Tetrahedron equation, 3D reflection equation and generalized quantum groups

Atsuo Kuniba

University of Tokyo

“New development in Teihumüler space theory; MCM2017”
OIST 30 November 2017
Key to integrability in 2D

Yang-Baxter equation
\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \]

Reflection equation
\[ R_{21} K_2 R_{12} K_1 = K_1 R_{21} K_2 R_{12} \]

\( R \) : 2 particle scattering

\( K \) : Reflection at boundary
What about 3D?

**Tetrahedron equation** (A.B. Zamolodchikov, 1980)

\[ R : F \otimes F \otimes F \to F \otimes F \otimes F \quad (3D \ R) \]

\[ R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123} \]

\[ R = \begin{cases} 
3 \text{ string scattering amplitude in (2+1)D} \\
\text{local Boltzmann weight of the vertex in 3D}
\end{cases} \]
Yang-Baxter eq. (2D)

- Infinitely many solutions constructed systematically by representation theory of the Drinfeld-Jimbo quantum affine algebra $U_q(\hat{g})$ ($\hat{g} = \text{affine Kac-Moody algebra}$).
Status of finding solutions and relevant maths

**Yang-Baxter eq. (2D)**
- Infinitely many solutions constructed systematically by representation theory of the Drinfeld-Jimbo quantum affine algebra $U_q(\hat{\mathfrak{g}})$ ($\hat{\mathfrak{g}}$ = affine Kac-Moody algebra).

**Tetrahedron eq. (3D)**
- A few classes of solutions are known.
- Systematic framework yet to be developed.
Status of finding solutions and relevant maths

Yang-Baxter eq. (2D)
- Infinitely many solutions constructed systematically by representation theory of the Drinfeld-Jimbo quantum affine algebra $U_q(\hat{g})$ ($\hat{g} =$ affine Kac-Moody algebra).

Tetrahedron eq. (3D)
- A few classes of solutions are known.
- Systematic framework yet to be developed.
- One such approach is by quantized algebra of functions $A_q(g)$ ($g =$ finite dimensional simple Lie algebra).
Status of finding solutions and relevant maths

**Yang-Baxter eq. (2D)**

- Infinitely many solutions constructed systematically by representation theory of the Drinfeld-Jimbo quantum affine algebra $U_q(\hat{g})$ ($\hat{g} =$ affine Kac-Moody algebra).

**Tetrahedron eq. (3D)**

- A few classes of solutions are known.
- Systematic framework yet to be developed.
- One such approach is by quantized algebra of functions $A_q(g)$ ($g =$ finite dimensional simple Lie algebra).
- $A_q(g)$ is the quantum group corresponding to the dual of $U_q(g)$. Studied by Drinfeld (87), Vaksman-Soibelman (89,91), Reshetikhin-Takhtajan-Faddeev (90), Noumi-Yamada-Mimachi (92), Kashiwara (93), Yakimov (2010), Geiss-Leclerc-Schröer (2011-) etc.
**Simplest example:**

\[
SL_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid [t_{ij}, t_{kl}] = 0, \ t_{11}t_{22} - t_{12}t_{21} = 1 \right\}.
\]
Simplest example:

\[
SL_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid [t_{ij}, t_{kl}] = 0, \ t_{11}t_{22} - t_{12}t_{21} = 1 \right\}.
\]

\(A_q(sl_2)\) is generated by \(t_{11}, t_{12}, t_{21}, t_{22}\) with the relations

\[
t_{11}t_{21} = qt_{21}t_{11}, \ t_{12}t_{22} = qt_{22}t_{12}, \ t_{11}t_{12} = qt_{12}t_{11}, \ t_{21}t_{22} = qt_{22}t_{21}, \ [t_{12}, t_{21}] = 0, \ [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \ t_{11}t_{22} - qt_{12}t_{21} = 1.
\]

Hopf algebra with coproduct \(\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}\).
Simplest example:

\[ SL_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mid [t_{ij}, t_{kl}] = 0, \; t_{11} t_{22} - t_{12} t_{21} = 1 \right\}. \]

\( A_q(sl_2) \) is generated by \( t_{11}, t_{12}, t_{21}, t_{22} \) with the relations

\[ t_{11} t_{21} = q t_{21} t_{11}, \; t_{12} t_{22} = q t_{22} t_{12}, \; t_{11} t_{12} = q t_{12} t_{11}, \; t_{21} t_{22} = q t_{22} t_{21}, \]

\[ [t_{12}, t_{21}] = 0, \; [t_{11}, t_{22}] = (q - q^{-1}) t_{21} t_{12}, \; t_{11} t_{22} - q t_{12} t_{21} = 1. \]

Hopf algebra with coproduct \( \Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj} \).

