarrow$ Tetrahedron equation

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Remark

(1) 3D R = "Quantization" of Miquel's theorem (1838)
(Bazhanov-Sergeev-Mangazeev 2008)

Summary so far (type A case)

$$\begin{aligned} \text{Weyl group elements} &\longleftrightarrow \text{“Multi-string states”} \\ \text{Cubic Coxeter relation} &\longleftrightarrow \text{3D } R \\ \text{Reduced words for longest element} &\longleftrightarrow \text{Tetrahedron equation} \end{aligned}$$

Remark

(1) 3D R = “Quantization” of Miquel’s theorem (1838)
(Bazhanov-Sergeev-Mangazeev 2008)

(2) Tropical (or combinatorial) analogue.

(3) Birational (or classical) analogue.

The next 3 pages demonstrate (2) and (3), which will lead to a Triad of the 3 versions of 3D Rs.

Tropical 3D R

Easy to show $R^{abc}_{ijk}|_{q=0} = \delta_i^{a+b-\min(a,c)} \delta_j^{\min(a,c)} \delta_k^{b+c-\min(a,c)}$.

So the tropical 3D R $\mathbf{R} := R|_{q=0} : \mathbb{Z}^3_{\geq 0} \rightarrow \mathbb{Z}^3_{\geq 0}$ defined by

$$\mathbf{R}(a,b,c) = (a+b-\min(a,c), \min(a,c), b+c-\min(a,c))$$

satisfies the tropical tetrahedron eq. (equality as maps on $\mathbb{Z}^6_{\geq 0}$) :

$$\begin{array}{ccccc}
 & & |314516\rangle & & \\
 & \mathbf{R}_{123} \swarrow & & \searrow \mathbf{R}_{356} & \\
|132516\rangle & & & & |311543\rangle \\
\mathbf{R}_{145} \downarrow & & & & \downarrow \mathbf{R}_{246} \\
|532156\rangle & & & & |351147\rangle \\
\mathbf{R}_{246} \downarrow & & & & \downarrow \mathbf{R}_{145} \\
|512354\rangle & & & & |151327\rangle \\
 & \mathbf{R}_{356} \searrow & & \swarrow \mathbf{R}_{123} & \\
 & & |515327\rangle & &
\end{array}$$

Birational 3D R

For the matrices generating a unipotent subgroup of SL_3

$$G_1(z) = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_2(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

the unique solution to the Lusztig type equation

$$G_1(a)G_2(b)G_1(c) = G_2(\tilde{a})G_1(\tilde{b})G_2(\tilde{c})$$

defines a birational map on 3 variables which we call birational 3D R:

$$\mathbf{R} : (a, b, c) \mapsto (\tilde{c}, \tilde{b}, \tilde{a}) = \left(\frac{ab}{a+c}, \, a+c, \, \frac{bc}{a+c} \right).$$

Its tropical limit reproduces the Tropical 3D R introduced before:

$$\mathcal{R} : \; (a, b, c) \mapsto (a + b - \min(a, c), \min(a, c), b + c - \min(a, c)).$$

Summary: Triad of 3D R

classical limit: Quantum R → Tropical $\mathbf{R}$

low temperature limit: Birational $\mathcal{R}$ → Tropical $\mathbf{R}$

Quantum R	Tropical $\mathbf{R}$	Birational $\mathcal{R}$
Intertwiner of $A_q(sl_3)$ modules	Bijection on $\mathbb{Z}^3_{\geq 0}$	Birational map on $\mathbb{C}^3$

All satisfying the tetrahedron equation.

Recent developments

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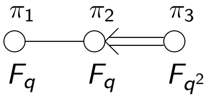
1. Type B, C, F_4 cases: 3D analogue of reflection equation.
2. Connection to the Poincaré-Birkhoff-Witt basis of $U_q^+(g)$.
3. Reduction to 2D YBE: Infinitely many quantum R matrices associated with generalized quantum groups.
4. Application to matrix product stationary states in integrable Markov processes such as totally asymmetric exclusion/zero-range processes.

Today in what follows: Mainly and will touch only briefly.

