30 Years of Box-Ball Systems

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Box-ball systems (BBS) brief chronicle

1990 First example by Takahashi-Satsuma

1996 Ultradiscretization

Classical integrability

1999 Connection to crystal base theory

Quantum (Yang-Baxter) integrability,
Quantum group theoretical generalizations,
Geometric crystals

2006 Combinatorial Bethe ansatz

Action-angle variables, KKR bijection, Fermionic formulas,
Solution of initial value problem

2018 Randomized BBS

Generalized Gibbs ensemble, Thermodynamic Bethe ansatz,
Limit shape of conserved Young diagrams

2019 Generalized hydrodynamics

Speed equation, Riemann problem,
Current fluctuations and large deviations

n-color Box-ball system (BBS)

$n = 3$ example.

$$\cdots 00000000\mathbf{3321}0000000000000000000000\cdots$$
$$\cdots 000000000000\mathbf{3321}00000000000000000\cdots$$
$$\cdots 0000000000000000\mathbf{3321}000000000000\cdots$$

0 =empty box, $\quad$ $1, 2, 3$ = balls with colors

- time evolution = (move 1) $\cdot$ (move 2) $\cdot$ (move 3)

(move i) $\cdot$ Pick the leftmost ball with color i and move it to the nearest right empty box.

$\cdot$ Do the same for the other color i balls.

- soliton=consecutive balls $i_1 \ldots i_a$ with color $i_1 \geq \cdots \geq i_a \geq 1$.
- velocity=amplitude.

- Collisions of 2 solitons

$$\cdots 000\mathbf{33211}0000\mathbf{322}0000000000000000000000\cdots$$
$$\cdots 00000000\mathbf{33211}00\mathbf{322}0000000000000000000\cdots$$
$$\cdots 000000000000\mathbf{3321132}00000000000000000\cdots$$
$$\cdots 0000000000000000\mathbf{2113332}000000000000\cdots$$
$$\cdots 00000000000000000000\mathbf{211}00\mathbf{3332}0000000\cdots$$
$$\cdots 000000000000000000000000\mathbf{211}0000\mathbf{3332}00\cdots$$

- Amplitudes are individually conserved.

- Two body scattering:

  Exchange of internal labels (colors) like quarks in hadrons

  Phase shift

Collision of 3 solitons

$$\cdots 00321003100002000000000000000\cdots$$
$$\cdots 00000321031000200000000000000\cdots$$
$$\cdots 00000000320311020000000000000\cdots$$
$$\cdots 00000000003200312100000000000\cdots$$
$$\cdots 00000000000032001032100000000\cdots$$
$$\cdots 00000000000000320100032100000\cdots$$
$$\cdots 00000000000000030210000321000\cdots$$

$$\cdots 00321000031002000000000000000\cdots$$
$$\cdots 00000321000310200000000000000\cdots$$
$$\cdots 00000000321003120000000000000\cdots$$
$$\cdots 00000000003210132000000000000\cdots$$
$$\cdots 00000000000003021321000000000\cdots$$
$$\cdots 00000000000000300210321000000\cdots$$
$$\cdots 00000000000000030002100321000\cdots$$

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

Double (classical and quantum) origin of integrability

(1) Ultra-Discretization (UD) of soliton equations

- Key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) + \exp(\frac{b}{\varepsilon}) \right) = \max(a, b)$$

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) \times \exp(\frac{b}{\varepsilon}) \right) = a + b$$

$$(+, \times) \longrightarrow (\max, +)$$

keeps distributive law:

$$AB + AC = A(B + C) \to \max(a + b, a + c) = a + \max(b, c)$$

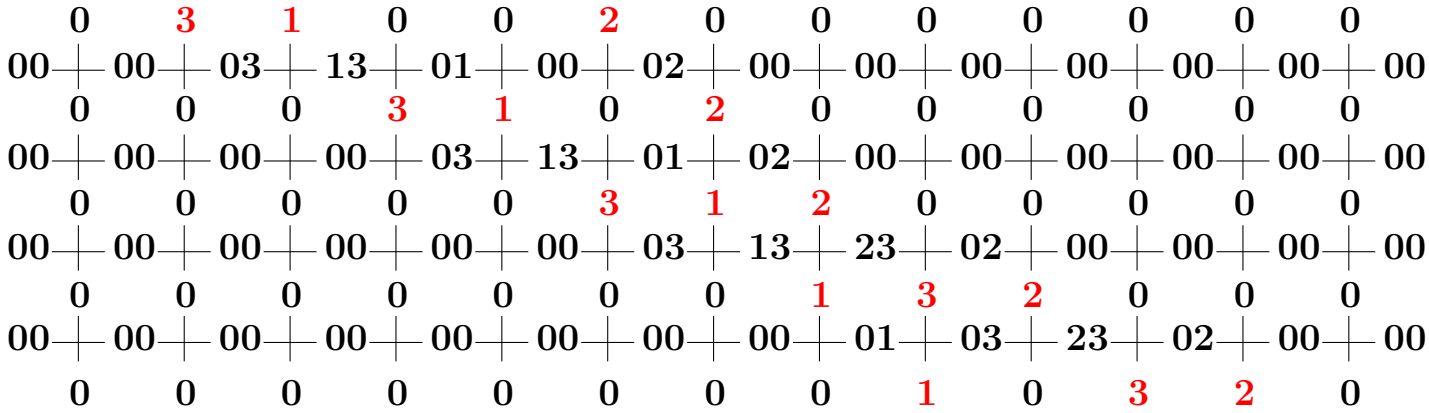
- UD of a discrete KdV equation gives an evolution equation of the $n = 1$ BBS (1996).

(2) Solvable lattice model at " Temperature 0 "

Time evolution pattern

$$\cdots 0310020000000 \cdots$$
$$\cdots 0003102000000 \cdots$$
$$\cdots 0000031200000 \cdots$$
$$\cdots 0000000132000 \cdots$$
$$\cdots 0000000010320 \cdots$$

emerges from a configuration of a 2D lattice model in statistical mechanics



by forgetting the hidden variables on the horizontal edges.

- n-color box-ball system

  = 2D solvable vertex model associated with quantum group

  $$U_q(\widehat{sl}_{n+1}) \text{ at } q = 0 \quad (q \sim \text{temperature})$$

- Row transfer matrix at $q = 0$

  = deterministic map (defined by the unique configuration surviving at $q = 0$)

  = time evolution of box-ball system (forming a commuting family $T_1, T_2, \ldots T_\infty$)

- Proper formulation uses *crystal base theory* (theory of quantum group at $q = 0$).

Some outcomes from such insight

- ∃ Integrable cellular automata with quantum group symmetry.

Example: $\widehat{\mathrm{so}}_{10}$-automaton

$$\cdots 000\bar{2}\bar{4}2110000\bar{1}\bar{1}\bar{4}000000000000000000000000000000\cdots$$
$$\cdots 00000000\bar{2}\bar{4}211001\bar{1}\bar{1}\bar{4}000000000000000000000000\cdots$$
$$\cdots 000000000000\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4}000000000000000000000000\cdots$$
$$\cdots 0000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4}000000000000000000000\cdots$$
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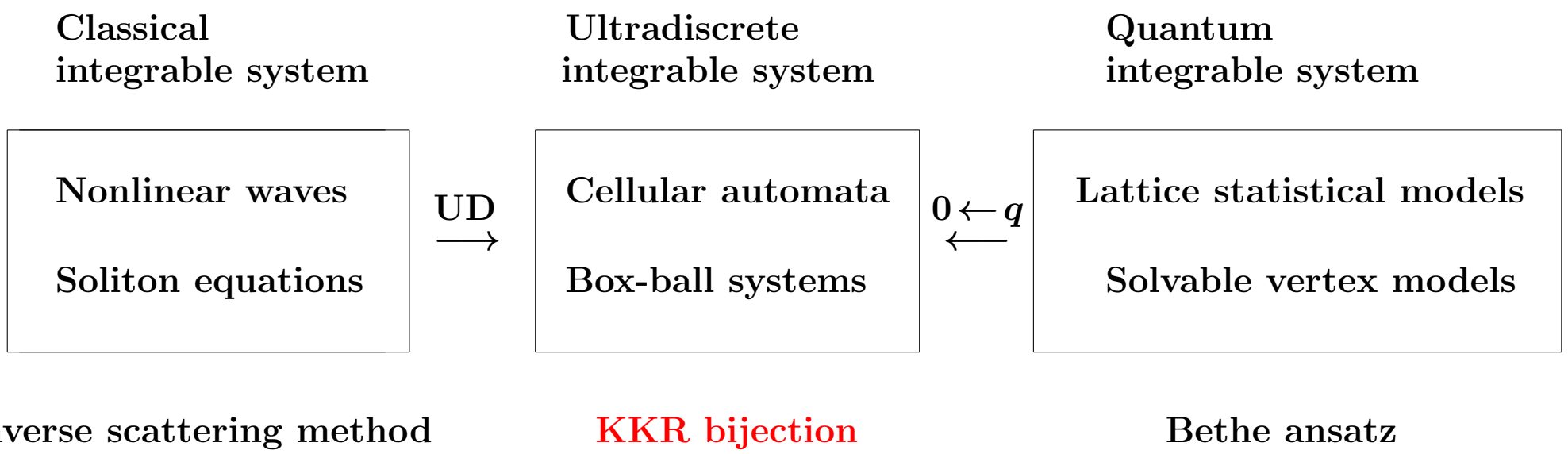
$$\cdots 000000000000000000000000311000\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}000\cdots$$

- Particles and antiparticles undergo pair-creations/annihilations.

- n-color BBS = $\widehat{sl}_{n+1}$-automaton = $\widehat{\mathrm{so}}_{2n+2}$-automaton in antipaticle-free sector.

- Soliton & scattering most naturally captured in quantum group framework.



- Kerov-Kirillov-Reshetikhin (KKR) bijection (1986) asserts "formal completeness" of the hypothetical string solutions to the Bethe equation at combinatorial level.

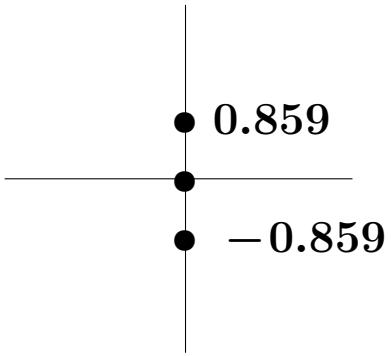
- Its remarkable connection to BBS was discovered in 2002.

