$A_q(A_3) \hookleftarrow A_q(A_2)$

$\Pi_{121321} \simeq \Pi_{321323}$ $\quad R : \Pi_{121} \simeq \Pi_{212}$ $\quad$ 2D Reduction

tetrahedron eq $\quad R\,LLL = LLL\,R \longrightarrow$ YBE

$\cdots$ quantized YBE

$A_q(F_4)$ $\quad A_q(B_3)$ $\hookleftarrow$ $\quad A_q(B_2) \simeq A_q(C_2)$

$\#$

$A_q(C_3)$

F_4 eq

$K : \Pi_{1212} \simeq \Pi_{2121}$

$\Pi_{123121323}$

$\simeq \Pi_{323121321}$ $\quad K(LGLG) = (GLGL)K \longrightarrow$ RE

3D Reflection eq $\quad \cdots$ quantized RE

$A_q(G_2)$

None $\hookleftarrow$ $F : \Pi_{121212} \simeq \Pi_{212121}$

$F(LJLJLJ) = (JLJLJL)F \longrightarrow G_2$ RE

$\cdots$ quantized G_2 RE

$A_q(SL_2)$ Quantized coordinate ring ... Hopf alg dual to $U_q(sl_2)$

generators

$$\left(\begin{tikzcd} t_{11} \arrow[r] \arrow[d] & t_{12} \arrow[d] \\ t_{21} \arrow[r] & t_{22} \end{tikzcd}\right)$$

Relations

$$xy = q\, yx \quad \text{for} \quad x \to y$$

$$[t_{12}, t_{21}] = 0$$

$$[t_{11}, t_{22}] = (q - q^{-1})\, t_{12} t_{21}$$

$$t_{11} t_{22} - q\, t_{12} t_{21} = 1$$

irreps. $\quad \downarrow \pi_1$

$$\begin{pmatrix} a^- & k \\ -qk & a^+ \end{pmatrix}$$

$$a^{\pm},\ k \in \mathrm{End}(F) \qquad F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$$

$$a^- |m\rangle = (1 - q^{2m})|m-1\rangle, \quad a^+|m\rangle = |m+1\rangle, \quad k|m\rangle = q^m |m\rangle$$

... q-oscillators

$$\Delta t_{ij} = \sum_{l} t_{il} \otimes t_{lj}$$

$A_q(sl_3)$ $\quad \overset{\pi_1}{\circ}$——$\overset{\pi_2}{\circ}$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} a^- & k & 0 \\ -qk & a^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & a^- & k \\ 0 & -qk & a^+ \end{pmatrix}}$$

<u>Th</u> (Soibelman '92)

(1) $\mathfrak{g}$: Classical finite type, rank n

$\exists$ irreducible representations

$\pi_1, \pi_2, \dots, \pi_n$ on $\mathcal{F}$ such that

$$\begin{tikzcd} A_q(\mathfrak{g}) \arrow[rr, "\pi_i"] \arrow[dr] & & \mathrm{End}(\mathcal{F}) \\ & A_q(sl_{2,i}) \arrow[ur] & \end{tikzcd}$$

π_1 for $A_q(sl_2)$ with $q \to q_i$

Weyl gr. $\qquad$ Simple reflections

(2) $W(\mathfrak{g}) = \langle s_1, \dots, s_n \rangle$

$\pi_{j_1, \dots, j_l}$

$$s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_l} \longrightarrow \pi_{i_1} \otimes \cdots \otimes \pi_{i_l} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_l}$$

is reduced $\qquad$ and they are irreducible

$A_q(sl_3)$ $\quad \overset{\pi_1}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{\pi_2}{\circ}$ $\qquad s_1 s_2 s_1 = s_2 s_1 s_2 \quad (s_i^2 = 1)$

$$\pi_{121} \simeq \pi_{212}$$

$$(\Phi\, \pi_{121})(f) = (\pi_{212}\, \Phi)(f) \qquad \forall f \in A_q(sl_3)$$

$$\Phi(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \qquad \Phi, R \in \mathrm{End}(\mathcal{F}^{\otimes 3})$$

$$\Phi = R P_{13} \qquad P_{13}(u \otimes v \otimes w) = w \otimes v \otimes u.$$

$$\begin{aligned}
t_{11}:&\quad R(\mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- - q\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{k}) = (1 \otimes \mathbf{a}^- \otimes 1)R,\\
t_{12}:&\quad R(\mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{a}^-) = (1 \otimes \mathbf{k} \otimes \mathbf{a}^-)R,\\
t_{13}:&\quad R(1 \otimes \mathbf{k} \otimes \mathbf{k}) = (1 \otimes \mathbf{k} \otimes \mathbf{k})R,\\
t_{21}:&\quad R(\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + \mathbf{a}^- \otimes 1 \otimes \mathbf{k}) = (\mathbf{a}^- \otimes \mathbf{k} \otimes 1)R,\\
t_{22}:&\quad R(\mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^+ - q\mathbf{k} \otimes 1 \otimes \mathbf{k}) = (\mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^- - q\mathbf{k} \otimes 1 \otimes \mathbf{k})R,\\
t_{23}:&\quad R(1 \otimes \mathbf{k} \otimes \mathbf{a}^+) = (\mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{a}^+)R,\\
t_{31}:&\quad R(\mathbf{k} \otimes \mathbf{k} \otimes 1) = (\mathbf{k} \otimes \mathbf{k} \otimes 1)R,\\
t_{32}:&\quad R(\mathbf{a}^+ \otimes \mathbf{k} \otimes 1) = (\mathbf{k} \otimes \mathbf{a}^+ \otimes \mathbf{a}^- + \mathbf{a}^+ \otimes 1 \otimes \mathbf{k})R,\\
t_{33}:&\quad R(1 \otimes \mathbf{a}^+ \otimes 1) = (\mathbf{a}^+ \otimes 1 \otimes \mathbf{a}^+ - q\mathbf{k} \otimes \mathbf{a}^+ \otimes \mathbf{k})R,
\end{aligned}$$

$$R(|i\rangle\otimes|j\rangle\otimes|k\rangle) = \sum_{abc} R^{abc}_{ijk}\,|a\rangle\otimes|b\rangle\otimes|c\rangle$$

$$\begin{aligned}
R^{041}_{314} &= -q^2(1-q^4)(1-q^6)(1-q^8),\\
R^{132}_{314} &= (1-q^6)(1-q^8)(1-q^4-q^6-q^8-q^{10}),\\
R^{223}_{314} &= q^2(1+q^2)(1+q^4)(1-q^6)(1-q^6-q^{10}),\\
R^{314}_{314} &= q^6(1+q^2+q^4-q^8-q^{10}-q^{12}-q^{14}),\\
R^{405}_{314} &= q^{12}.
\end{aligned}$$