$A_q(C_3) = \langle t_{ij} \rangle_{i,j=1}^{6}$: (Reshetikhin-Takhtajan-Faddeev 1990)



$\pi_k(t_{ij})$ are given as follows.

$$\pi_1: \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} \\ 0 & 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 \\ 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} & 0 \\ 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_3: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \langle \mathbf{A}^\pm, \mathbf{K} \rangle = \langle \mathbf{a}^\pm, \mathbf{k} \rangle|_{q \to q^2}.$$

$W(C_3) = \langle s_1, s_2, s_3 \rangle$

$$s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2.$$

Write simply as $\pi_{i_1,\ldots,i_r} := \pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$. Then,

Equivalence	Intertwiner	
$\pi_{13} \simeq \pi_{31}$,	$P_{12}(x \otimes y) = y \otimes x$,	(trivial)
$\pi_{121} \simeq \pi_{212}$,	$\Phi = R P_{13}$	(same as type A),
$\pi_{2323} \simeq \pi_{3232}$,	$\Psi = K P_{14} P_{23}$	(new).

$$K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q), \quad R \in \mathrm{End}(F_q^{\otimes 3}).$$

Matrix elements

$$K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K^{cmdn}_{a\,i\,b\,j} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle.$$

$$K^{cmdn}_{a\,i\,b\,j} = 0 \text{ unless } c+m+d = a+i+b, \quad d+n-c = b+j-a.$$

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Prop. (K-Okado '12, A more structural formula in K-Maruyama '15)

$$K^{c,m,0,n}_{a,i,0,j} = \sum_{\lambda\geq 0}(-1)^{m+\lambda}\frac{(q^4)_{c+\lambda}}{(q^4)_c} q^{\phi_2} \begin{bmatrix} i,j \\ \lambda, j-\lambda, m-\lambda, i-m+\lambda \end{bmatrix},$$

$$\phi_2 = (a+c+1)(m+j-2\lambda)+m-j.$$

$$K^{cmdn}_{a\,i\,b\,j} = \frac{(q^4)_a}{(q^4)_c} \sum_{\alpha,\beta,\gamma\geq 0} \frac{(-1)^{\alpha+\gamma}}{(q^4)_{d-\beta}} q^{\phi_1} K^{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma}_{c,m+d-\alpha-\beta-\gamma,0,n+d-\alpha-\beta-\gamma}$$

$$\times \begin{bmatrix} b, d-\beta, i+b-\alpha-\beta, j+b-\alpha-\beta \\ \alpha, \beta, \gamma, m-\alpha, n-\alpha, b-\alpha-\beta, d-\beta-\gamma \end{bmatrix},$$

$$\phi_1 = \alpha(\alpha+2d-2\beta-1)+(2\beta-d)(m+n+d)+\gamma(\gamma-1)-b(i+j+b).$$

Example

$$
\begin{aligned}
K^{1300}_{2110} &= q^8(1-q^8),\\
K^{2110}_{2110} &= -q^4(1-q^8+q^{14}),\\
K^{2201}_{2110} &= -q^6(1+q^2)(1-q^2+q^4-q^6-q^{10}),\\
K^{3011}_{2110} &= 1-q^8+q^{14},\\
K^{3102}_{2110} &= -q^{10}(1-q+q^2)(1+q+q^2),\\
K^{4003}_{2110} &= q^4.
\end{aligned}
$$

Properties

- $\forall K^{abcd}_{ijkl} \in \mathbb{Z}[q], \qquad K^{abcd}_{ijkl}|_{q=0} \in \{0,1\}$,
- For any $(i,j,k,l) \in \mathbb{Z}^4_{\geq 0}$, there is unique $(a,b,c,d) \in \mathbb{Z}^4_{\geq 0}$ such that $K^{abcd}_{ijkl}|_{q=0} = 1$.

Theorem (K-Okado 2012)

R and K yield the first nontrivial solution to the **3D reflection equation** proposed by Isaev-Kulish in 1997:

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

Theorem (K-Okado 2012)

R and K yield the first nontrivial solution to the **3D reflection equation** proposed by Isaev-Kulish in 1997:

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

- An equality in $\mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q \otimes F_q \otimes F_q \otimes F_{q^2} \otimes F_q \otimes F_q)$.
- The proof is parallel with type A.
- Uses the reduced expressions of the longest element

$$s_1s_2s_3s_2s_1s_2s_3s_2s_3 = s_3s_2s_3s_2s_1s_2s_3s_2s_1 \in W(C_3).$$

- The 2 sides come from the 2 ways to construct the intertwiner for

$$\pi_{123212323} \simeq \pi_{323212321}$$

from R and K.