Fock representation \( \pi_1 : A_q(sl_2) \to \text{End}(F_q) \)

\[ F_q = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle : q\)-boson Fock space

\[ \pi_1 : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} a^- & k \\ -qk & a^+ \end{pmatrix} \]

\[ k|m\rangle = q^m|m\rangle, \; a^+|m\rangle = |m + 1\rangle, \; a^-|m\rangle = (1 - q^{2m})|m - 1\rangle. \]
Theorem (Classification of irreducible representations. Soibelman 1991)

Irreducible reps. $i_1$ are elements of the Weyl group $W(g)$ (up to a "torus degree of freedom").

Set $i :=$ the irreducible rep. for the simple connection $s_i W(g)$ ($i : a vertex of the Dynkin diagram of g$).

The irreducible rep. corresponding to the reduced expression $s_i v s_i r W(g)$ is realized as the tensor product $i_1 \otimes i_r$.

Crucial Corollary

If $s_i v s_i r = s_j v s_j r$ are 2 different reduced expressions, then $i_1 \otimes i_r \cong j_1 \otimes j_r$.

Exists the unique map called intertwiner such that $(i_1 \otimes i_r) \circ = \circ (j_1 \otimes j_r)$.
Theorem (Classification of irreducible representations. Soibelman 1991)

1. Irreducible reps. $\leftrightarrow^{1:1}$ elements of the Weyl group $W(g)$ (up to a “torus degree of freedom”).
Theorem (Classification of irreducible representations. Soibelman 1991)

1. Irreducible reps. $\xleftarrow{1:1}\rightarrow$ elements of the Weyl group $W(g)$ (up to a “torus degree of freedom”).

Set $\pi_i :=$ the irreducible rep. for the simple reflection $s_i \in W(g)$ ($i :$ a vertex of the Dynkin diagram of $g$).
Theorem (Classification of irreducible representations. Soibelman 1991)

1. Irreducible reps. \( \overset{1:1}{\longleftrightarrow} \) elements of the Weyl group \( W(g) \) (up to a “torus degree of freedom”).
   
   Set \( \pi_i := \) the irreducible rep. for the simple reflection \( s_i \in W(g) \) (\( i : \) a vertex of the Dynkin diagram of \( g \)).

2. The irreducible rep. corresponding to the reduced expression \( s_{i_1} \cdots s_{i_r} \in W(g) \) is realized as the tensor product \( \pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \).
Theorem (Classification of irreducible representations. Soibelman 1991)

1. Irreducible reps. $\overset{1:1}{\leftrightarrow}$ elements of the Weyl group $W(g)$ (up to a “torus degree of freedom”).

Set $\pi_i :=$ the irreducible rep. for the simple reflection $s_i \in W(g)$ ($i$ : a vertex of the Dynkin diagram of $g$).

2. The irreducible rep. corresponding to the reduced expression $s_{i_1} \cdots s_{i_r} \in W(g)$ is realized as the tensor product $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$.

Crucial Corollary

If $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_r}$ are 2 different reduced expressions, then

$$\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}.$$ 

$\implies$ Exists the unique map $\Phi$ called intertwiner such that

$$(\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}) \circ \Phi = \Phi \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_r})$$
Example

\[ A_q(sl_3) = \langle t_{ij}\rangle_{i,j=1}^3 \]

Fock representations

\[
\begin{pmatrix}
  t_{11} & t_{12} & t_{13} \\
  t_{21} & t_{22} & t_{23} \\
  t_{31} & t_{32} & t_{33}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  a^- & k & 0 \\
  -qk & a^+ & 0 \\
  0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & a^- & k \\
  0 & -qk & a^+
\end{pmatrix}
\]
Example

\[ A_q(sl_3) = \langle t_{ij} \rangle_{i,j=1}^{3} \]

Fock representations

\[
\begin{pmatrix}
  t_{11} & t_{12} & t_{13} \\
  t_{21} & t_{22} & t_{23} \\
  t_{31} & t_{32} & t_{33}
\end{pmatrix}
\mapsto
\begin{pmatrix}
  a^- & k & 0 \\
  -qa & a^+ & 0 \\
  0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & a^- & k \\
  0 & -qa & a^+
\end{pmatrix}
\]

\[ W(sl_3) = \langle s_1, s_2 \rangle. \quad s_2 s_1 s_2 = s_1 s_2 s_1 \text{ (Coxeter relation)} \]

\[ \pi_2 \otimes \pi_1 \otimes \pi_2 \cong \pi_1 \otimes \pi_2 \otimes \pi_1 \] as representations on \((F_q)^{\otimes 3}\)