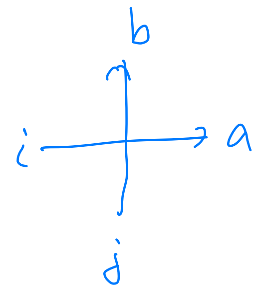
$$R^{abc}_{ijk} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\, q^{ik+b} \oint \frac{du}{2\pi \mathrm{i} u^{b+1}} \frac{(-q^{2+a+c}u;q^2)_\infty(-q^{-i-k}u;q^2)_\infty}{(-q^{a-c}u;q^2)_\infty(-q^{c-a}u;q^2)_\infty}.$$

$$= \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{\lambda+\mu=b} (-1)^\lambda q^{ik+b+\lambda(c-a)+\mu(\mu-i-k-1)} \binom{\lambda+a}{a}_{q^2} \binom{i}{\mu}_{q^2}$$

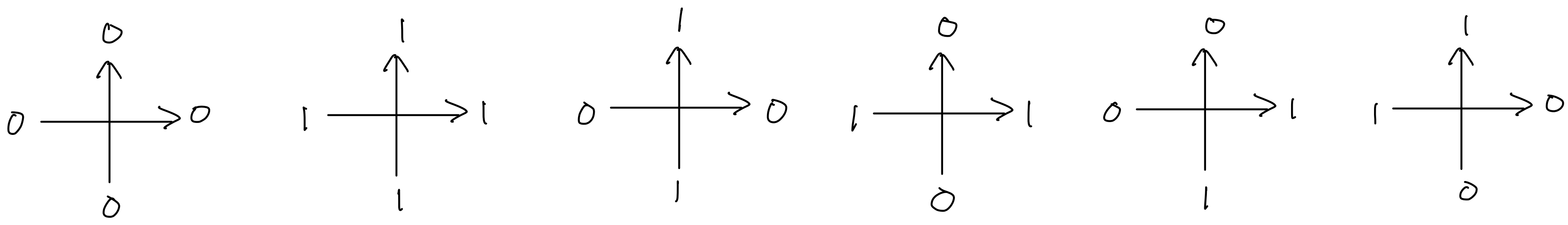
$$= \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\ (-1)^b q^{ik+b(k-i+1)} \binom{a+b}{a}_{q^2} {}_2\phi_1\left(\begin{matrix} q^{-2b}, q^{-2i} \\ q^{-2a-2b}\end{matrix}; q^2, q^{-2c}\right)$$

Quantized YBE

$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \qquad L \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_q)$



$$L(v_i \otimes v_j \otimes |m\rangle) = \sum_{a,b} v_a \otimes v_b \otimes L^{ab}_{ij}|m\rangle$$

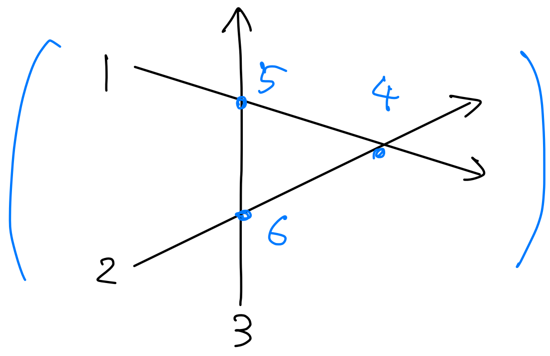


$1 \qquad 1 \qquad -qk \qquad k \qquad a^+ \qquad a^-$

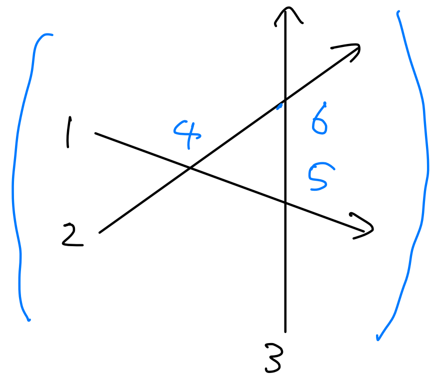
$\cdots$ q-oscillator valued 6 vertex

<u>Prop</u> (RLLL = LLLR)

$R_{456}L_{124}L_{135}L_{236}$ $\qquad R\circ$



$=$



$\circ R$

$= L_{236}L_{135}L_{124}R_{456}$

$(\because \iff \Phi\,\Pi_{121} = \Pi_{212}\,\Phi)$

Compatibility $\quad A_q(sl_4) \;(\longleftrightarrow A_q(sl_3))$

$$\overset{\pi_1}{\circ}\!-\!\!-\!\!-\overset{\pi_2}{\circ}\!-\!\!-\!\!-\overset{\pi_3}{\circ}$$

$$\begin{pmatrix} t_{11} & \cdots & t_{14} \\ \vdots & & \vdots \\ t_{41} & \cdots & t_{44} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} a^- & k & & \\ -qk & a^+ & & \\ & & 1 & \\ & & & 1 \end{pmatrix}},\ \overset{\pi_2}{\begin{pmatrix} 1 & & & \\ & a^- & k & \\ & -qk & a^+ & \\ & & & 1 \end{pmatrix}},\ \overset{\pi_3}{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & a^- & k \\ & & -qk & a^+ \end{pmatrix}}$$

$$\Phi\, \pi_{121} = \pi_{212}\, \Phi \qquad \Phi = R\, P_{13} \quad \text{same } R \text{ as } A_q(sl_3)$$

$$\Phi\, \pi_{232} = \pi_{323}\, \Phi$$

$$P\, \pi_{13} = \pi_{31}\, P \qquad P(u \otimes v) = v \otimes u$$

$$W(sl_4) \ni s_1 s_2 s_1 s_3 s_2 s_1 = s_3 s_2 s_1 s_3 s_2 s_3 \quad \cdots \text{ longest}$$

$A_q(sl_4)$ $\quad \Pi_{121321} \simeq \Pi_{321323}$

$$
\begin{tikzcd}[column sep=small]
 & 121321 \arrow[dl, "\Phi_{123}"'] \arrow[dr, "P_{34}"] & \\
212321 \arrow[d, "\Phi_{345}"'] & & 123121 \arrow[d, "\Phi_{456}"] \\
213231 \arrow[d, "P_{23}P_{56}"'] & & 123212 \arrow[d, "\Phi_{234}"] \\
231213 \arrow[d, "\Phi_{345}"'] & & 132312 \arrow[d, "P_{12}P_{45}"] \\
232123 \arrow[d, "\Phi_{123}"'] & & 312132 \arrow[d, "\Phi_{234}"] \\
323123 \arrow[dr, "P_{34}"'] & & 321232 \arrow[dl, "\Phi_{456}"] \\
 & 321323 &
\end{tikzcd}
$$