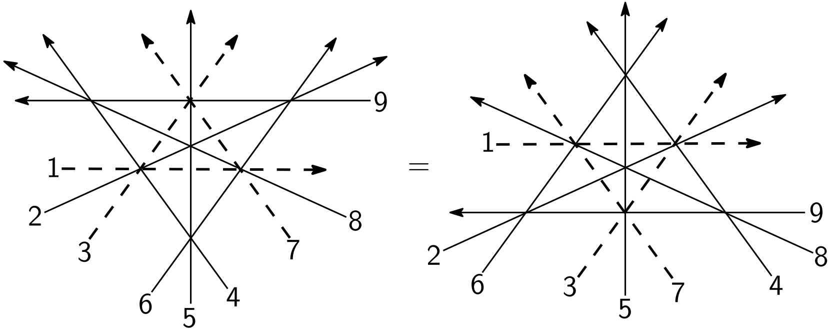
Physical (geometric) interpretation of the 3D reflection eq.

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$

is a "factorization" of 3 string scattering with boundary reflections.

R : Scattering amplitude of 3 strings.

K: Reflection amplitude with boundary freedom signified by spaces $1, 3, 7$.



Classical version: Birational 3D K

Introduce generators of the unipotent subgroup of Sp_4:

$$X_1(z) = \begin{pmatrix} 1 & z & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -z \\ & & & 1 \end{pmatrix}, \quad X_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 2z & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

Given 4 parameters (a, b, c, d), it is easy to check that the matrix equation

$$X_2(a)X_1(b)X_2(c)X_1(d) = X_1(\tilde{a})X_2(\tilde{b})X_1(\tilde{c})X_2(\tilde{d})$$

for $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ has the unique and totally positive solution

$$\tilde{a} = \frac{bcd}{A}, \quad \tilde{b} = \frac{A^2}{B}, \quad \tilde{c} = \frac{B}{A}, \quad \tilde{d} = \frac{ab^2c}{B},$$
$$A = ab + ad + cd, \quad B = ab^2 + 2abd + ad^2 + cd^2.$$

In terms of the $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$, define the birational 3D $\mathcal{K}$ to be the map

$$\mathcal{K} : (a, b, c, d) \mapsto (\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}).$$

(By the definition $\mathcal{K}^{-1} = \mathcal{K}$.)

Together with the birational 3D $\mathcal{R}$, it satisfies the 3D reflection eq:

$$\mathcal{R}_{456}\mathcal{R}_{489}\mathcal{K}_{3579}\mathcal{R}_{269}\mathcal{R}_{258}\mathcal{K}_{1678}\mathcal{K}_{1234} = \mathcal{K}_{1234}\mathcal{K}_{1678}\mathcal{R}_{258}\mathcal{R}_{269}\mathcal{K}_{3579}\mathcal{R}_{489}\mathcal{R}_{456},$$

which is an equality of birational maps on 9 variables.

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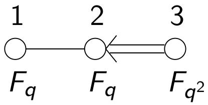
One can also define Tropical 3D **K** which forms the

Triad of Quantum, Birational and Tropical 3D K's

in the same way as the type A case.

B, F_4 cases

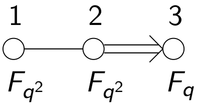
C_3



$R : 121 = 212$

$K : 2323 = 3232$

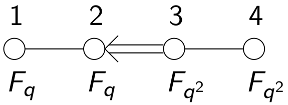
B_3



$S : 121 = 212$

$J : 2323 = 3232$

F_4



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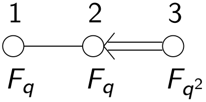
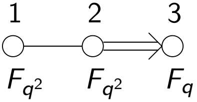
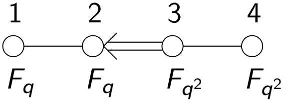
B, F_4 cases