Exists the intertwiner \(\Phi : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}\) such that
\[
(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi = \Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1).
\]
Explicit form

\[ R := \Phi P_{13}, \quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x, \]
\[ R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R_{ijk}^{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle. \]

\[ R_{ijk}^{abc} = \delta_{i+j,a+b} \delta_{j+k,b+c} \sum_{\lambda,\mu \geq 0, \lambda+\mu=b} (-1)^\lambda q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \]
\[ \times \left[ \begin{array}{ccc} i, j, c + \mu \\ \mu, \lambda, i - \mu, j - \lambda, c \end{array} \right]. \]

\[ (q)_m = \prod_{j=1}^{m} (1 - q^j), \quad \left[ \begin{array}{c} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{array} \right] = \frac{\prod_{m=1}^{r} (q^2)^{i_m}}{\prod_{m=1}^{s} (q^2)^{j_m}} \]
Example

\[
R^{041}_{314} = -q^2(1 - q^4)(1 - q^6)(1 - q^8),
\]
\[
R^{132}_{314} = (1 - q^6)(1 - q^8)(1 - q^4 - q^6 - q^8 - q^{10}),
\]
\[
R^{223}_{314} = q^2(1 + q^2)(1 + q^4)(1 - q^6)(1 - q^6 - q^{10}),
\]
\[
R^{314}_{314} = q^6(1 + q^2 + q^4 - q^8 - q^{10} - q^{12} - q^{14}),
\]
\[
R^{405}_{314} = q^{12}.
\]

As these examples indicate, all the matrix elements of $R$ are polynomials in $q$ with integer coefficients.
**Example**

\[
R_{314}^{041} = -q^2(1 - q^4)(1 - q^6)(1 - q^8),
\]

\[
R_{314}^{132} = (1 - q^6)(1 - q^8)(1 - q^4 - q^6 - q^8 - q^{10}),
\]

\[
R_{314}^{223} = q^2(1 + q^2)(1 + q^4)(1 - q^6)(1 - q^6 - q^{10}),
\]

\[
R_{314}^{314} = q^6(1 + q^2 + q^4 - q^8 - q^{10} - q^{12} - q^{14}),
\]

\[
R_{314}^{405} = q^{12}.
\]

As these examples indicate, all the matrix elements of \( R \) are polynomials in \( q \) with integer coefficients.

Furthermore, for any \((i, j, k) \in \mathbb{Z}_0^3\), there is a unique \((a, b, c)\) such that

\[
R_{ijk}^{abc} |_{q=0} = 1.
\]

This property will be utilized later to define the **tropical 3D \( R \)**.
Theorem (Kapranov-Voevodsky 1994)

\[ R \text{ satisfies the tetrahedron eq. } R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123}. \]
Theorem (Kapranov-Voevodsky 1994)

$R$ satisfies the tetrahedron eq. $R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$. 

Essence of proof. Consider $A_q(sl_4)$ and $W(sl_4) = \langle s_1, s_2, s_3 \rangle$.

$s_2s_1s_2 = s_1s_2s_1$, $s_3s_2s_3 = s_2s_3s_2$, $s_1s_3 = s_3s_1$, $s_1s_2s_3s_1s_2s_1 = s_3s_2s_3s_1s_2s_3$ (longest element)

The intertwiner for the last one is constructed in 2 different ways as

\[
\begin{align*}
123121 & \quad \Phi_{456} & 123121 & \quad P_{34} \\
123212 & \quad \Phi_{234} & 121321 & \quad \Phi_{123} \\
132312 & \quad P_{12}P_{45} & 212321 & \quad \Phi_{345} \\
312132 & \quad \Phi_{234} & 213231 & \quad P_{23}P_{56} \\
321232 & \quad \Phi_{456} & 231213 & \quad \Phi_{345} \\
321323 & \quad P_{34} & 232123 & \quad \Phi_{123} \\
323123 & & 323123 & \\
\end{align*}
\]

Equate the 2 sides, substitute $\Phi_{ijk} = R_{ijk}P_{ik}$ and cancel $P_{ij}$'s. □
Summary so far (type A case)

Weyl group elements $\leftrightarrow$ “Multi-string states”
Cubic Coxeter relation $\leftrightarrow$ 3D $R$
Reduced words for longest element $\leftrightarrow$ Tetrahedron equation
Summary so far (type A case)

Weyl group elements $\leftrightarrow$ “Multi-string states”
Cubic Coxeter relation $\leftrightarrow$ 3D $R$
Reduced words for longest element $\leftrightarrow$ Tetrahedron equation

Remark

(1) 3D $R =$ “Quantization” of Miquel’s theorem (1838)
(Bazhanov-Sergeev-Mangazeev 2008)
Summary so far (type A case)

Weyl group elements $\longleftrightarrow$ “Multi-string states”
Cubic Coxeter relation $\longleftrightarrow$ 3D $R$
Reduced words for longest element $\longleftrightarrow$ Tetrahedron equation

Remark

(1) 3D $R =$ “Quantization” of Miquel’s theorem (1838)
    (Bazhanov-Sergeev-Mangazeev 2008)
(2) Tropical (or combinatorial) analogue.
(3) Birational (or classical) analogue.