Commutativity

$\downarrow$

$$P_{34}\Phi_{123}\Phi_{345}P_{23}P_{56}\Phi_{345}\Phi_{123}$$

$$= \Phi_{456}\Phi_{234}P_{12}P_{45}\Phi_{234}\Phi_{456}P_{34}$$

$\downarrow$ $\quad \Phi_{ijk} = R_{ijk}P_{ik}$

$$R_{124}R_{135}R_{236}R_{456}$$

$$= R_{456}R_{236}R_{135}R_{124}$$

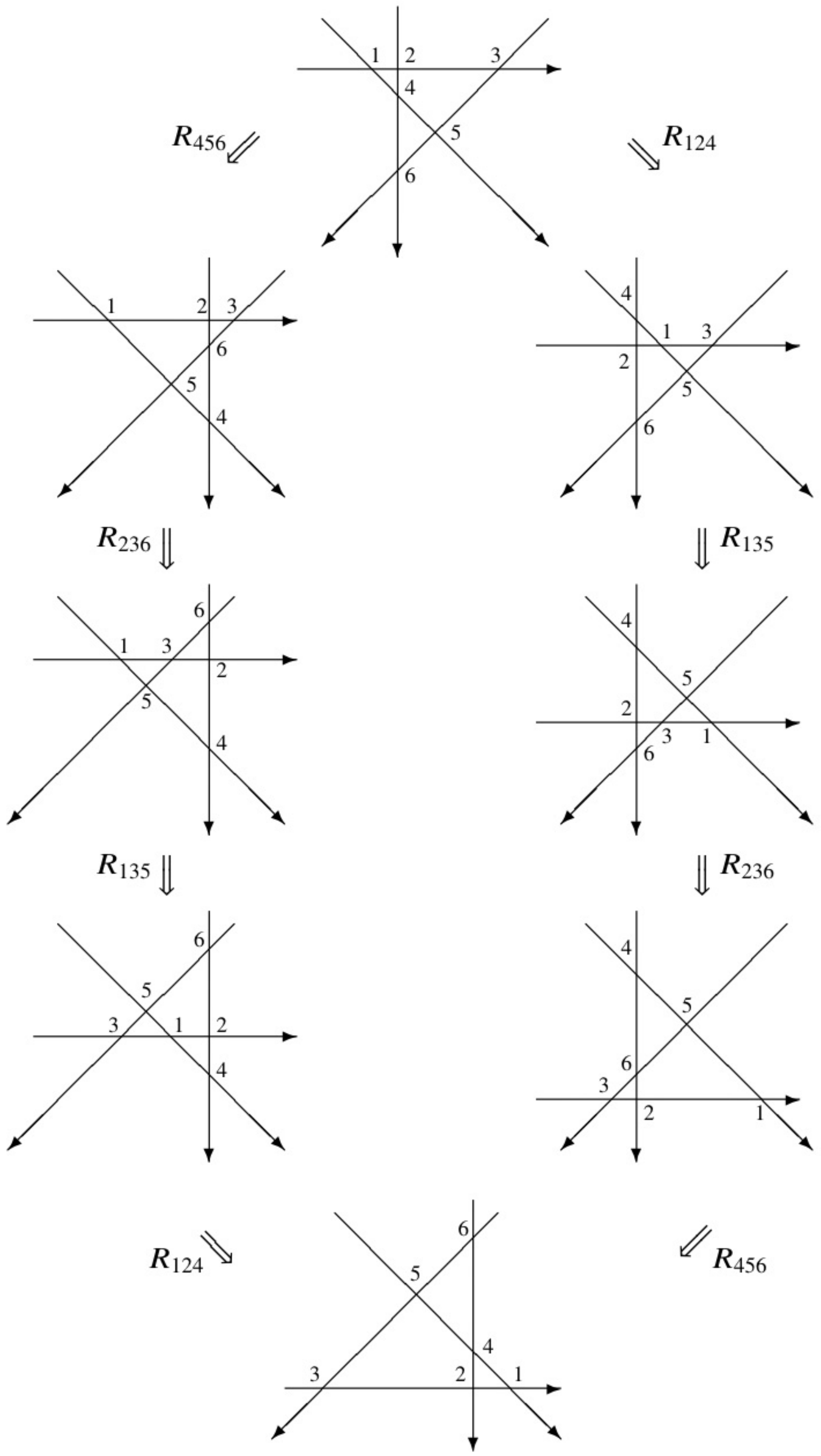
... Tetrahedron eq

Derivation based on

Quantized YBE

RLLL = LLLR

$$R_{124} R_{135} R_{236} R_{456} = R_{456} R_{236} R_{135} R_{124}$$



$A_q(C_2)$

$$\begin{array}{ccc} \pi_1 & & \pi_2 \\ \circ & \Longleftarrow & \circ \\ q_1 = q & & q_2 = q^2 \end{array}$$

$A^{\pm} = a^{\pm}\big|_{q \to q^2} \qquad K = k\big|_{q \to q^2}$

$$\begin{pmatrix} t_{11} & \cdots & t_{14} \\ \vdots & & \vdots \\ t_{41} & \cdots & t_{44} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} a^- & k & & \\ -qk & a^+ & & \\ & & a^- & -k \\ & & qk & a^+ \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & & & \\ & A^- & K & \\ & -q^2K & A^+ & \\ & & & 1 \end{pmatrix}}$$

$$A^-|m\rangle = (1-q^{4m})|m-1\rangle, \quad A^+|m\rangle = |m+1\rangle, \quad K|m\rangle = q^{2m}|m\rangle$$

$$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \quad \longrightarrow \quad \Phi\, \pi_{1212} = \pi_{2121}\, \Phi$$

$$\Phi = K P_{14} P_{23} \qquad \Phi,\ K \in \mathrm{End}(\mathcal{F}^{\otimes 4})$$

$$t_{11}: \quad [1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{a}^- - q1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{k},\ K] = 0, \tag{5.41}$$

$$t_{12}: \quad (1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+)K$$
$$= K(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^- - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k}), \tag{5.42}$$

$$t_{13}: \quad (1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-)K = K(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^-), \tag{5.43}$$

$$t_{14}: \quad [1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k}, K] = 0, \tag{5.44}$$

$$t_{21}: \quad (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^- - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k})K$$
$$= K(1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+), \tag{5.45}$$

$$t_{22}: \quad [\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+,\ K] = 0, \tag{5.46}$$

$$t_{23}: \quad (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k})K$$
$$= K(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k}), \tag{5.47}$$

$$t_{24}: \quad (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+)K = K(1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+), \tag{5.48}$$

$$t_{31}: \quad (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^-)K = K(1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-), \tag{5.49}$$

$$t_{32}: \quad (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k})K$$
$$= K(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k}), \tag{5.50}$$

$$t_{33}: \quad [\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^-,\ K] = 0, \tag{5.51}$$

$$t_{34}: \quad (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k})K$$
$$= K(1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-), \tag{5.52}$$

$$t_{41}: \quad [1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k},\ K] = 0 \quad \text{(same as } t_{14}\text{)}, \tag{5.53}$$

$$t_{42}: \quad (1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+)K = K(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+), \tag{5.54}$$

$$t_{43}: \quad (1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-)K$$
$$= K(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k}), \tag{5.55}$$

$$t_{44}: \quad [1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{a}^+ - q1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{k},\ K] = 0. \tag{5.56}$$

$$K(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle) = \sum_{abcd} K^{abcd}_{ijkl} |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle$$

$$K^{0000}_{0000} = 1$$

$$K^{1300}_{2110} = q^8(1 - q^8),$$
$$K^{2110}_{2110} = -q^4(1 - q^8 + q^{14}),$$
$$K^{2201}_{2110} = -q^6(1 + q^2)(1 - q^2 + q^4 - q^6 - q^{10}),$$
$$K^{3011}_{2110} = 1 - q^8 + q^{14},$$
$$K^{3102}_{2110} = -q^{10}(1 - q + q^2)(1 + q + q^2),$$
$$K^{4003}_{2110} = q^4.$$