- **Example. Spin $\frac{1}{2}$ periodic Heisenberg chain**

$$\boldsymbol{H = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)}$$

For $L = 6$ sites in 3 down-spin sector, the Bethe equation reads

$$\begin{aligned}
\left(\frac{u_1 + i}{u_1 - i}\right)^6 &= \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)}, \\
\left(\frac{u_2 + i}{u_2 - i}\right)^6 &= \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)}, \\
\left(\frac{u_3 + i}{u_3 - i}\right)^6 &= \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.
\end{aligned}$$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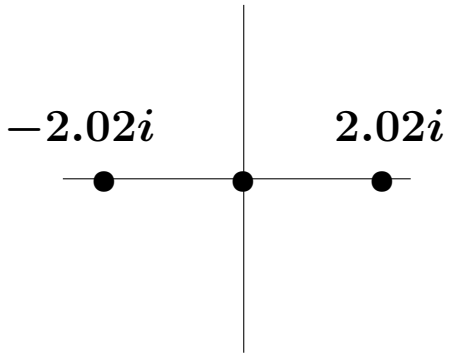
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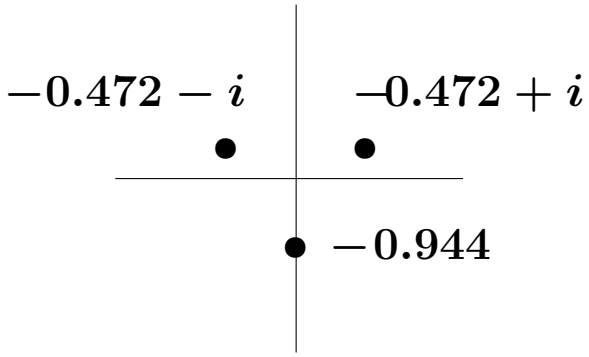
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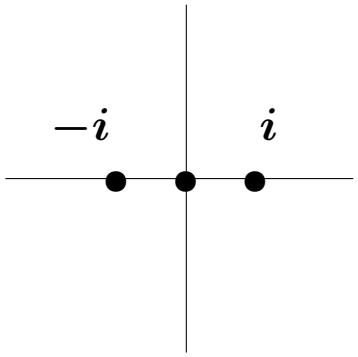
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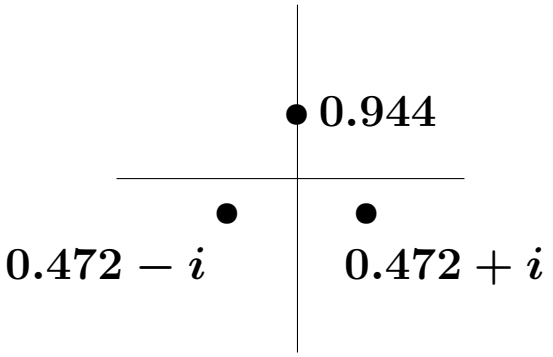
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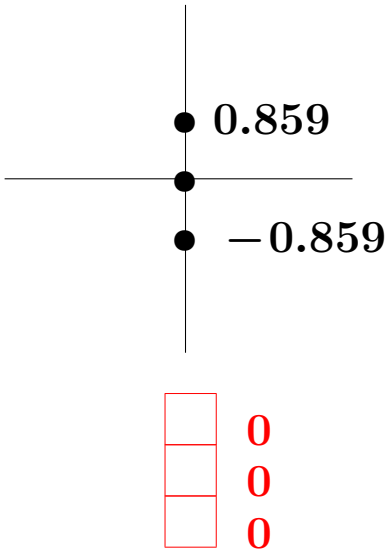
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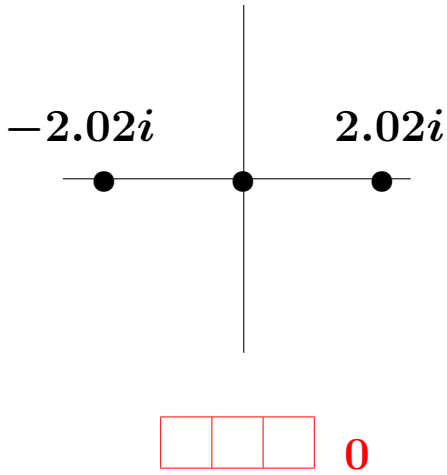
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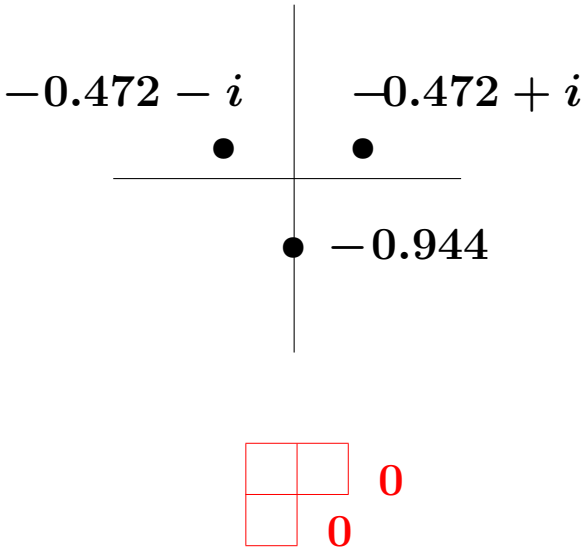
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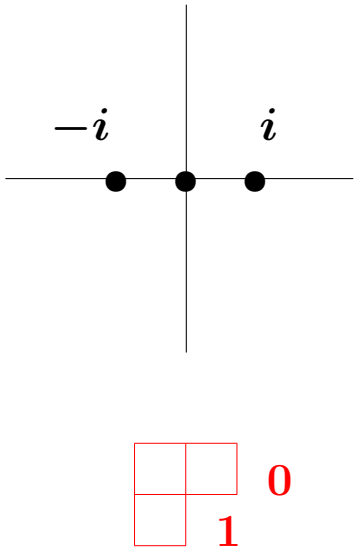
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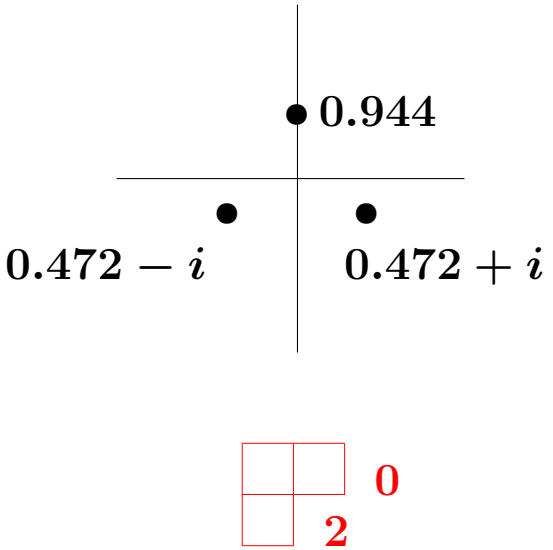
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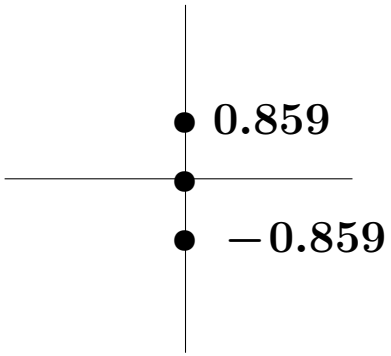
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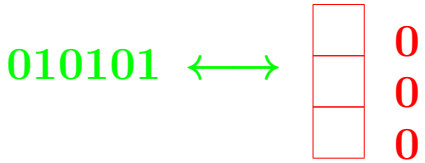
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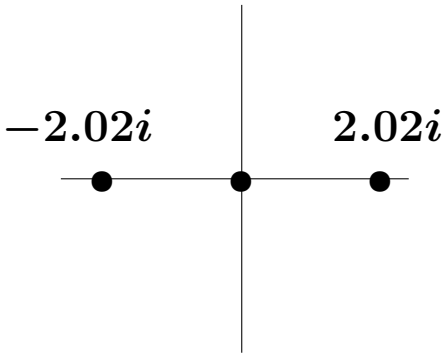
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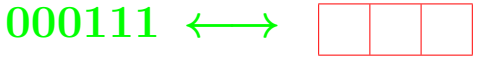
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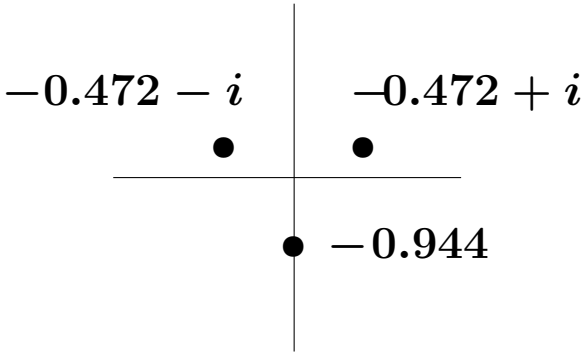
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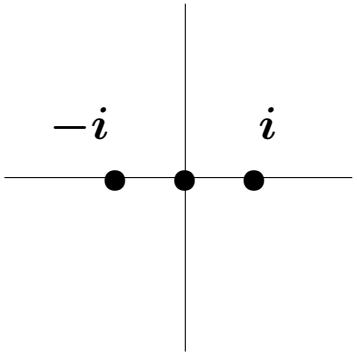
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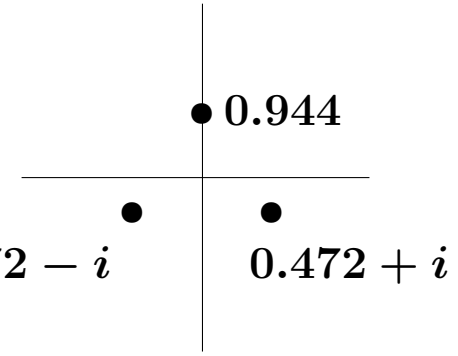
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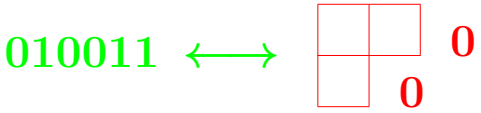
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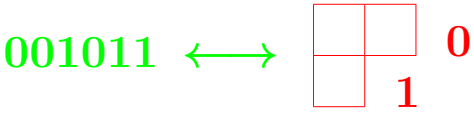
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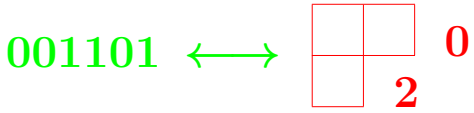
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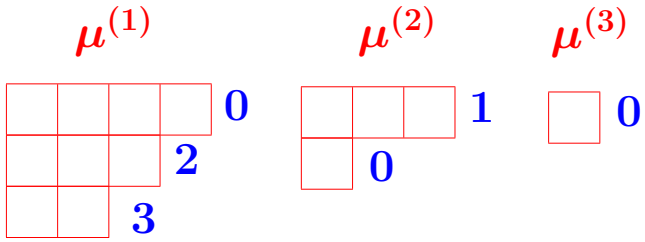
KKR bijection for sl_{n+1}

$$\{\text{highest states}\} \overset{1:1}{\longleftrightarrow} \{\text{rigged configurations}\}$$