The next 3 pages demonstrate (2) and (3), which will lead to a Triad of the 3 versions of 3D Rs.
Tropical 3D $R$

Easy to show $R_{ijk}^{abc} |_{q=0} = \delta_i^{a+b-\min(a,c)} \delta_j^{\min(a,c)} \delta_k^{b+c-\min(a,c)}$.

So the tropical 3D $R := R|_{q=0} : \mathbb{Z}_\geq 0^3 \to \mathbb{Z}_\geq 0^3$ defined by

$$R(a, b, c) = (a + b - \min(a, c), \min(a, c), b + c - \min(a, c))$$

satisfies the tropical tetrahedron eq. (equality as maps on $\mathbb{Z}_\geq 0^6$):

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R_{123} \Downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
|314516\rangle \quad \downarrow \quad |311543\rangle
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R_{356} \Downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
|132516\rangle \quad \Downarrow \quad |351147\rangle
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R_{145} \Downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
|532156\rangle \quad \Downarrow \quad |151327\rangle
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R_{246} \Downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
|512354\rangle
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R_{356} \Downarrow
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
|515327\rangle \quad \Downarrow \quad |155327\rangle
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R_{123} \Downarrow
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}}
For the matrices generating a unipotent subgroup of $SL_3$,

\[
G_1(z) = \begin{pmatrix}
1 & z & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad G_2(z) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & z \\
0 & 0 & 1 \\
\end{pmatrix},
\]

the unique solution to the Lusztig type equation

\[G_1(a)G_2(b)G_1(c) = G_2(\tilde{a})G_1(\tilde{b})G_2(\tilde{c})\]

defines a birational map on 3 variables which we call **birational 3D $R$**:

\[R : (a, b, c) \mapsto (\tilde{c}, \tilde{b}, \tilde{a}) = \left(\frac{ab}{a + c}, a + c, \frac{bc}{a + c}\right).\]

Its tropical limit reproduces the Tropical 3D $R$ introduced before:

\[\mathcal{R} : (a, b, c) \mapsto (a + b - \min(a, c), \min(a, c), b + c - \min(a, c)).\]
Summary: Triad of 3D $R$

Classical limit $\rightarrow$ Tropical $R$ $\rightarrow$ Birational $R$

Intertwiner of $A_q(sl_3)$ modules

Bijection on $\mathbb{Z}^3_{\geq 0}$

Birational map on $\mathbb{C}^3$

All satisfying the tetrahedron equation.
Type B, C, $F_4$ cases: 3D analogue of reflection equation.
Recent developments

1. Type B, C, F_4 cases: 3D analogue of reflection equation.

2. Connection to the Poincaré-Birkhoff-Witt basis of $U_q^+(g)$.
Recent developments

1. Type B, C, $F_4$ cases: 3D analogue of reflection equation.

2. Connection to the Poincaré-Birkhoff-Witt basis of $U_q^+(g)$.

3. Reduction to 2D YBE: Infinitely many quantum R matrices associated with generalized quantum groups.
Recent developments

1. Type B, C, F₄ cases: 3D analogue of reflection equation.

2. Connection to the Poincaré-Birkhoff-Witt basis of $U_q^+(g)$.

3. Reduction to 2D YBE: Infinitely many quantum R matrices associated with generalized quantum groups.

4. Application to matrix product stationary states in integrable Markov processes such as totally asymmetric exclusion/zero-range processes.