$$K_{ijkl}^{abcd} = \delta_{i+j+k}^{a+b+c}\,\delta_{j+2k+l}^{b+2c+d}\,\frac{(q^4)_i}{(q^4)_a}\sum_{\alpha,\beta,\gamma\geq 0}\frac{(-1)^{\alpha+\gamma}}{(q^4)_{c-\beta}}q^{\phi_1}$$
$$\times K_{a,b+c-\alpha-\beta-\gamma,0,c+d-\alpha-\beta-\gamma}^{i,j+k-\alpha-\beta-\gamma,0,k+l-\alpha-\beta-\gamma}\left\{\begin{matrix} k, c-\beta, j+k-\alpha-\beta, k+l-\alpha-\beta \\ \alpha,\beta,\gamma, b-\alpha, d-\alpha, k-\alpha-\beta, c-\beta-\gamma\end{matrix}\right\}_{q^2},$$

$$\phi_1 = \alpha(\alpha+2c-2\beta-1)+(2\beta-c)(b+c+d)+\gamma(\gamma-1)-k(j+k+l),$$

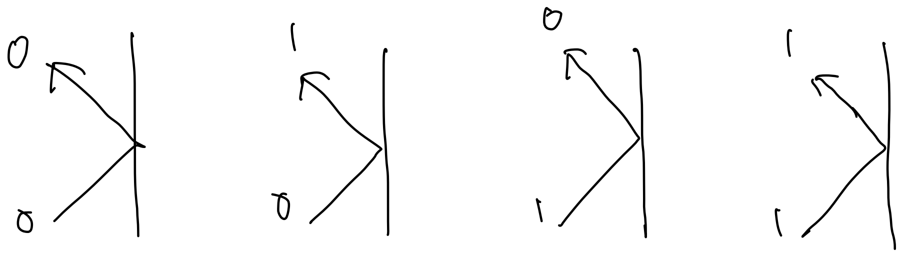
where the special case $K_{\bullet,\bullet,0,\bullet}^{\bullet,\bullet,0,\bullet}$ *appearing in the sum is given by*

$$K_{ij0l}^{ab0d} = \delta_{i+j}^{a+b}\,\delta_{j+l}^{b+d}\sum_{\lambda\geq 0}(-1)^{b+\lambda}\frac{(q^4)_{a+\lambda}}{(q^4)_a}q^{\phi_2}\binom{l}{\lambda}_{q^2}\binom{j}{b-\lambda}_{q^2},$$
$$\phi_2 = (i+a+1)(b+l-2\lambda)+b-l,$$

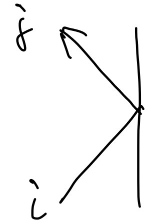
$$\begin{Bmatrix} i_1 \cdots i_l \\ j_1 \cdots j_k\end{Bmatrix}_{q^2} = \frac{\prod_\alpha (q^2;q^2)_{i_\alpha}}{\prod_\beta (q^2;q^2)_{j_\beta}} \qquad \binom{m}{n}_{q^2} = \begin{Bmatrix} m \\ n, m-n\end{Bmatrix}_{q^2}$$

Quantized Reflection Eq.

$G \in \mathrm{End}(V \otimes \mathcal{F})$ $\qquad G(v_i \otimes |m\rangle) = \sum_j v_j \otimes G_i^j |m\rangle$



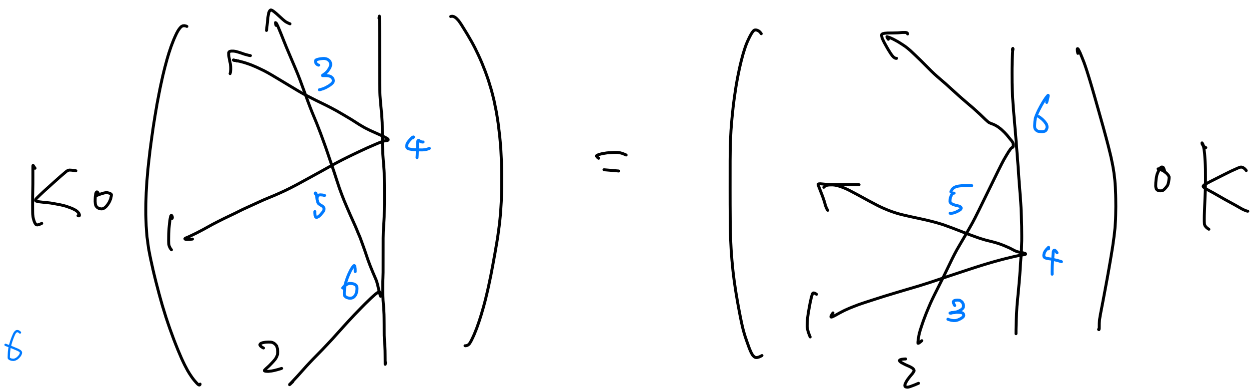
$a^+ \qquad k \qquad -qk \qquad a^-$



Prop ($KLGLG = GLGLK$) $\qquad$ ($\leftarrow L = L^{\text{type}A}|_{q \to q^2}$)

$$K_{3456} L_{213} G_{14} L_{125} G_{26}$$

$$= G_{26} L_{215} G_{14} L_{123} K_{3456}$$



$(\because \iff \Phi\, \Pi_{1212} = \Pi_{2121}\, \Phi)$

$A_q(C_3)$ $\quad \underbrace{\overset{\pi_1}{\circ} - \overset{\pi_2}{\circ}}_{R} \underbrace{\Leftarrow \overset{\pi_3}{\circ}}_{K}$

$W(C_3) \ni s_1 s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_3 \quad (= -1)$

$\cdots$ longest

$\Phi\, \pi_{121} = \pi_{212}\, \Phi \qquad \Phi = R P_{13}$

$\Psi\, \pi_{2323} = \pi_{3232}\, \Psi \qquad \Psi = K P_{14} P_{23}$

$P\, \pi_{13} = \pi_{31}\, P$

$\pi_{123121323}$

Φ

Ψ

P

$\circlearrowright$

$\pi_{323121321}$

$\Downarrow$

3D Reflection eqs.