$n = 3$ example

$$0000111021132200 \quad 0 \longleftrightarrow$$



"Bethe vectors" "Bethe roots"

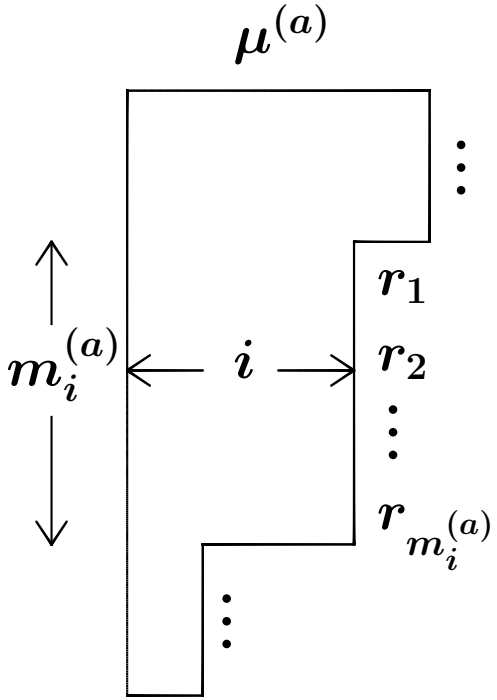
- highest states $= i_1 i_2 \dots i_L$ $(0 \le i_k \le n)$ satisfying the highest condition:

$$\#_0\{i_1, \dots, i_k\} \ge \#_1\{i_1, \dots, i_k\} \ge \cdots \ge \#_n\{i_1, \dots, i_k\} \quad (\forall k)$$

- rigged configuration: $((\mu^{(1)}, r^{(1)}), \dots, (\mu^{(n)}, r^{(n)}))$

$\mu^{(1)}, \dots, \mu^{(n)}$: configuration $=$ n-tuple of Young diagrams
$r^{(1)}, \dots, r^{(n)}$: rigging $=$ integers assigned to each row

$\Big\}$ + selection rule (next page)



$m_i^{(a)} = \#(\text{length } i \text{ rows in } \mu^{(a)})$

$0 \le r_1 \le \cdots \le r_{m_i^{(a)}} \le h_i^{(a)}$

$\cdots$ "Fermionic" selection rule

$h_i^{(a)} = L\delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j \ge 1} \min(i,j) m_j^{(b)}$

$\cdots$ vacancy for *holes*

(C_{ab}) $\cdots$ Cartan matrix of sl_{n+1}

$$\#\text{ of rigging choices for a fixed configuration} = \prod_{a=1}^{n} \prod_{i \ge 1} \begin{pmatrix} h_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{pmatrix}$$

This is an sl_{n+1} generalization of Bethe's formula for # of string solutions (1931), which yields the so-called Fermionic character formula for KR modules.

hat also eine Möglichkeit weniger, die des letzten Komplexes von n Wellen, λ_{q_n}, kann schließlich nur noch

$$Q_n' - (q_n - 1) = Q_n + 1$$

verschiedene Werte annehmen, wo

$$Q_n(N, q_1\, q_2 \ldots) = N - 2\sum_{p<n} p\, q_p - 2\sum_{p\geqq n} n\, q_p. \tag{44}$$

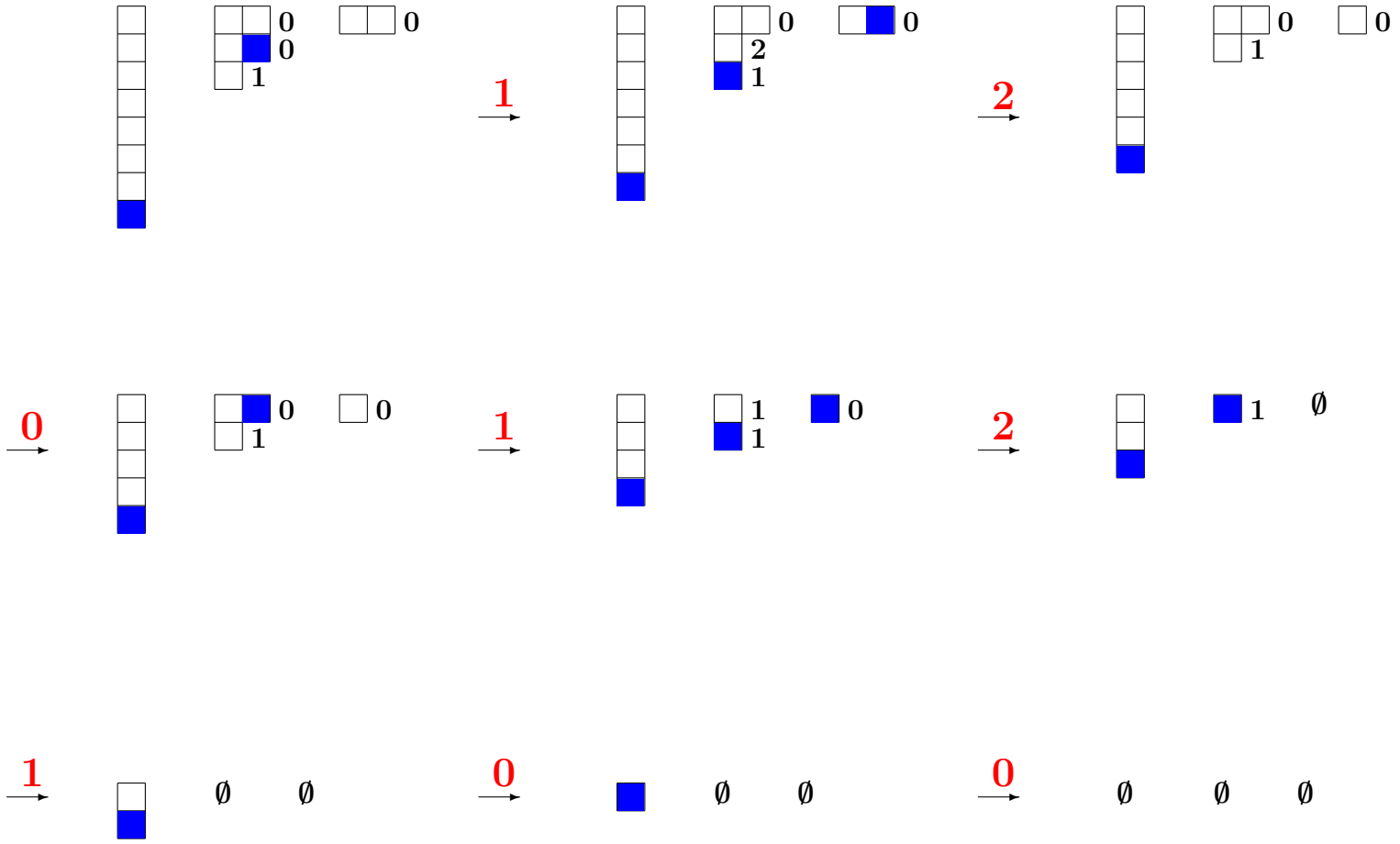
Schließlich ist zu berücksichtigen, daß Vertauschung der λ der verschiedenen Wellenkomplexe mit gleicher Anzahl n von Wellen nicht zu neuen Lösungen führt. Die gesamte Zahl unserer Lösungen wird somit

$$z(N, q_1\, q_2 \ldots) = \prod_{n=1}^{\infty} \frac{(Q_n + q_n)\ldots(Q_n + 1)}{q_n!} = \prod_n \binom{Q_n + q_n}{q_n}, \tag{45}$$

wo die Q_n durch (44) definiert sind.