Today in what follows: Mainly and will touch only briefly.
\(A_q(C_3) = \langle t_{ij} \rangle_{i,j=1}^6: \) (Reshetikhin-Takhtajan-Faddeev 1990)

\[
\begin{array}{ccc}
\pi_1 & \pi_2 & \pi_3 \\
\circ & \circ & \circ \\
F_q & F_q & F_{q^2}
\end{array}
\]

\(\pi_k(t_{ij})\) are given as follows.

\[
\pi_1: \begin{pmatrix}
    a^- & k & 0 & 0 & 0 & 0 \\
    -qk & a^+ & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & a^- & -k \\
    0 & 0 & 0 & 0 & qk & a^+
\end{pmatrix}, \quad \pi_2: \begin{pmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & a^- & k & 0 & 0 & 0 \\
    0 & -qk & a^+ & 0 & 0 & 0 \\
    0 & 0 & 0 & a^- & -k & 0 \\
    0 & 0 & 0 & qk & a^+ & 0 \\
    0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\pi_3: \begin{pmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & A^- & K & 0 & 0 \\
    0 & 0 & -q^2K & A^+ & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \langle A^{\pm}, K \rangle = \langle a^{\pm}, k \rangle_{q \rightarrow q^2}.
\]

Atsuo Kuniba (University of Tokyo)  Tetrahedron equation, 3D reflection equation
\[ \mathcal{W}(C_3) = \langle s_1, s_2, s_3 \rangle \]

\[ s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2. \]

Write simply as \[ \pi_{i_1, \ldots, i_r} := \pi_{i_1} \otimes \cdots \otimes \pi_{i_r}. \] Then,

**Equivalence**  
\[ \pi_{13} \simeq \pi_{31}, \quad \pi_{121} \simeq \pi_{212}, \quad \pi_{2323} \simeq \pi_{3232}, \]

**Intertwiner**  
\[ P_{12}(x \otimes y) = y \otimes x, \quad (\text{trivial}) \]
\[ \phi = R P_{13} \]
\[ \psi = K P_{14} P_{23} \quad (\text{same as type A}), \]

\[ K \in \text{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q), \quad R \in \text{End}(F_{q^3}). \]
Matrix elements

\[ K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K_{a \, i \, b \, j}^{c \, m \, d \, n} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle. \]

\[ K_{a \, i \, b \, j}^{c \, m \, d \, n} = 0 \text{ unless } c + m + d = a + i + b, \quad d + n - c = b + j - a. \]
Matrix elements

\[ K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K_{aibj}^{cmdn} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle. \]

\[ K_{aibj}^{cmdn} = 0 \text{ unless } c+m+d = a+i+b, \ d+n-c = b+j-a. \]

Prop. (K-Okado ’12, A more structural formula in K-Maruyama ’15)

\[ K_{a,i,0,j}^{c,m,0,n} = \sum_{\lambda \geq 0} (-1)^{m+\lambda} (q^4)_{c+\lambda} q^{\phi_2} \left[ \begin{array}{c} i, j \\ \lambda, j - \lambda, m - \lambda, i - m + \lambda \end{array} \right], \]

\[ \phi_2 = (a + c+1)(m+j-2\lambda)+m-j. \]

\[ K_{aibj}^{cmdn} = \frac{(q^4)_a}{(q^4)_c} \sum_{\alpha,\beta,\gamma \geq 0} (-1)^{\alpha+\gamma} (q^4)_d^{-\beta} q^{\phi_1} K_{c,m+d-\alpha-\beta-\gamma,0,j+b-\alpha-\beta}^{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma} \]

\[ \times \left[ \begin{array}{c} b, d - \beta, i + b - \alpha - \beta, j + b - \alpha - \beta \\ \alpha, \beta, \gamma, m - \alpha, n - \alpha, b - \alpha - \beta, d - \beta - \gamma \end{array} \right], \]

\[ \phi_1 = \alpha(\alpha+2d-2\beta-1)+(2\beta-d)(m+n+d)+\gamma(\gamma-1)-b(i+j+b). \]
Example

\[ K_{2110}^{1300} = q^8(1 - q^8), \]
\[ K_{2110}^{2110} = -q^4(1 - q^8 + q^{14}), \]
\[ K_{2110}^{2201} = -q^6(1 + q^2)(1 - q^2 + q^4 - q^6 - q^{10}), \]
\[ K_{2110}^{3011} = 1 - q^8 + q^{14}, \]
\[ K_{2110}^{3102} = -q^{10}(1 - q + q^2)(1 + q + q^2), \]
\[ K_{2110}^{4003} = q^4. \]

Properties

- \( \forall K_{ijkl}^{abcd} \in \mathbb{Z}[q], \quad K_{ijkl}^{abcd}|_{q=0} \in \{0, 1\}, \)

- For any \((i, j, k, l) \in \mathbb{Z}_{\geq 0}^4\), there is unique \((a, b, c, d) \in \mathbb{Z}_{\geq 0}^4\) such that \( K_{ijkl}^{abcd}|_{q=0} = 1. \)
Theorem (K-Okado 2012)