$A_q(B_3)$ $\quad \underbrace{\overset{\pi_1}{\circ} - \overset{\pi_2}{\circ}}_{S} \underbrace{\Rightarrow \overset{\pi_3}{\circ}}_{K^{op}}$

$S = R|_{q \to q^2} \qquad K^{op} = P_{14} P_{13} K P_{14} P_{13}$

C_3 :

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

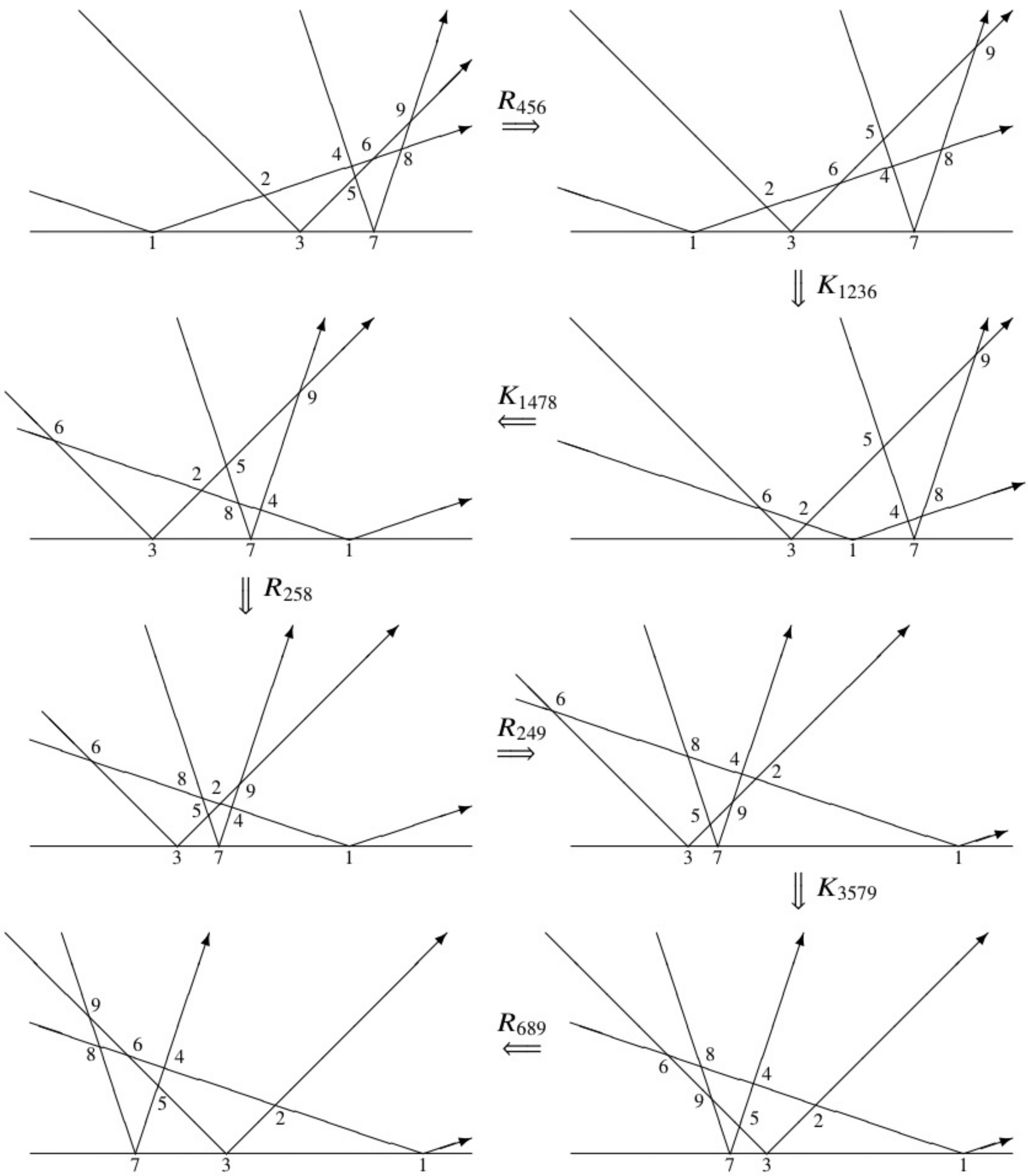
B_3 :

$$S_{689}K_{9753}S_{249}S_{258}K_{8741}K_{6321}S_{456} = S_{456}K_{6321}K_{8741}S_{258}S_{249}K_{9753}S_{689}.$$

Derivation based on

Quantized RE: $K\,LGLG = GLGL\,K$

" YBE: $R\,LLL = LLL\,R$



$$
\begin{array}{ccccc}
R_{\text{quantum}} & \xrightarrow{\quad} & R_{\text{combinatorial}} & \xleftarrow{\quad} & R_{\text{birational}} \\
 & q \longrightarrow 0 & & \text{Ultradiscretization} & \\
K_{\text{quantum}} & \xrightarrow{\quad} & K_{\text{combinatorial}} & \xleftarrow{\quad} & K_{\text{birational}}
\end{array}
$$

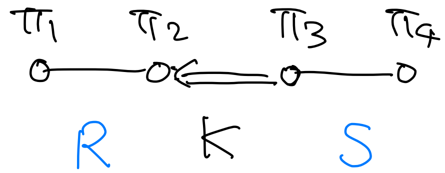
Type C

$$
\begin{tikzcd}
 & (211202341) \arrow[dl, "R_{456}"'] \arrow[dr, "R_{689}"] & \\
(211020341) \arrow[d, "K_{1236}"'] & & (211205314) \arrow[d, "K_{3579}"] \\
(301021341) \arrow[d, "K_{1478}"'] & & (213205118) \arrow[d, "R_{249}"] \\
(301021341) \arrow[d, "R_{258}"'] & & (223105119) \arrow[d, "R_{258}"] \\
(321001361) \arrow[d, "R_{249}"'] & & (213115109) \arrow[d, "K_{1478}"] \\
(311101360) \arrow[d, "K_{3579}"'] & & (313015119) \arrow[d, "K_{1236}"] \\
(313101164) \arrow[dr, "R_{689}"'] & & (313015119) \arrow[dl, "R_{456}"] \\
 & (313106119) &
\end{tikzcd}
$$

Type B

$$
\begin{tikzcd}
 & (211202341) \arrow[dl, "R^B_{456}"'] \arrow[dr, "R^B_{689}"] & \\
(211020341) \arrow[d, "K^B_{1236}"'] & & (211205314) \arrow[d, "K^B_{3579}"] \\
(401021341) \arrow[d, "K^B_{1478}"'] & & (213205116) \arrow[d, "R^B_{249}"] \\
(301021431) \arrow[d, "R^B_{258}"'] & & (223105117) \arrow[d, "R^B_{258}"] \\
(321001451) \arrow[d, "R^B_{249}"'] & & (213115107) \arrow[d, "K^B_{1478}"] \\
(311101450) \arrow[d, "K^B_{3579}"'] & & (413015117) \arrow[d, "K^B_{1236}"] \\
(314101153) \arrow[dr, "R^B_{689}"'] & & (314014117) \arrow[dl, "R^B_{456}"] \\
 & (314105117) &
\end{tikzcd}
$$

$A_q(F_4)$



$\Phi\, \pi_{121} = \pi_{212}\, \Phi$ $\qquad \Phi = R\, P_{13}$ $\qquad P\, \pi_{ij} = \pi_{ji}\, P \quad |i-j| \geq 2$