8. Wir werden nun nachweisen, daß wir die richtige Anzahl Lösungen erhalten haben.

Example of KKR algorithm



Top left rigged configuration $\overset{\text{KKR}}{\longmapsto}$ 00121021

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 0000**1111**00000**22**10**32**0000000000000000000000000

$t = 1$: 00000000**1111**0000**22**1**32**0000000000000000000000

$t = 2$: 000000000000**1111**000**2213****2**0000000000000000000

$t = 3$: 0000000000000000**1111**00**2132****2**0000000000000000

$t = 4$: 00000000000000000000**111**0**2113****22**00000000000000

$t = 5$: 0000000000000000000000**11**00**21132****21**0000000000

$t = 6$: 000000000000000000000000**11**000**211**0**321**000000000

$t = 7$: 0000000000000000000000000**11**0000**211**00**321**000000

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 0000**1111**00000**22**1**0**0**32**0000000000000000000000000000

$t = 1$: 00000000**1111**0000**22**1**0**3**2**0000000000000000000000000

$t = 2$: 000000000000**1111**000**22**1**3**2**2**0000000000000000000000

$t = 3$: 0000000000000000**1111**00**2**1**3**2**2**0000000000000000000

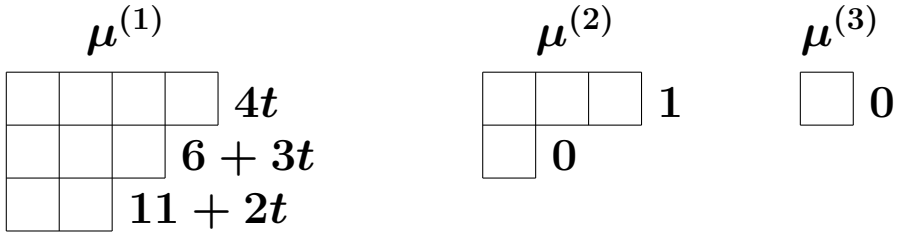
$t = 4$: 00000000000000000000**111**0**2**1**1**3**22**0000000000000000

$t = 5$: 000000000000000000000000**11**00**2**1**1**3**22**1**0**0000000000

$t = 6$: 0000000000000000000000000**11**000**2**1**1**0**3**2**2**1**0**000000000

$t = 7$: 00000000000000000000000000**11**0000**2**1**1**00**3**2**2**1**0**00000

$\xrightarrow{\text{KKR}}$



How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 0000**1111**00000**221**00**32**000000000000000000000000000

$t = 1$: 00000000**1111**0000**221****32**000000000000000000000000

$t = 2$: 000000000000**1111**000**2213****2**000000000000000000000

$t = 3$: 0000000000000000**1111**00**2132****2**000000000000000000

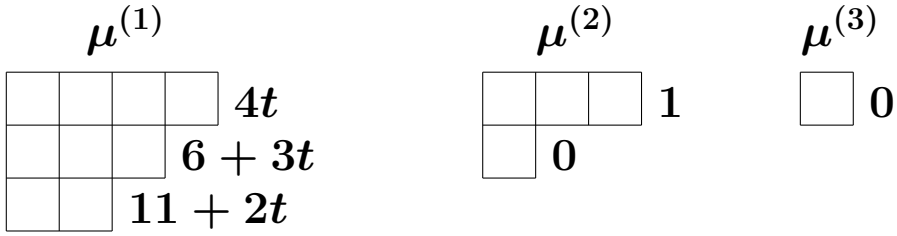
$t = 4$: 000000000000000000**111****02113****22**000000000000000

$t = 5$: 00000000000000000000**110****02113****221**0000000000000

$t = 6$: 0000000000000000000000**11****000****211****0322**10000000000

$t = 7$: 000000000000000000000000**11****0000****211****00322****1**000000

$\xrightarrow{\text{KKR}}$



- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 0000**1111**0000**22**1**0**0**32**0000000000000000000000000

$t = 1$: 00000000**1111**0000**22**1**0**3**2**0000000000000000000000

$t = 2$: 0000000000001**111**000**22**1**3**2000000000000000000000

$t = 3$: 00000000000000001**111**00**2**1**3**2**2**000000000000000000

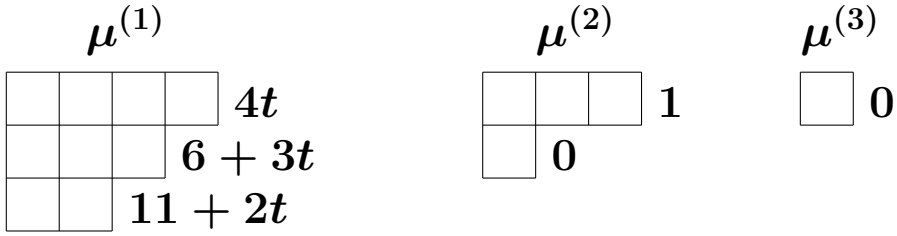
$t = 4$: 000000000000000000**111**0**2**1**13**2**2**000000000000000

$t = 5$: 0000000000000000000001**1**00**2**1**13**2**2**1000000000000

$t = 6$: 00000000000000000000000**11**000**2**1**1**0**3**2**2**1000000000

$t = 7$: 000000000000000000000000**11**0000**2**1**1**00**3**2**2**1000000

$\xrightarrow{\text{KKR}}$



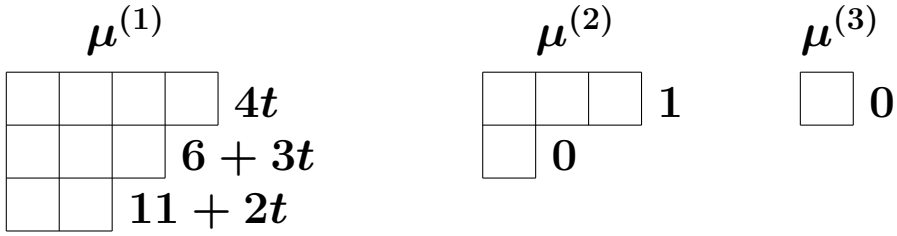
- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

Rigged configuration = action angle variable of BBS!

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 0000**1111**0000**22**10**3**2000000000000000000000000000
$t = 1$: 00000000**1111**0000**221032**000000000000000000000000
$t = 2$: 0000000000001**1111**000**22132**0000000000000000000000
$t = 3$: 000000000000000**1111**00**21322**00000000000000000000
$t = 4$: 00000000000000000001**110211322**000000000000000000
$t = 5$: 0000000000000000000001**100211322**10000000000000
$t = 6$: 000000000000000000000001**10002110322**1000000000
$t = 7$: 0000000000000000000000001**100002110032221**000000

KKR $\longrightarrow$



- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

Rigged configuration = action angle variable of BBS!

$\mu^{(1)}$ = list of amplitude of solitons (Soliton/String correspondence)

$(\mu^{(1)}, \ldots, \mu^{(n)})$ will be called a soliton content.

Randomized box-ball system

BBS state		Soliton content
$i_1 i_2 \dots i_L 00000 \dots$	$\overset{\text{KKR}}{\longmapsto}$	$(\mu^{(1)}, \dots, \mu^{(n)})$

Randomize $i_1 i_2 \dots i_L$ by introducing the i.i.d. measure on the set of states:

$$\text{Prob}(\text{local state} = i) = p_i \qquad (p_0 + \cdots + p_n = 1).$$

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Limit shape Problem

Determine the scaling form of the most probable $(\mu^{(1)}, \dots, \mu^{(n)})$ when $L \to \infty$.

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Limit shape Problem

Determine the scaling form of the most probable $(\mu^{(1)}, \dots, \mu^{(n)})$ when $L \to \infty$.

This can be done by TBA minimizing the free energy F associated with

$$\text{Prob}(\mu^{(1)}, \dots, \mu^{(n)}) \propto \mathrm{e}^{-\beta_1 |\mu^{(1)}| - \cdots - \beta_n |\mu^{(n)}|} \prod_{a=1}^{n} \prod_{i \geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}},$$

$$\mathrm{e}^{\beta_a} := p_{a-1}/p_a,$$

Introduce the scaled string and hole densities $\rho_i^{(a)}, \sigma_i^{(a)}$ by

$$m_i^{(a)} \simeq L\rho_i^{(a)}, \quad h_i^{(a)} \simeq L\sigma_i^{(a)}, \quad \sigma_i^{(a)} = \delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \rho_j^{(b)},$$

Assume $p_0 \geq \cdots \geq p_n$ in the rest.

The condition $\frac{\delta F}{\delta \rho_i^{(a)}} = 0$ leads to the TBA equation

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \log(1 + (Y_j^{(b)})^{-1})$$

in terms of $Y_i^{(a)} = \frac{\sigma_i^{(a)}}{\rho_i^{(a)}}$ with the boundary condition $\lim_{i \to \infty} \frac{1 + Y_{i+1}^{(a)}}{1 + Y_i^{(a)}} = \mathrm{e}^{\beta_a}$.

This is equivalent to the (constant) Y-system

$$\left(Y_i^{(a)}\right)^2 = \frac{(1 + Y_{i-1}^{(a)})(1 + Y_{i+1}^{(a)})}{\left(1 + (Y_i^{(a-1)})^{-1}\right)\left(1 + (Y_i^{(a+1)})^{-1}\right)}$$

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Solution (Rare case for which an exact formula can be given)

$$Y_i^{(a)} = \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{Q_i^{(a-1)} Q_i^{(a+1)}},$$

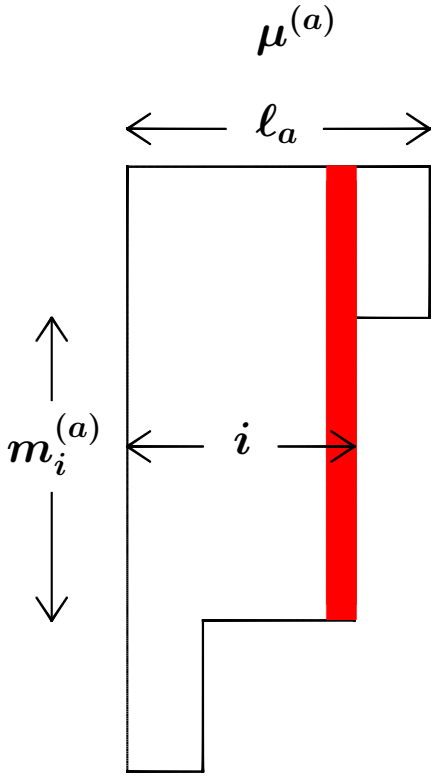
$$Q_i^{(a)} = Q_i^{(a)}(p_0, \ldots, p_n) = \frac{\det(p_k^{\lambda_j + n - j})_{j,k=0}^{n}}{\det(p_k^{n-j})_{j,k=0}^{n}} \qquad \left((\lambda_0, \ldots, \lambda_n) = (\overbrace{i \ldots i}^{a} \overbrace{0, \ldots, 0}^{n+1-a}) \right)$$

$$= \text{Schur function for } a \times i \text{ rectangular Young diagram} \quad \overbrace{\square}^{i} \Big\} a$$

Result. The limit shape of soliton content $(\mu^{(1)}, \ldots, \mu^{(n)})$ is given by

$$\eta_i^{(a)} := \lim_{L\to\infty} \frac{1}{L}(\text{Length of the } i\text{ th column of } \mu^{(a)}) = \frac{Q_{i-1}^{(a-1)} Q_i^{(a+1)}}{Q_i^{(a)} Q_{i-1}^{(a)} Q_1^{(1)}}$$

$$\text{width } \ell_a \text{ of } \mu^{(a)} \simeq \frac{\log L}{\log \frac{p_{a-1}}{p_a}} \quad (L \to \infty \text{ if } p_0 > \cdots > p_n)$$



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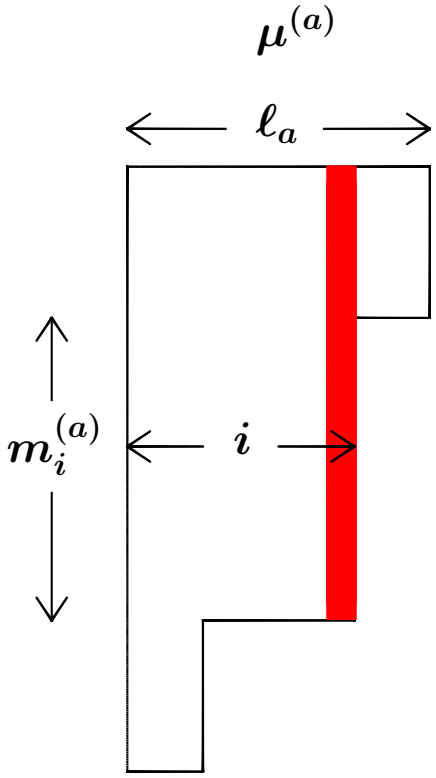
Special case $p_a = \frac{q^a}{1+q+\cdots+q^n}$ $(0 < q \leq 1)$.