\( R \) and \( K \) yield the first nontrivial solution to the 3D reflection equation proposed by Isaev-Kulish in 1997:

\[
R_{489} K_{3579} R_{269} R_{258} K_{1678} K_{1234} R_{654} = R_{654} K_{1234} K_{1678} R_{258} R_{269} K_{3579} R_{489}.
\]
Theorem (K-Okado 2012)

$R$ and $K$ yield the first nontrivial solution to the 3D reflection equation proposed by Isaev-Kulish in 1997:

$$R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234}R_{654} = R_{654}K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}.$$ 

- An equality in $\text{End}(F_q^2 \otimes F_q \otimes F_q^2 \otimes F_q \otimes F_q \otimes F_q \otimes F_q^2 \otimes F_q \otimes F_q).$
- The proof is parallel with type $A$.
- Uses the reduced expressions of the longest element

$$s_1s_2s_3s_2s_1s_2s_3s_2s_3 = s_3s_2s_3s_2s_1s_2s_3s_2s_1 \in W(C_3).$$

- The 2 sides come from the 2 ways to construct the intertwiner for

$$\pi_{123212323} \simeq \pi_{323212321}$$

from $R$ and $K$. 
Physical (geometric) interpretation of the 3D reflection eq.

\[ R_{489} K_{3579} R_{269} R_{258} K_{1678} K_{1234} R_{654} = R_{654} K_{1234} K_{1678} R_{258} R_{269} K_{3579} R_{489}. \]

is a “factorization” of 3 string scattering with boundary reflections.

\( R \): Scattering amplitude of 3 strings.
\( K \): Reflection amplitude with **boundary freedom** signified by spaces 1, 3, 7.
Introduce generators of the unipotent subgroup of $Sp_4$:

\[
X_1(z) = \begin{pmatrix} 1 & z & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -z & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad X_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2z & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.
\]

Given 4 parameters $(a, b, c, d)$, it is easy to check that the matrix equation

\[
X_2(a)X_1(b)X_2(c)X_1(d) = X_1(\tilde{a})X_2(\tilde{b})X_1(\tilde{c})X_2(\tilde{d})
\]

for $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ has the unique and totally positive solution

\[
\tilde{a} = \frac{bcd}{A}, \quad \tilde{b} = \frac{A^2}{B}, \quad \tilde{c} = \frac{B}{A}, \quad \tilde{d} = \frac{ab^2c}{B},
\]

\[
A = ab + ad + cd, \quad B = ab^2 + 2abd + ad^2 + cd^2.
\]
In terms of the $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$, define the **birational 3D $\mathcal{K}$** to be the map

$$\mathcal{K} : (a, b, c, d) \mapsto (\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}).$$

(By the definition $\mathcal{K}^{-1} = \mathcal{K}$.)

Together with the birational 3D $\mathcal{R}$, it satisfies the 3D reflection eq:

$$\mathcal{R}_{456} \mathcal{R}_{489} \mathcal{K}_{3579} \mathcal{R}_{269} \mathcal{R}_{258} \mathcal{K}_{1678} \mathcal{K}_{1234} = \mathcal{K}_{1234} \mathcal{K}_{1678} \mathcal{R}_{258} \mathcal{R}_{269} \mathcal{K}_{3579} \mathcal{R}_{489} \mathcal{R}_{456},$$

which is an equality of birational maps on 9 variables.
In terms of the \((\check{a}, \check{b}, \check{c}, \check{d})\), define the **birational 3D \(K\)** to be the map

\[
K : (a, b, c, d) \mapsto (\check{d}, \check{c}, \check{b}, \check{a}).
\]

(By the definition \(K^{-1} = K\).)

Together with the birational 3D \(R\), it satisfies the 3D reflection eq:

\[
R_{456}R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234} = K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}R_{456},
\]

which is an equality of birational maps on 9 variables.

One can also define **Tropical 3D \(K\)** which forms the

Triad of Quantum, Birational and Tropical 3D \(K\)'s

in the same way as the type A case.
$B, F_4$ cases

\begin{align*}
\text{C}_3 & : \\
& \begin{array}{c}
1 \quad 2 \quad 3 \\
\circ \quad \circ \quad \circ \\
F_q \quad F_q \quad F_{q^2}
\end{array} \\
& \begin{array}{c}
R : 121 = 212 \\
K : 2323 = 3232
\end{array}

\text{B}_3 & : \\
& \begin{array}{c}
1 \quad 2 \quad 3 \\
\circ \quad \circ \quad \circ \\
F_{q^2} \quad F_{q^2} \quad F_q
\end{array} \\
& \begin{array}{c}
S : 121 = 212 \\
J : 2323 = 3232
\end{array}