$\Psi\, \pi_{1212} = \pi_{2121}\, \Psi$ $\qquad \Psi = K\, P_{14} P_{23}$

$\Upsilon\, \pi_{343} = \pi_{434}\, \Upsilon$ $\qquad \Upsilon = S\, P_{13}$

$w_0 = s_{i_1} \cdots s_{i_{24}}$ longest word $(= -1)$

$\overline{w_0} = s_{i_{24}} \cdots s_{i_1}$ another longest word

$\pi_{i_1, \dots, i_{24}}$

Φ^{16}

Ψ^{18}

$P,\ \Upsilon^{16}$

$\pi_{i_{24} \cdots i_1}$

$\rightarrow$ F_4 analogue of TE

$$
\begin{array}{rll}
\mathbf{w}_0: & \underline{434}23\underline{42}3212342312\underline{341}2321 & P_{6,7}P_{18,19}P_{19,20}\Upsilon^{-1}_{1,2,3} \\
 & 34\underline{3232}432123423\underline{121}342321 & \Psi^{-1}_{3,4,5,6}\Phi_{16,17,18} \\
 & 3\underline{42}32\underline{343212}3423212342321 & P_{2,3}P_{10,11}P_{9,10}P_{8,9}\Upsilon_{6,7,8} \\
 & 3243\underline{24}3212\underline{434}232123\underline{42}321 & P_{5,6}P_{20,21}\Upsilon^{-1}_{11,12,13} \\
 & 32\underline{434}2321234\underline{32321}2324321 & \Upsilon^{-1}_{3,4,5}P_{16,17}\Psi^{-1}_{13,14,15,16} \\
 & 3234\underline{32321}23\underline{42}321\underline{3232}4321 & P_{8,9}\Psi^{-1}_{5,6,7,8}P_{12,13}\Psi^{-1}_{17,18,19,20} \\
 & 323\underline{42}321\underline{3232}432123\underline{234321} & P_{4,5}\Psi^{-1}_{9,10,11,12}P_{19,20}P_{23,24}P_{22,23}\Upsilon_{20,21,22} \\
 & \underline{3232}4321232\underline{343212}3423214 & \Psi^{-1}_{1,2,3,4}P_{16,17}P_{15,16}P_{14,15}\Upsilon_{12,13,14} \\
 & 232\underline{343212}3\underline{24}3212\underline{434}23214 & \Upsilon^{-1}_{17,18,19}P_{11,12}P_{8,9}P_{7,8}P_{6,7}\Upsilon_{4,5,6} \\
 & 23243212\underline{434}232123\underline{432321}4 & \Upsilon^{-1}_{9,10,11}P_{18,19}P_{22,23}\Psi^{-1}_{19,20,21,22} \\
 & 23243\underline{2123432321}232432134 & P_{9,10}P_{8,9}\Phi^{-1}_{6,7,8}P_{14,15}\Psi^{-1}_{11,12,13,14} \\
 & 2324\underline{31}23412321\underline{3232}432134 & P_{5,6}\Psi^{-1}_{15,16,17,18} \\
 & 232\underline{41}3234123\underline{2123}2\underline{34321}34 & P_{4,5}P_{15,16}\Phi^{-1}_{13,14,15}P_{21,22}P_{20,21}\Upsilon_{18,19,20} \\
 & 23214323412\underline{31}23124321434 & P_{12,13} \\
 & 232143234\underline{121}323124321434 & \Phi_{10,11,12} \\
 & 23214323\underline{4212323}124321434 & P_{10,11}P_{9,10}\Psi_{12,13,14,15} \\
 & 23214\underline{32321}4323\underline{21243}21434 & P_{9,10}\Psi^{-1}_{6,7,8,9}P_{18,19}P_{17,18}\Phi^{-1}_{15,16,17} \\
 & 2321\underline{42}321\underline{3432}3\underline{124}3\underline{121}434 & P_{5,6}P_{12,13}\Upsilon_{10,11,12}P_{15,16}P_{16,17}\Phi_{19,20,21} \\
 & 23\underline{212}432\underline{143}2\underline{4341}23212434 & \Phi^{-1}_{3,4,5}P_{10,11}P_{9,10}P_{15,16}\Upsilon^{-1}_{13,14,15} \\
 & 2\underline{31}2143\underline{24}312\underline{34132321}2434 & P_{2,3}P_{8,9}P_{13,14}P_{14,15}P_{19,20}\Psi^{-1}_{16,17,18,19} \\
 & 21321\underline{434}23\underline{121}3\underline{42}32132434 & \Upsilon^{-1}_{6,7,8}\Phi_{11,12,13}P_{15,16} \\
 & 2132\underline{13432321}232432132434 & P_{6,7}P_{5,6}P_{11,12}\Psi^{-1}_{8,9,10,11} \\
 & 21323412321\underline{3232}432132434 & \Psi^{-1}_{12,13,14,15} \\
 & 213234123\underline{2123}2\underline{34321}32434 & P_{12,13}\Phi^{-1}_{10,11,12}P_{18,19}P_{17,18}\Upsilon_{15,16,17} \\
 & 21323412\underline{31}23124321432434 & P_{9,10} \\
 & 213234\underline{121}323124321432434 & \Phi_{7,8,9} \\
 & 21323\underline{4212323}124321432434 & P_{7,8}P_{6,7}\Psi_{9,10,11,12} \\
 & 21\underline{32321}4323\underline{21243}21432434 & P_{6,7}\Psi^{-1}_{3,4,5,6}P_{15,16}P_{14,15}\Phi^{-1}_{12,13,14} \\
 & \underline{2123}21\underline{3432}3\underline{124}3\underline{121}432434 & P_{3,4}\Phi^{-1}_{1,2,3}P_{9,10}\Upsilon_{7,8,9}P_{12,13}P_{13,14}\Phi_{16,17,18} \\
 & 123\underline{1214}32\underline{4341}23212432434 & P_{6,7}\Phi_{4,5,6}P_{12,13}\Upsilon^{-1}_{10,11,12} \\
 & 123214\underline{23234}1\underline{32321}2432434 & P_{10,11}\Psi_{7,8,9,10}P_{16,17}\Psi^{-1}_{13,14,15,16} \\
 & 12321432\underline{342123}2132432434 & P_{9,10}P_{10,11}P_{13,14}\Phi^{-1}_{11,12,13} \\
 & 1232143213\underline{42}3\underline{121}32432434 & P_{11,12}\Phi_{14,15,16} \\
\tilde{\mathbf{w}}_0: & 123214321324321232432434. &
\end{array}
$$

F_4 analogue of tetrahedron equation

$$
\begin{aligned}
&R_{14,15,16}R_{9,11,16}K_{7,8,10,16}K_{17,15,13,9}R_{4,5,16}S_{7,12,17}R_{1,2,16}S_{6,10,17}R_{9,14,18}\\
&\times K_{17,5,3,1}R_{11,15,18}K_{6,8,12,18}R_{1,4,18}R_{1,8,15}S_{7,13,19}K_{19,11,6,1}K_{19,15,12,4}S_{3,10,19}\\
&\times R_{4,8,11}K_{20,14,7,1}R_{2,5,18}S_{6,13,20}S_{3,12,20}R_{1,9,21}K_{20,15,10,2}R_{4,14,21}K_{3,8,13,21}\\
&\times R_{2,11,21}R_{2,8,14}S_{6,7,22}K_{22,4,3,2}R_{5,15,21}K_{22,14,13,11}S_{10,12,22}K_{23,9,6,2}S_{3,7,23}\\
&\times S_{19,20,22}K_{22,18,17,16}S_{10,13,23}K_{23,14,12,5}S_{3,6,24}K_{23,21,19,16}K_{24,9,7,4}S_{17,20,23}\\
&\times K_{24,11,10,5}S_{12,13,24}S_{17,19,24}K_{24,21,20,18}R_{5,8,9}S_{22,23,24}\\
&= \text{product in reverse order.}
\end{aligned}
$$