Scaled column length of $\mu^{(a)}$

$$\eta_i^{(a)} = \frac{q^{i+a-1}(1-q)(1-q^a)(1-q^{n+1-a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})}$$

Strings

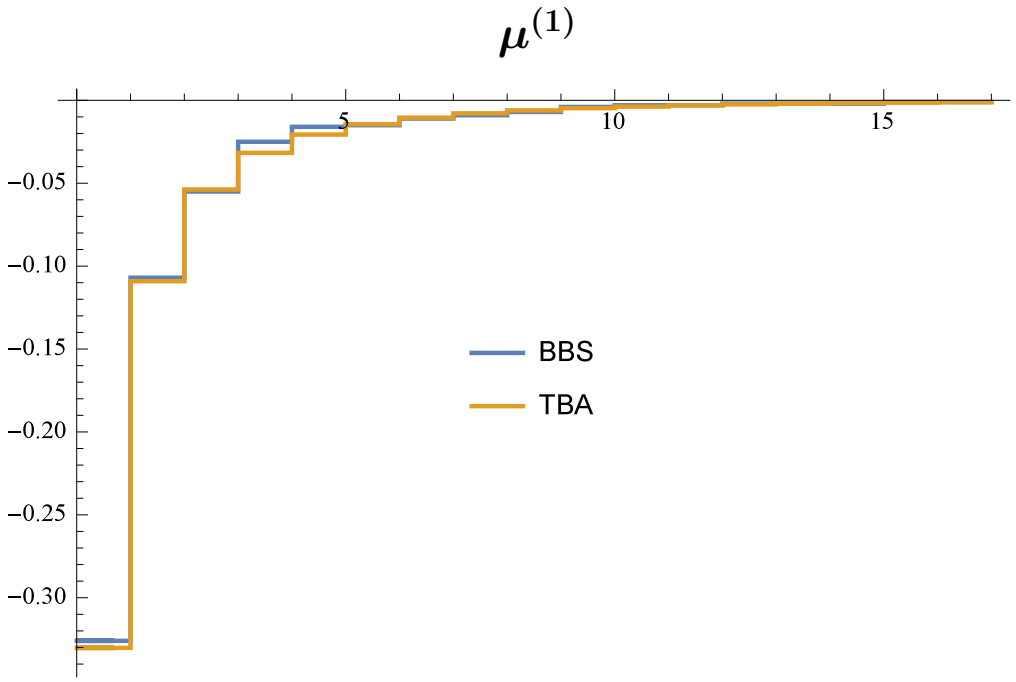
$$\rho_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} m_i^{(a)} = \frac{q^{i+a-1}(1-q)^2(1-q^a)(1-q^{n+1-a})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

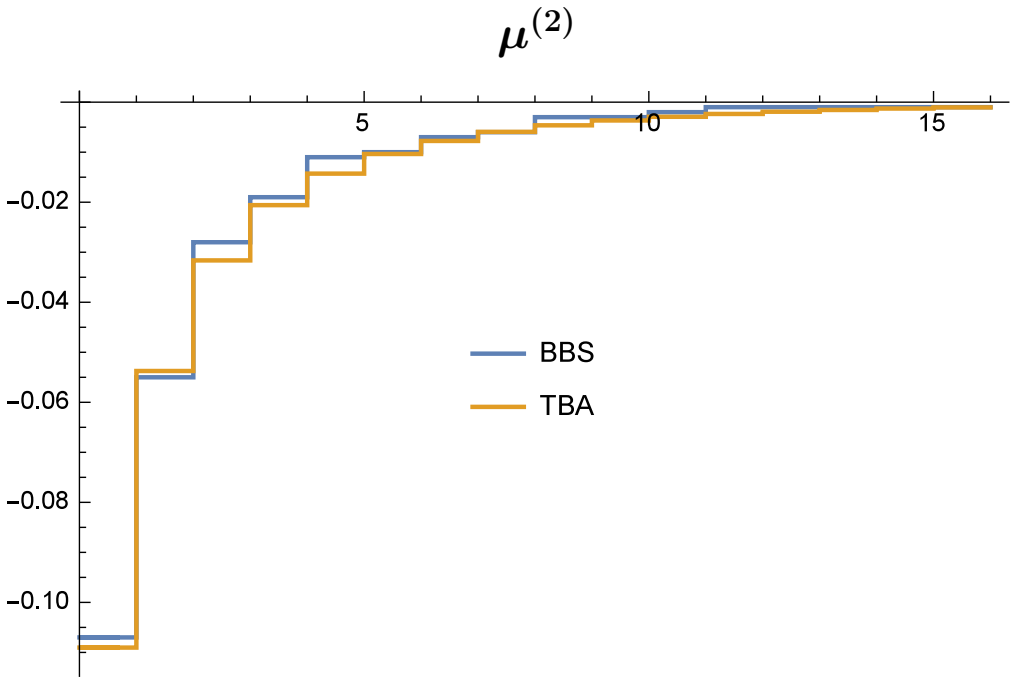
Holes

$$\sigma_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} h_i^{(a)} = \frac{q^{a-1}(1-q)^2(1-q^i)(1-q^{n+i+1})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

2-color BBS with $L = 1000$ sites with distribution $(p_0, p_1, p_2) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$.

Vertically L^{-1} scaled soliton contents.





Generalized hydrodynamics, GHD (from here 1-color BBS only)

[Castro-Alvaredo, Doyon, Yoshimura, Bertini, Collura, De Nardis, Fagotti, ... 2016∼]

Densities: ρ_i (i-soliton), σ_i (i-hole), $(\rho_i, \sigma_i) = (\rho_i^{(1)}, \sigma_i^{(1)})$ in previous pages

Speed equation: $v_i = i + \sum_j 2\min(i,j)(v_i - v_j)\rho_j$ $\left(\begin{array}{c} v_i = \text{effective speed} \\ 2\min(i,j) = \text{phase shift} \end{array}\right)$

[Croydon,Sasada]

Solution for homogeneous BBS : $v_i = i\dfrac{1+q}{1-q} - \dfrac{2q(1+q)(1-q^i)}{(1-q)^2(1+q^{i+1})}$

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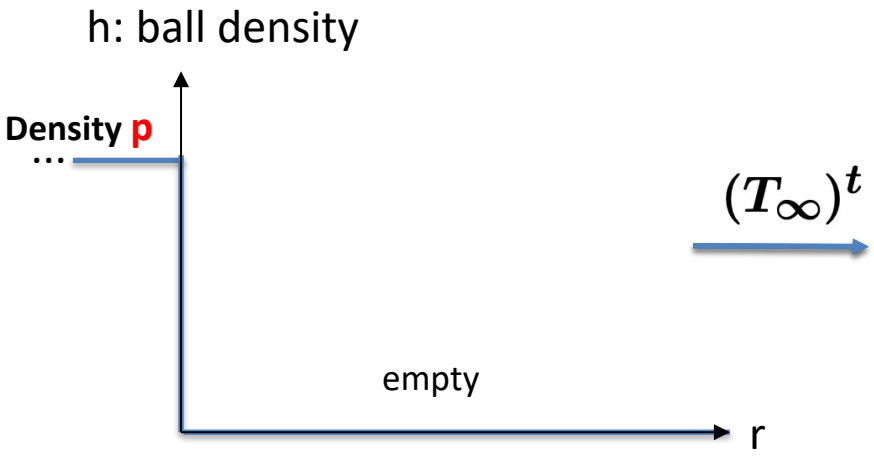
GHD: assumes weak (x,t) dependence of ρ_i, σ_i,
postulates continuity equation and local entropy maximum

$\Longrightarrow$ Y-variable $Y_i := \frac{\rho_i}{\sigma_i}$ plays the role of normal mode satisfying the separated equation $\frac{\partial Y_i}{\partial t} + v_i \frac{\partial Y_i}{\partial x} = 0$ for each i.

One of its most fruitful application is the Riemann problem:

"Describe the profile of the fluid starting from the *homogeneous* initial state except a *single discontinuity* at some point."