\text{F}_4 & : \\
& \begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \\
\circ \quad \circ \quad \circ \quad \circ \\
F_q \quad F_q \quad F_{q^2} \quad F_{q^2}
\end{array} \\
& \begin{array}{c}
R : 121 = 212 \\
K : 2323 = 3232 \\
S : 434 = 343
\end{array}
\end{align*}
$B, F_4$ cases

- **$C_3$**
  - 1 2 3
  - $F_q$ $F_q$ $F_{q^2}$
  - $R : 121 = 212$
  - $K : 2323 = 3232$

- **$B_3$**
  - 1 2 3 4
  - $F_{q^2}$ $F_{q^2}$ $F_q$
  - $S : 121 = 212$
  - $J : 2323 = 3232$

- **$F_4$**
  - 1 2 3 4
  - $F_q$ $F_q$ $F_{q^2}$ $F_{q^2}$
  - $R : 121 = 212$
  - $K : 2323 = 3232$
  - $S : 434 = 343$

$R \in \text{End}(F_q \otimes F_q \otimes F_q)$, \quad $K \in \text{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$
### $B, F_4$ cases

<table>
<thead>
<tr>
<th></th>
<th>$C_3$</th>
<th>$B_3$</th>
<th>$F_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 3</td>
<td>1 2 3</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td></td>
<td>$F_q$ $F_q$ $F_{q^2}$</td>
<td>$F_{q^2}$ $F_{q^2}$ $F_q$</td>
<td>$F_q$ $F_q$ $F_{q^2}$ $F_{q^2}$</td>
</tr>
<tr>
<td>$R$ $K$</td>
<td>$121 = 212$</td>
<td>$2323 = 3232$</td>
<td>$121 = 212$</td>
</tr>
<tr>
<td></td>
<td>$2323 = 3232$</td>
<td>$2323 = 3232$</td>
<td>$2323 = 3232$</td>
</tr>
</tbody>
</table>

\[
R \in \text{End}(F_q \otimes F_q \otimes F_q), \quad K \in \text{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)
\]

\[
S = R\big|_{q \rightarrow q^2} \in \text{End}(F_{q^2} \otimes F_{q^2} \otimes F_{q^2})
\]

\[
J = P_{14}P_{23}KP_{23}P_{14} \in \text{End}(F_q \otimes F_{q^2} \otimes F_q \otimes F_{q^2}).
\]
\( B, F_4 \) cases

\[
\begin{array}{c|c|c}
\text{C}_3 & \text{B}_3 & \text{F}_4 \\
\hline
1 & 2 & 3 \\
F_q & F_q & F_{q^2} \\
\hline
1 & 2 & 3 \\
F_{q^2} & F_{q^2} & F_q \\
\hline
1 & 2 & 3 & 4 \\
F_q & F_q & F_{q^2} & F_{q^2} \\
\end{array}
\]

\[
R : 121 = 212 \\
K : 2323 = 3232 \\
S : 121 = 212 \\
J : 2323 = 3232 \\
R : 121 = 212 \\
K : 2323 = 3232 \\
S : 434 = 343 \\
\]

\[
R \in \text{End}(F_q \otimes F_q \otimes F_q), \quad K \in \text{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)
\]

\[
S = R|_{q \rightarrow q^2} \in \text{End}(F_{q^2} \otimes F_{q^2} \otimes F_{q^2})
\]

\[
J = P_{14}P_{23}KP_{23}P_{14} \in \text{End}(F_q \otimes F_{q^2} \otimes F_q \otimes F_{q^2}).
\]

Both \((R, K)\) and \((S, J)\) satisfy the 3D reflection equation.
A reduced expression of the longest element of $W(F_4)$ is

$$s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1 \ (= -1, \text{ length } 24).$$
A reduced expression of the longest element of $W(F_4)$ is

$$s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1$$

(= $-1$, length 24).

The intertwiner for

$$\pi_{434234232123423123412321} \simeq \pi_{\text{reverse order}}$$

can be constructed by composition of $R, K, S$ in two ways, which must coincide.
A reduced expression of the longest element of $W(F_4)$ is

$$s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1$$

($= -1$, length 24).