50 operators on each side

Th

This is attributed to $(\text{3DRE for } B_3)^{12}$ and $(\text{3DRE for } C_3)^{12}$

$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \\ 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 39, 40, 41, 43, 45, 46, 37, 38, 42, 44, 47, 48, 50, 49 \end{pmatrix}$$

$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \\ 26, 27, 28, 29, 30, 31, 32, 35, 36, 41, 43, 33, 34, 39, 40, 45, 46, 50, 48, 47, 44, 42, 38, 37, 49 \end{pmatrix}$$

$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \\ 26, 27, 28, 29, 32, 30, 31, 35, 36, 41, 43, 50, 46, 45, 40, 49, 39, 34, 33, 48, 47, 44, 42, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 1, 2, 3, 4, 5, 6, 15, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 19, 18, 20, 21, 22, 23, 24 \\ 25, 26, 27, 28, 29, 43, 41, 36, 35, 31, 30, 46, 32, 45, 40, 49, 39, 34, 33, 48, 47, 44, 42, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 1, 2, 3, 5, 19, 4, 6, 15, 7, 8, 17, 10, 16, 14, 9, 13, 20, 24, 26, 43, 18, 12, 22, 23 \\ 27, 41, 46, 36, 11, 21, 25, 28, 32, 45, 48, 29, 35, 40, 49, 47, 39, 44, 42, 31, 30, 34, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 1, 2, 3, 5, 19, 4, 6, 15, 7, 8, 17, 10, 16, 14, 9, 13, 20, 24, 26, 43, 18, 12, 22, 23 \\ 27, 41, 46, 48, 36, 45, 47, 32, 28, 25, 21, 29, 35, 40, 49, 39, 44, 42, 31, 30, 34, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 1, 2, 3, 5, 19, 4, 6, 15, 7, 8, 17, 10, 16, 14, 9, 13, 20, 24, 26, 43, 48, 46, 18, 41 \\ 27, 23, 22, 36, 45, 47, 32, 28, 25, 12, 21, 29, 35, 40, 49, 39, 44, 42, 31, 30, 34, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 48, 1, 2, 3, 5, 19, 4, 6, 15, 17, 43, 46, 7, 8, 10, 16, 14, 26, 24, 20, 13, 18, 41, 45 \\ 27, 23, 22, 36, 47, 32, 28, 25, 9, 12, 21, 29, 35, 40, 49, 39, 44, 42, 31, 30, 34, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 48, 1, 2, 3, 5, 19, 4, 6, 15, 17, 43, 26, 46, 7, 8, 10, 16, 18, 41, 45, 47, 14, 24, 27 \\ 32, 49, 20, 23, 36, 40, 28, 22, 25, 35, 39, 29, 13, 21, 44, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 48, 1, 2, 3, 5, 19, 4, 6, 15, 17, 43, 46, 47, 26, 45, 41, 18, 7, 16, 10, 14, 24, 27, 32 \\ 49, 20, 23, 36, 40, 28, 8, 22, 25, 35, 39, 44, 29, 13, 21, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 48, 47, 46, 1, 2, 3, 5, 19, 43, 45, 41, 17, 15, 6, 26, 18, 7, 16, 4, 10, 14, 24, 27, 32 \\ 49, 20, 23, 36, 40, 28, 8, 22, 25, 35, 39, 44, 29, 13, 21, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 48, 47, 46, 1, 2, 3, 5, 19, 43, 45, 49, 41, 17, 15, 6, 26, 18, 7, 16, 32, 27, 24, 14, 10 \\ 20, 23, 36, 40, 28, 4, 8, 22, 25, 35, 39, 44, 29, 13, 21, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 48, 47, 46, 1, 2, 3, 5, 19, 43, 45, 49, 41, 17, 15, 26, 18, 7, 16, 32, 27, 24, 14, 6, 10 \\ 20, 23, 36, 40, 44, 39, 28, 35, 42, 25, 22, 29, 13, 21, 31, 8, 12, 30, 34, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 48, 47, 44, 46, 1, 2, 3, 5, 19, 43, 45, 49, 41, 17, 15, 26, 18, 7, 16, 32, 27, 24, 14, 40 \\ 36, 23, 20, 10, 6, 39, 28, 35, 42, 25, 22, 29, 13, 21, 31, 8, 12, 30, 34, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 43, 41, 19, 1, 5, 17, 3, 15, 26, 32, 18, 40, 27, 36, 39, 2, 7, 16 \\ 24, 28, 35, 42, 14, 23, 20, 25, 22, 29, 10, 13, 21, 31, 6, 8, 12, 30, 34, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 43, 41, 19, 1, 5, 17, 26, 32, 40, 3, 15, 18, 27, 36, 39, 42, 35, 28 \\ 24, 16, 7, 14, 23, 20, 25, 22, 29, 10, 13, 21, 31, 6, 8, 12, 30, 34, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 43, 41, 19, 1, 5, 17, 26, 32, 40, 42, 39, 36, 27, 18, 15, 35, 28, 24 \\ 16, 3, 7, 14, 23, 20, 25, 22, 29, 10, 13, 21, 31, 6, 8, 12, 30, 34, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 15, 16 \\ 5, 1, 3, 7, 14, 20, 25, 29, 22, 10, 13, 21, 31, 6, 8, 12, 30, 34, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 15, 16 \\ 5, 29, 25, 20, 14, 7, 3, 22, 10, 13, 21, 31, 6, 8, 12, 30, 34, 1, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 29, 25 \\ 15, 16, 20, 14, 5, 7, 22, 10, 13, 21, 31, 3, 6, 8, 12, 30, 34, 38, 33, 37, 11, 9, 4, 2, 1 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 29, 25 \\ 15, 16, 20, 22, 14, 5, 7, 10, 13, 21, 31, 38, 34, 30, 33, 37, 12, 11, 8, 6, 3, 9, 4, 2, 1 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 38, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 17, 18, 23, 29, 25 \\ 31, 34, 24, 15, 16, 20, 22, 30, 33, 37, 21, 14, 13, 12, 11, 9, 10, 8, 7, 5, 6, 3, 4, 2, 1 \end{pmatrix}$$

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 38, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 29, 17, 18, 23, 25 \\ 31, 34, 37, 33, 30, 22, 24, 20, 16, 15, 21, 14, 13, 12, 11, 9, 10, 8, 7, 5, 6, 3, 4, 2, 1 \end{pmatrix}$$