Density Plateaux emerging from domain wall initial condition



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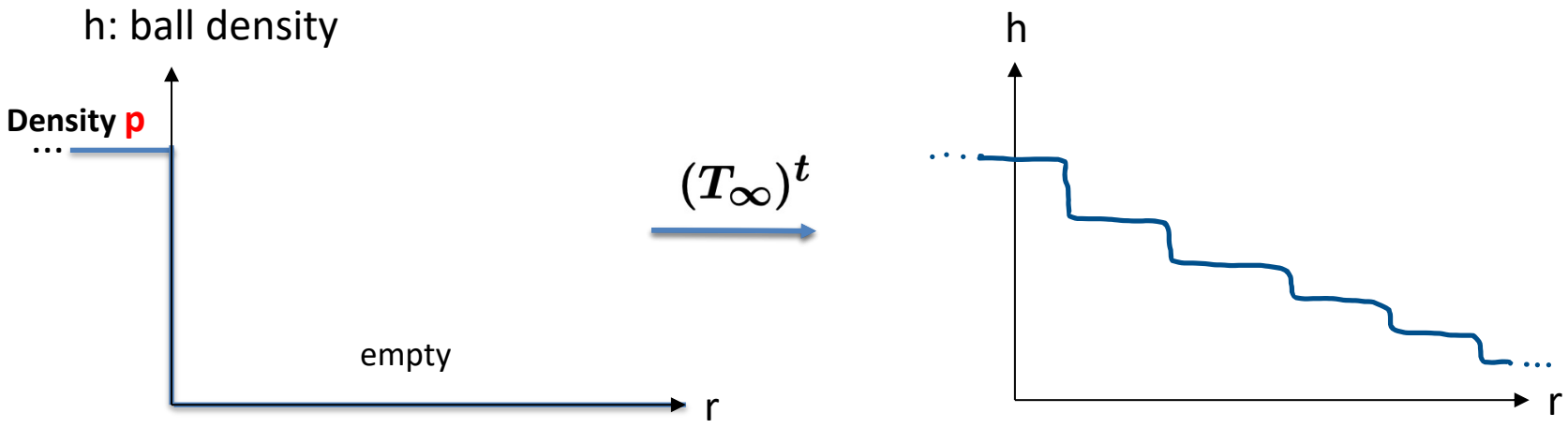

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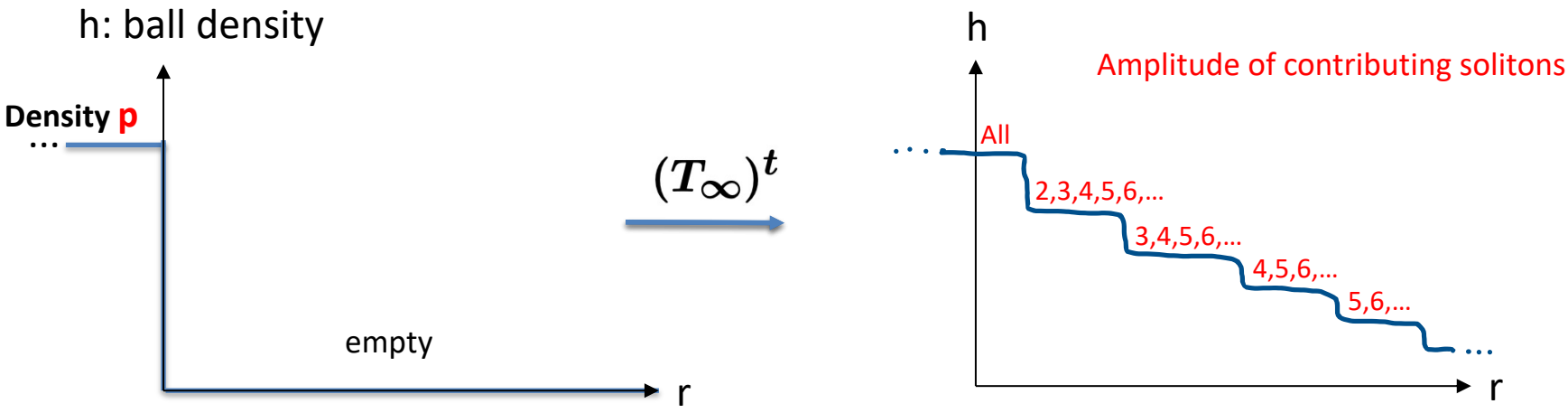

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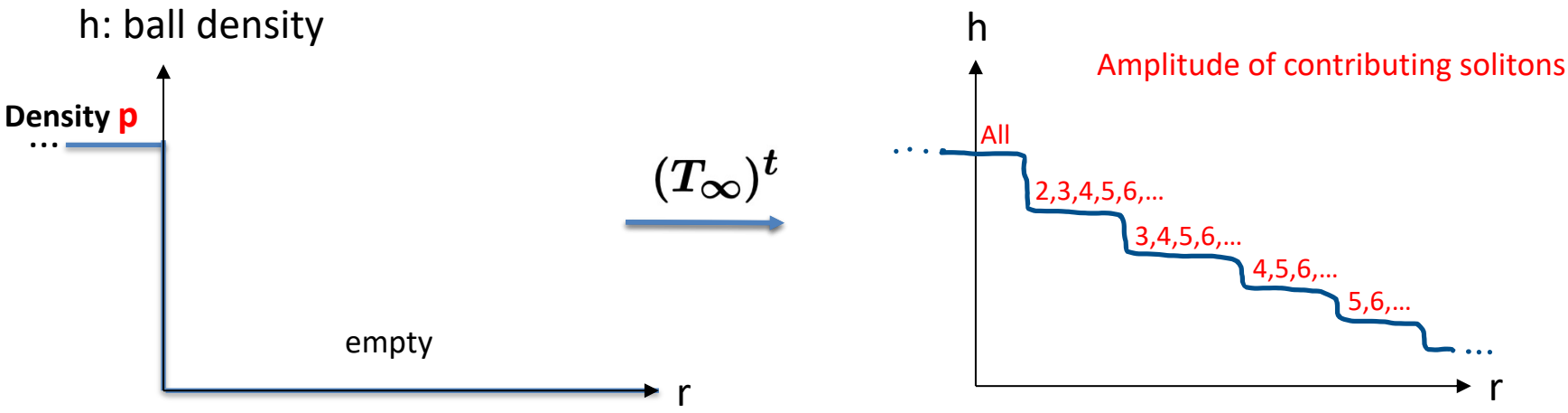
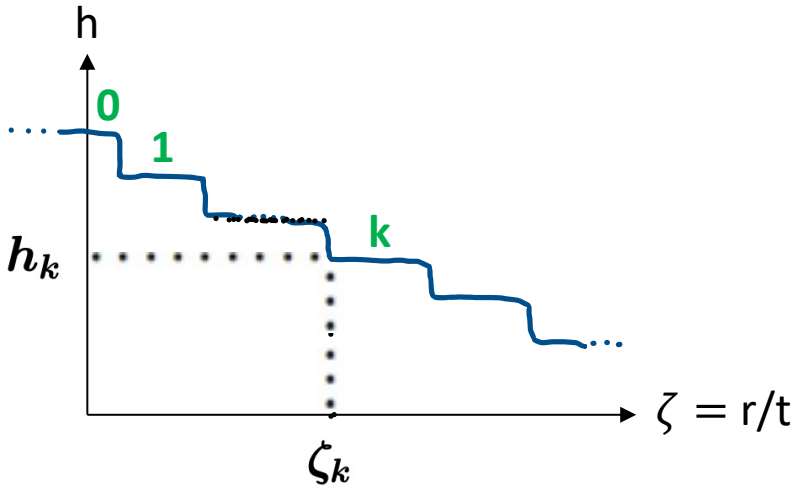


Plateaux grow linearly in time t. The plot against $\zeta = r/t$ collapses into a single curve.



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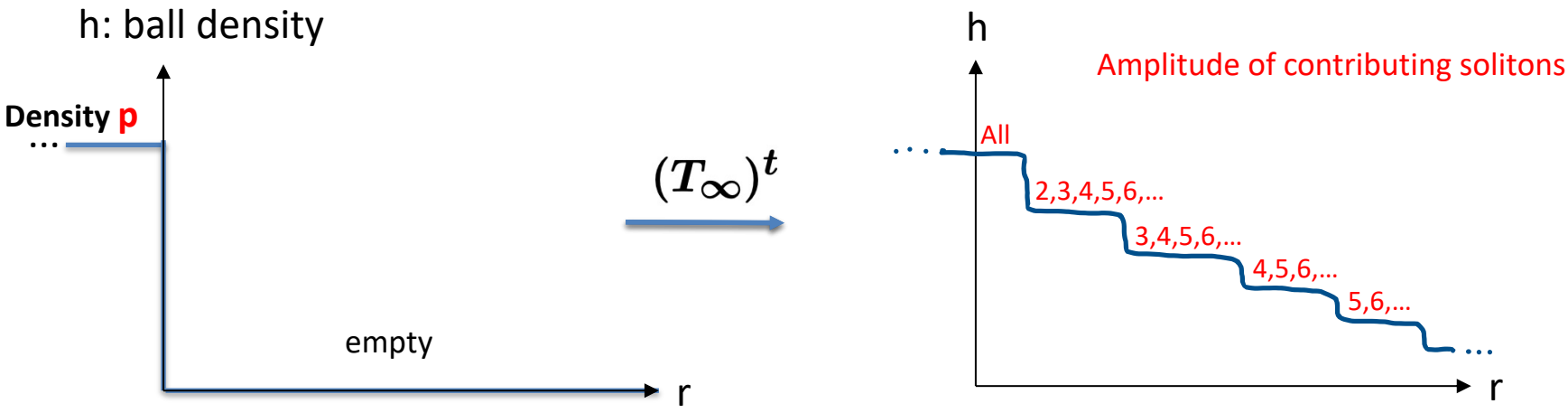

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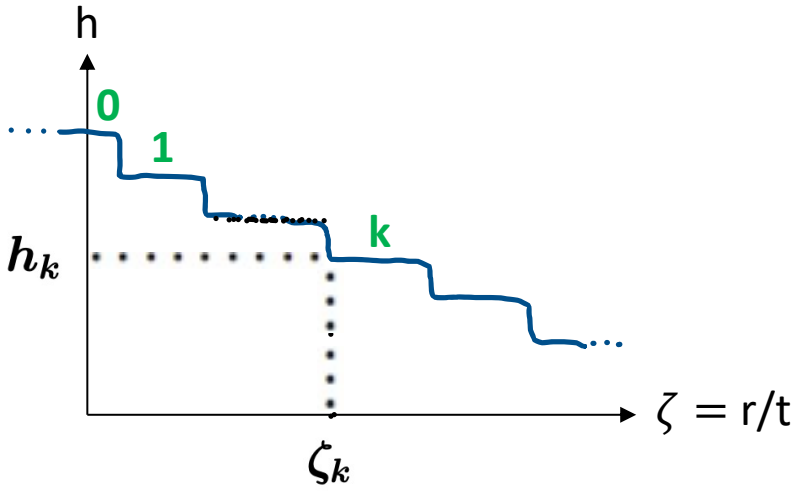