The intertwiner for

$$\pi_{434234232123423123412321} \simeq \pi_{\text{reverse order}}$$

can be constructed by composition of $R, K, S$ in two ways, which must coincide. This leads to the $F_4$-analogue of the tetrahedron equation:

$$16R's, 16S's \text{ and } 18K's \text{ acting on } F_{q_{i_1}} \otimes \cdots \otimes F_{q_{i_{24}}}.$$
Another aspect: Connection with PBW basis

\[ U_q^+(sl_3) = \langle e_1, e_2 \rangle: \text{Subalgebra of } U_q(sl_3) \text{ obeying the } q\text{-Serre relation:} \]

\[ e_1^2 e_2 - (q + q^{-1})e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad e_2^2 e_1 - (q + q^{-1})e_2 e_1 e_2 + e_1 e_2^2 = 0. \]
Another aspect: Connection with PBW basis

\( U_q^+(sl_3) = \langle e_1, e_2 \rangle \): Subalgebra of \( U_q(sl_3) \) obeying the \( q \)-Serre relation:

\[
e_1^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad e_2^2 e_1 - (q + q^{-1}) e_2 e_1 e_2 + e_1 e_2^2 = 0.
\]

Two PBW bases: \( \{ E^{a,b,c} \}_{(a,b,c) \in \mathbb{Z}_3^{\geq 0}} \), \( \{ E'_{a,b,c} \}_{(a,b,c) \in \mathbb{Z}_3^{\geq 0}} \)

\[
E^{a,b,c} = e_1^a ([e_2,e_1]_q)^b e_2^c \frac{[a]![b]![c]!}{[a]![b]![c]!}, \quad E'_{a,b,c} = E^{a,b,c} |_{e_1 \leftrightarrow e_2},
\]

([a]! := \prod_{i=1}^{a} \frac{q^i - q^{-i}}{q - q^{-1}} )
Another aspect: Connection with PBW basis

\(U_q^+(sl_3) = \langle e_1, e_2 \rangle\): Subalgebra of \(U_q(sl_3)\) obeying the \(q\)-Serre relation:

\[ e_1^2 e_2 - (q + q^{-1})e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad e_2^2 e_1 - (q + q^{-1})e_2 e_1 e_2 + e_1 e_2^2 = 0. \]

Two PBW bases: \( \{ E^{a,b,c} \}_{(a,b,c) \in \mathbb{Z}^3_{\geq 0}} \), \( \{ E'^{a,b,c} \}_{(a,b,c) \in \mathbb{Z}^3_{\geq 0}} \)

\[ E^{a,b,c} = \frac{e_1^a ([e_2, e_1]_q)^b e_2^c}{[a]![b]![c]!}, \quad E'^{a,b,c} = E^{a,b,c} |_{e_1 \leftrightarrow e_2}, \]

\([a]! := \prod_{i=1}^{a} q^i - q^{-i} \]

**Theorem (Sergeev 2009)**

\[ E^{a,b,c} = \sum_{ijk} R_{i,j,k}^a b c E'^{k,j,i}. \]

Namely, \( 3D \ R = \) Transition matrix of the PBW bases of \( U_q^+(sl_3) \)!
Generalizations

For arbitrary classical simple Lie algebra $g$, let $w_0$ be the longest element of its Weyl group.

- $\Phi :=$ Intertwiner of the irreducible $A_q(g)$ modules labeled by $w_0$.
- $\Gamma :=$ Transition matrix of the PBW bases of $U_q^+(g)$.
Generalizations

For arbitrary classical simple Lie algebra $g$, let $w_0$ be the longest element of its Weyl group.

- $\Phi :=$ Intertwiner of the irreducible $A_q(g)$ modules labeled by $w_0$.
- $\Gamma :=$ Transition matrix of the PBW bases of $U_q^+(g)$.

The both $\Phi$ and $\Gamma$ are associated with a pair of reduced expressions of $w_0$. Take the pair common.
Generalizations

For arbitrary classical simple Lie algebra $g$, let $w_0$ be the longest element of its Weyl group.

- $\Phi := \text{Intertwiner of the irreducible } A_q(g) \text{ modules labeled by } w_0$.
- $\Gamma := \text{Transition matrix of the PBW bases of } U_q^+(g)$.

The both $\Phi$ and $\Gamma$ are associated with a pair of reduced expressions of $w_0$. Take the pair common.

**Theorem (K-Okado-Yamada 2013)**

$$\Phi = \Gamma.$$

More aspects have been explored in [Tanisaki 2014] and [Y. Saito 2014].
Generalizations

For arbitrary classical simple Lie algebra $g$, let $w_0$ be the longest element of its Weyl group.

- $\Phi :=$ Intertwiner of the irreducible $A_q(g)$ modules labeled by $w_0$.
- $\Gamma :=$ Transition matrix of the PBW bases of $U_q^+(g)$.

The both $\Phi$ and $\Gamma$ are associated with a pair of reduced expressions of $w_0$. Take the pair common.

Theorem (K-Okado-Yamada 2013)

$$\Phi = \Gamma.$$  

More aspects have been explored in [Tanisaki 2014] and [Y. Saito 2014].

「いっぺーにふぇーでーびたん！」
("Thank you so much!" in Okinawan)