C_3	B_3	F_4
		
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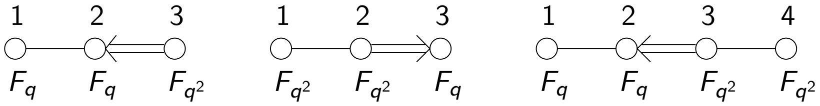
$$R \in \mathrm{End}(F_q \otimes F_q \otimes F_q), \qquad K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$$

B, F_4 cases

C_3 $\qquad\qquad$ B_3 $\qquad\qquad$ F_4



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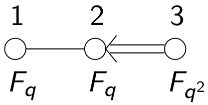
C_3



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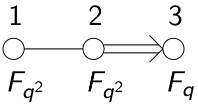

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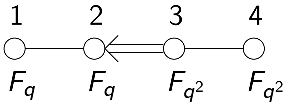

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Both (R, K) and (S, J) satisfy the 3D reflection equation.

A reduced expression of the longest element of $W(F_4)$ is

$$s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1 \quad (= -1, \text{ length } 24).$$

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can be constructed by composition of R, K, S in two ways, which must coincide. This leads to the F_4-analogue of the tetrahedron equation:

$$\begin{aligned} & S_{14,15,16} S_{9,11,16} K_{16,10,8,7} K_{9,13,15,17} S_{4,5,16} R_{7,12,17} S_{1,2,16} R_{6,10,17} S_{9,14,18} K_{1,3,5,17} \\ \times\ & S_{11,15,18} K_{18,12,8,6} S_{1,4,18} S_{1,8,15} R_{7,13,19} K_{1,6,11,19} K_{4,12,15,19} R_{3,10,19} S_{4,8,11} K_{1,7,14,20} \\ \times\ & S_{2,5,18} R_{6,13,20} R_{3,12,20} S_{1,9,21} K_{2,10,15,20} S_{4,14,21} K_{21,13,8,3} S_{2,11,21} S_{2,8,14} R_{6,7,22} \\ \times\ & K_{2,3,4,22} S_{5,15,21} K_{11,13,14,22} R_{10,12,22} K_{2,6,9,23} R_{3,7,23} R_{19,20,22} K_{16,17,18,22} R_{10,13,23} \\ \times\ & K_{5,12,14,23} R_{3,6,24} K_{16,19,21,23} K_{4,7,9,24} R_{17,20,23} K_{5,10,11,24} R_{12,13,24} R_{17,19,24} \\ \times\ & K_{18,20,21,24} S_{5,8,9} R_{22,23,24} \ = \ \text{product in reverse order.} \end{aligned}$$

$16R$'s, $16S$'s and $18K$'s acting on $F_{q_{i_1}} \otimes \cdots \otimes F_{q_{i_{24}}}$.

Another aspect: Connection with PBW basis

$U_q^+(sl_3) = \langle e_1, e_2 \rangle$: Subalgebra of $U_q(sl_3)$ obeying the q-Serre relation:

$$e_1^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad e_2^2 e_1 - (q + q^{-1}) e_2 e_1 e_2 + e_1 e_2^2 = 0.$$

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Two PBW bases: $\{E^{a,b,c}\}_{(a,b,c)\in\mathbb{Z}^3_{\geq 0}}$, $\{E'^{a,b,c}\}_{(a,b,c)\in\mathbb{Z}^3_{\geq 0}}$

$$E^{a,b,c} = \frac{e_1^a([e_2, e_1]_q)^b e_2^c}{[a]![b]![c]!}, \quad E'^{a,b,c} = E^{a,b,c}|_{e_1 \leftrightarrow e_2},$$

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Theorem (Sergeev 2009)

$$E^{a,b,c} = \sum_{ijk} R^{abc}_{i,j,k} E'^{k,j,i}.$$

Namely, 3D R = Transition matrix of the PBW bases of $U_q^+(sl_3)$!

Generalizations

For arbitrary classical simple Lie algebra g, let w_0 be the longest element of its Weyl group.

- $\Phi :=$ Intertwiner of the irreducible $A_q(g)$ modules labeled by w_0.
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「いっぺーにふぇーでーびたん！」

(“Thank you so much!” in Okinawan)