GHD prediction

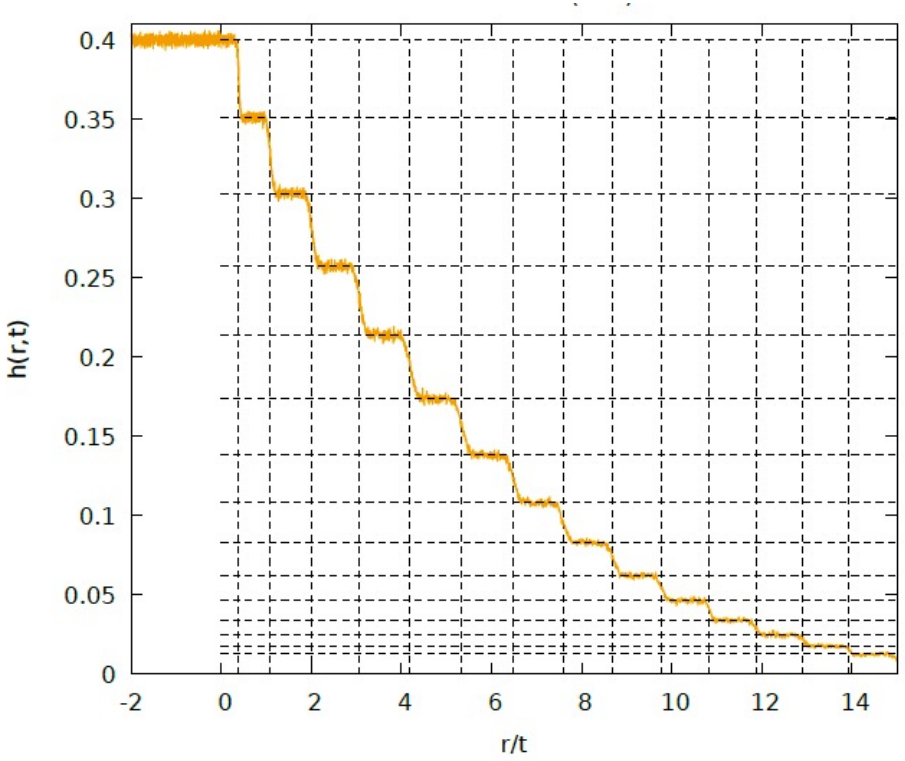
$$h_k = \frac{q^{k+1}(1 - q^{k+2} + k(1-q))}{1 - q^{2k+3} + (2k+1)(1-q)q^{k+1}}$$

$$\zeta_k = \frac{k(1-q^{k+1})}{1+q^{k+1}} \qquad \left(p = \frac{q}{1+q}\right)$$

Simulation with $N_{samples}$ = 50000

(Plots of ball density vs $\zeta = r/t$. Dotted lines are GHD predictions)

p=0.4, q=0.666…, t=500.



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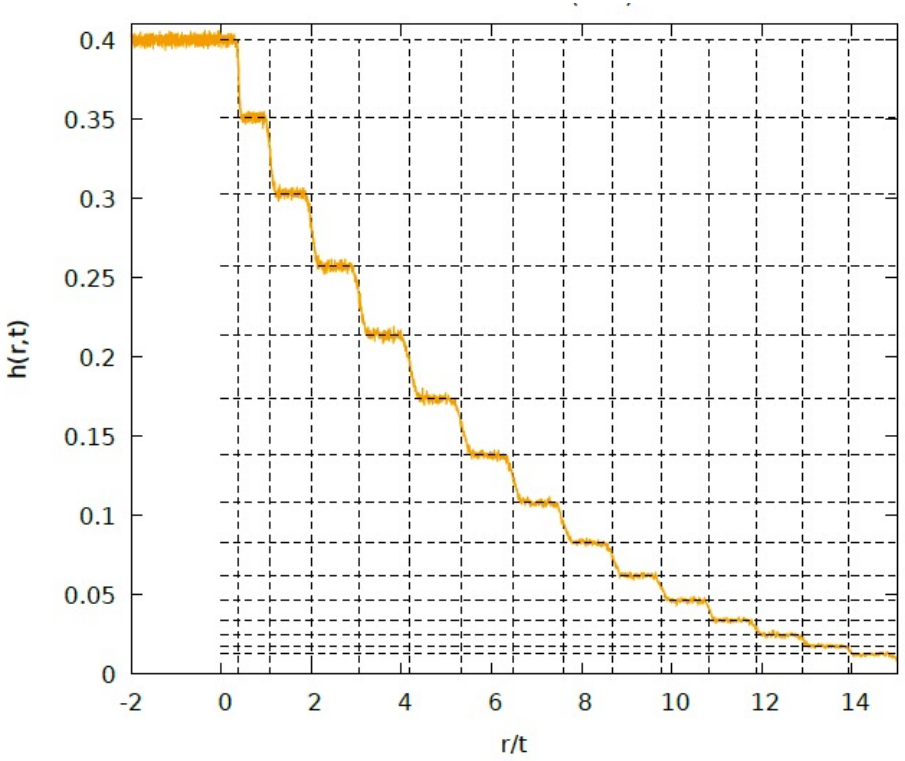


Actual plateau edges are not strict and exhibit some **broadening**.

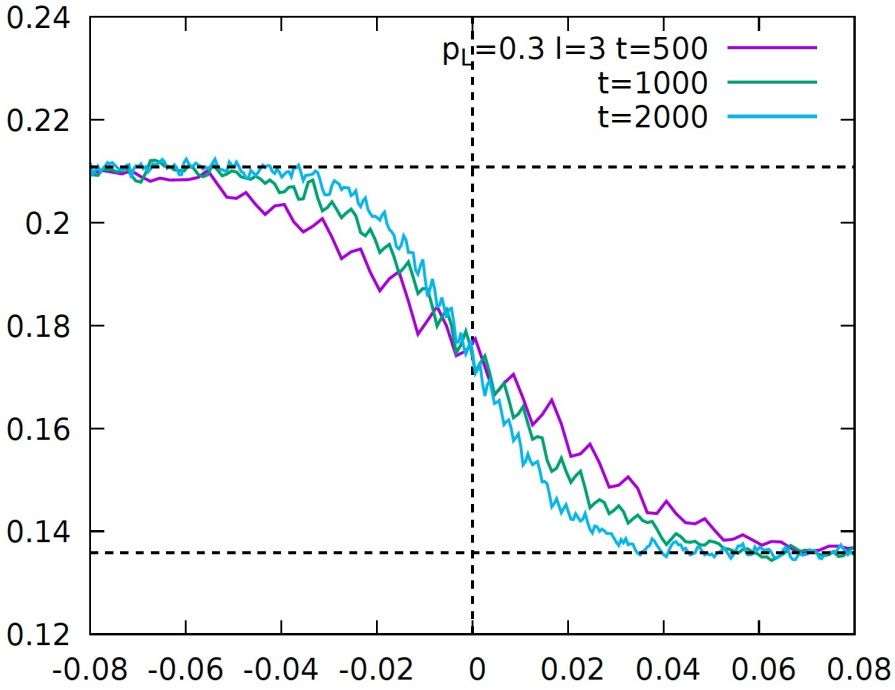
This is due to **diffusive** correction to the **ballistic** picture,
which may be viewed as a finite t effect.

Analytical description of the **diffusive broadening** of plateau edges

Position of plateau edge - ζ

fluctuates over the scale

$$\frac{1}{\sqrt{(\textbf{Diffusion const})t}}$$



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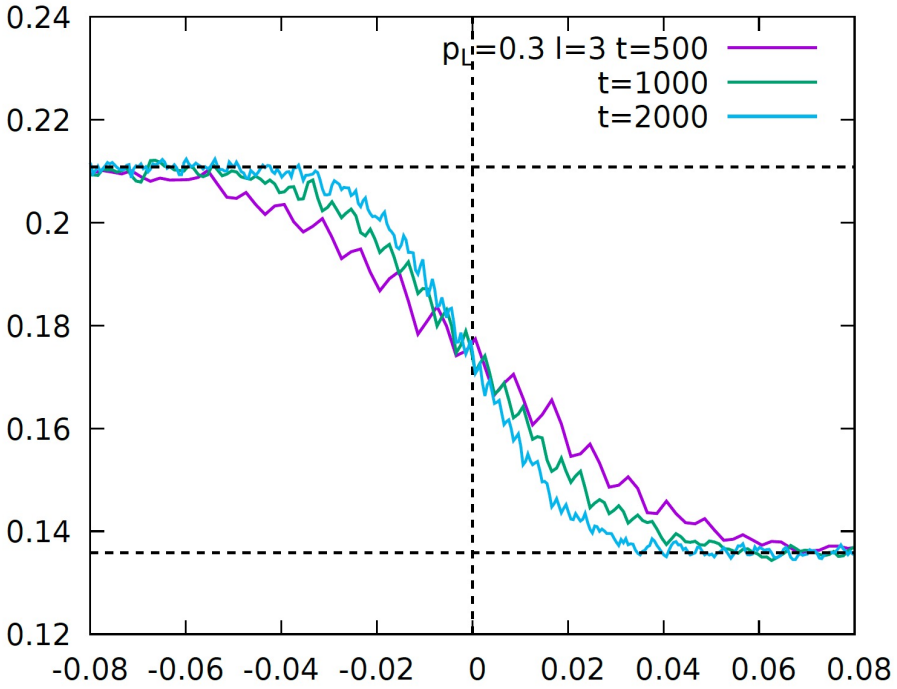

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \mathrm{e}^{-s^2} ds$$

$$\langle \boldsymbol{\rho}_j(\boldsymbol{r}, \boldsymbol{t}) \rangle = \frac{\mathbf{1}}{\mathbf{2}} \left(\boldsymbol{\rho}_j(\boldsymbol{k}-\mathbf{1}) - \boldsymbol{\rho}_j(\boldsymbol{k}) \right) \mathbf{erfc} \left(\frac{\boldsymbol{r} - \boldsymbol{\zeta}(\boldsymbol{k})\boldsymbol{t}}{\sqrt{\mathbf{2}\boldsymbol{t}}\, \boldsymbol{\Sigma}_{\boldsymbol{k}}^{(l)}} \right) + \boldsymbol{\rho}_j(\boldsymbol{k})$$

amplitude j-soliton density around the k th plateau edge $r = \zeta(k)t$ under the time evolution T_l.