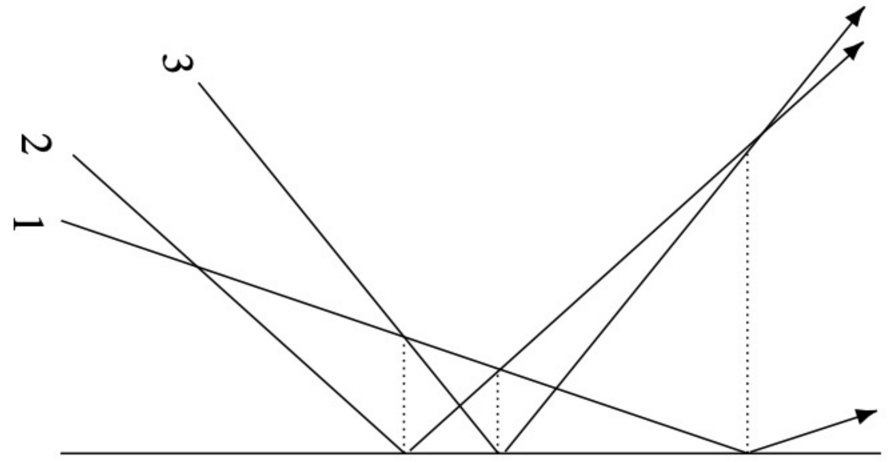
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 43, 41, 44, 42, 40, 39, 38, 37, 34, 36, 35, 32, 19, 26, 27, 28, 29, 31, 33 \\ 25, 23, 18, 30, 22, 24, 20, 17, 16, 15, 21, 14, 13, 12, 11, 9, 10, 8, 7, 5, 6, 3, 4, 2, 1 \end{pmatrix}$$

B_3 type RE

C_3 type RE

$\square$

G_2 reflection equation



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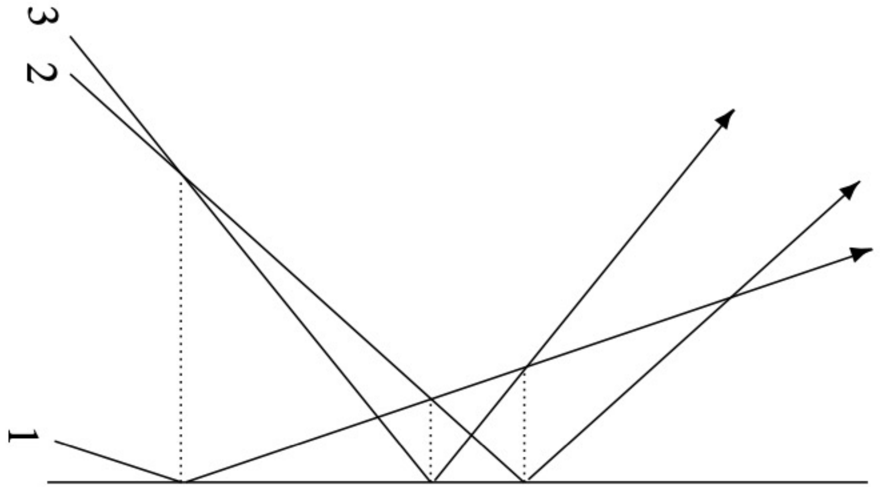


$$X_{231}(y)R_{13}(xy^3)X_{123}(xy^2)R_{32}(x^2y^3)X_{312}(xy)R_{21}(x) \quad = \quad R_{12}(x)X_{132}(xy)R_{23}(x^2y^3)X_{213}(xy^2)R_{31}(xy^3)X_{321}(y)$$

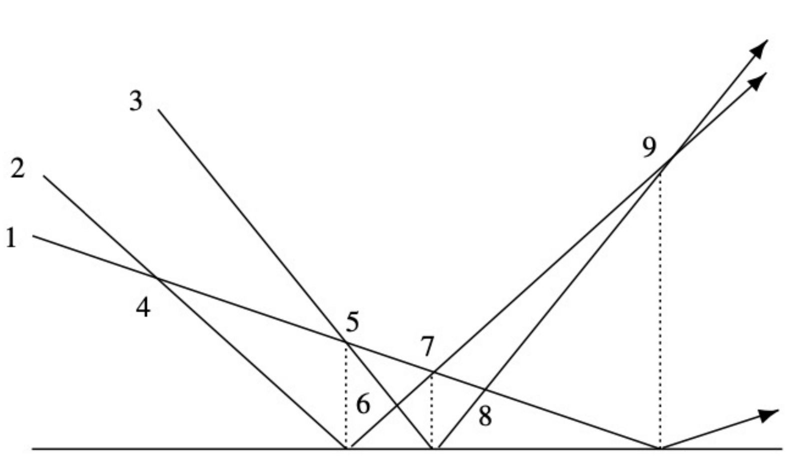
X= Amplitude of **special 3 particle event** in which 2 particle collision happens exactly at the same instant as the boundary reflection of the 3rd particle.

{Spectral parameters} = {positive roots of G2}

Elementary geometric consistency of the above diagram is due to **Desargues-Pappus** theorem

Quantized G_2 reflection equation

$F_{456789} \circ$



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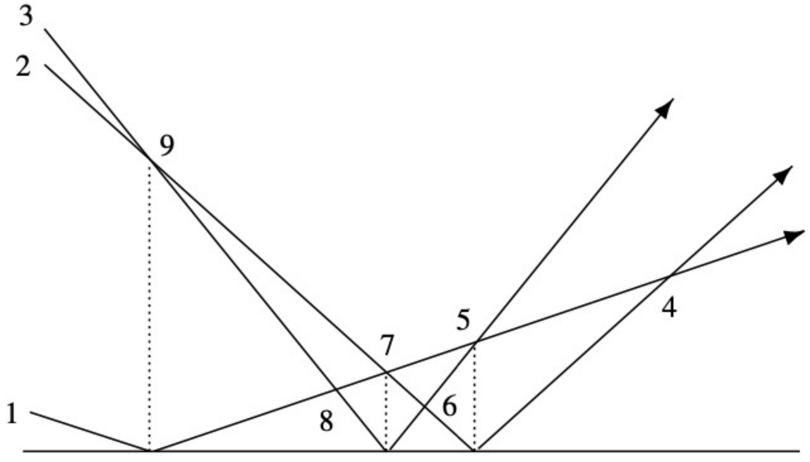

$\circ F_{456789}$

$$F_{456789}J_{2319}L_{138}J_{1237}L_{326}J_{3125}L_{214} \quad = \quad L_{124}J_{1325}L_{236}J_{2137}L_{318}J_{3219}F_{456789}$$

More symbolically, G_2 reflection equation **up to conjugation**:

$$F(J_{231}L_{13}J_{123}L_{32}J_{312}L_{21}) \quad = \quad (L_{12}J_{132}L_{23}J_{213}L_{31}J_{321})F$$

$A_q(G_2)$ $\quad$ $\overset{\pi_1}{\circ} \Lleftarrow \overset{\pi_2}{\circ}$ $\quad$ $s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$

(*) $\quad \Xi \, \pi_{121212} = \pi_{212121} \, \Xi$, $\quad \Xi, F \in \mathrm{End}(\mathcal{F}_q^{\otimes 6})$

$\Xi = F P_{16} P_{25} P_{34}$

Result

- (*) $\Longleftrightarrow$ Quantized G_2 Reflection equation
  by elaborate choice of quantized amplitude
  for the special 3 particle event J.

- Construction of Solutions to G_2 Reflection eq
  in Matrix Product forms

$X(z) = \mathrm{tr}(z^h J J \cdots J)$

$X(z) = \langle \xi | z^h J J \cdots J | \xi \rangle$