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Position of plateau edge - ζ fluctuates over the scale

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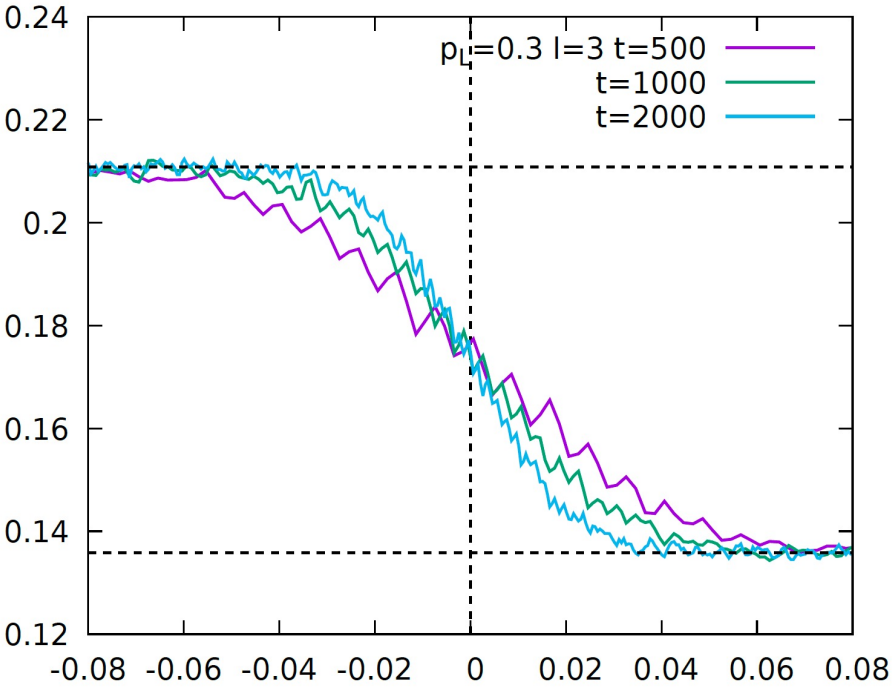

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$$\left(\boldsymbol{\Sigma}_{\boldsymbol{k}}^{(\boldsymbol{l})}\right)^{\boldsymbol{2}} = \frac{\boldsymbol{4k^2 q^{k+1}(1-q^{k+1})(1-q^{l-k})(1+q^{l+k+2})}}{\boldsymbol{(1+q^{k+1})^3 (1-q^{l+1})^2}}$$

GHD
(Bethe ansatz)

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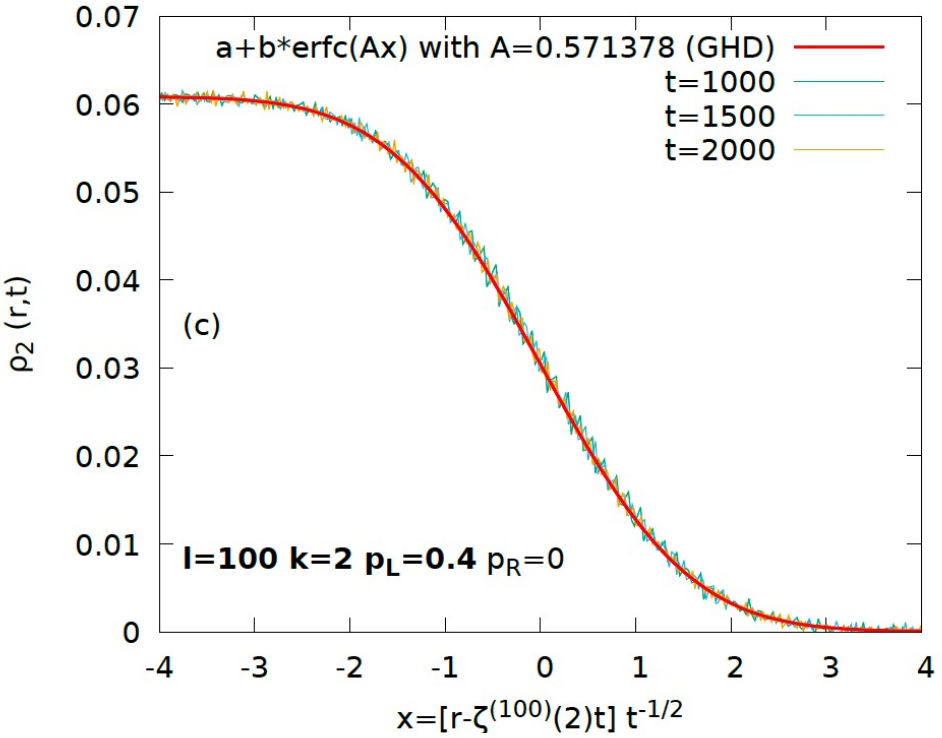

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GHD
(Bethe ansatz)

Current fluctuation and Large deviation

N_t = number of balls crossing $x = 0$ during time interval $[0, t]$.

Scaled cumulant generating function under time evolution T_l:

$$F(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log\langle e^{\lambda N_t}\rangle = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \left(\lim_{t\to\infty} \frac{1}{t} \langle N_t^k \rangle^c \right)$$
$$= \log\left(\frac{1-(qe^{\lambda})^{l+1}}{1-qe^{\lambda}}\right) - \log\left(\frac{1-q^{l+1}}{1-q}\right)$$

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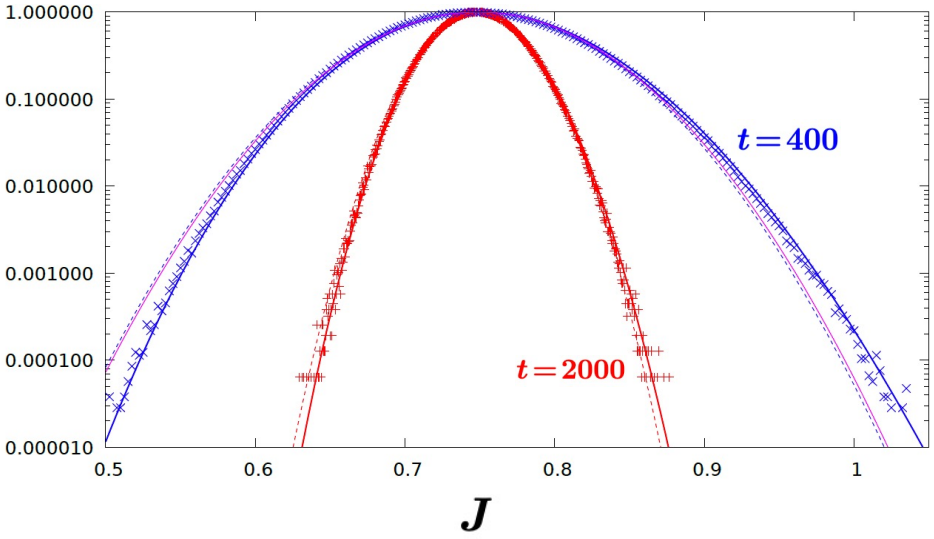
Large deviation rate function:

$$G(J) = \text{Legendre transformation of } F(\lambda)$$



Plot : $\dfrac{\text{Prob}(N_t/t = J)}{\text{Prob}_{\max}}$ ($10^{6.5}$ samples)

Solid curves: $\exp(-tG(J))$

Dotted curves: Gaussian fit

(ball density =0.3, time evolution = T_{10})

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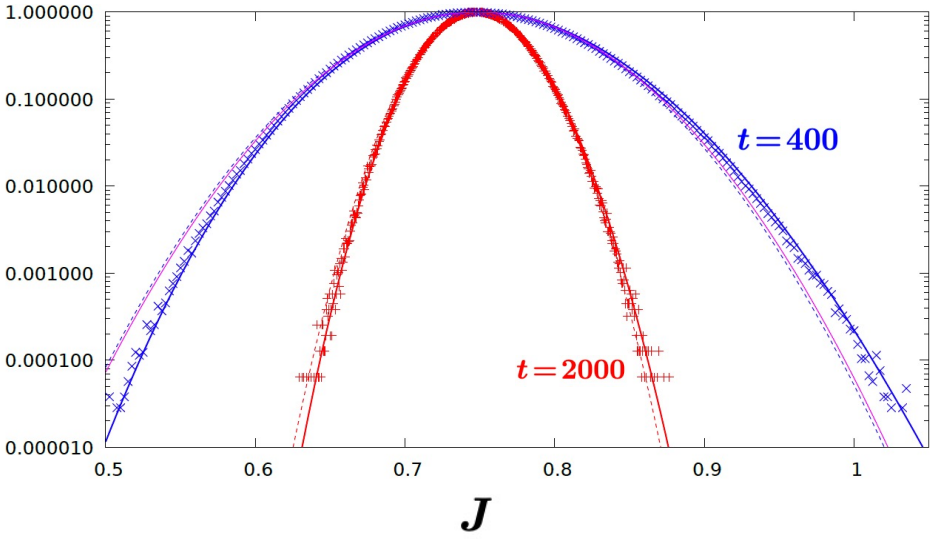

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(ball density =0.3, time evolution = T_{10})

F and G have been obtained also for non-i.i.d. 2-temperature GGE

Summary

1. **BBS is a Yang-Baxter integrable cellular automaton with quantum group symmetry and explicit action-angle variables originating in Bethe strings.**

2. **Limit shape of soliton content in randomized BBS is determined by TBA.**

3. **It serves as an essential input to GHD, which provides an analytical description of the density plateaux emerging from domain wall initial conditions and the large deviation principle of the current.**

Reference

Review part:

Inoue-K-Takagi
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Limit shape problem:

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Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz, NPB(2018)

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Generalized hydrodynamics in box-ball system, JPA(2020)
Generalized hydrodynamics in complete box-ball system for $U_q(\widehat{sl}_n)$, SciPostPhys (2021)
Current fluctuations and large deviations in a box-ball system, in preparation