Randomized box-ball systems, limit shape of soliton distributions and thermodynamic Bethe ansatz

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IPhT Saclay, 18 November 2019

Contents

Box-ball system (short review)

Time evolution, solitons, integrability,

KKR bijection, action-angle variables.

Randomized box-ball system

Soliton contents, limit shape problem,

Induced measure, Fermionic formula,

Thermodynamic Bethe ansatz, results.

n-color Box-ball system (BBS)

$n = 3$ example.

$\cdots$ 0000000033211000000000000000000000000 $\cdots$
$\cdots$ 0000000000000332110000000000000000000 $\cdots$
$\cdots$ 0000000000000000003321100000000000000 $\cdots$

0 =empty box, $1, 2, 3$ = balls with colors

- time evolution = (move 1) $\cdot$ (move 2) $\cdot$ (move 3)

(move i) $\cdot$ Pick the leftmost ball with color i and move it to the nearest right empty box.

$\cdot$ Do the same for the other color i balls.

- soliton=consecutive balls $i_1 \ldots i_a$ with color $i_1 \geq \cdots \geq i_a \geq 1$.
- velocity=amplitude.

- **Collisions of 2 solitons**

$\cdots$000**33211**0000**322**00000000000000000000000$\cdots$
$\cdots$00000000**33211**00**322**00000000000000000000$\cdots$
$\cdots$0000000000000**33211322**00000000000000000$\cdots$
$\cdots$000000000000000000**21133322**000000000000$\cdots$
$\cdots$000000000000000000000**211**00**33322**0000000$\cdots$
$\cdots$000000000000000000000000**211**0000**33322**00$\cdots$

- **Amplitudes are individually conserved.**

- **Two body scattering:**

  Exchange of internal labels (colors) like quarks in hadrons

  Phase shift

Collision of 3 solitons

$\cdots$ 00321003100002000000000000000 $\cdots$
$\cdots$ 00000321031000200000000000000 $\cdots$
$\cdots$ 00000000320311020000000000000 $\cdots$
$\cdots$ 00000000003200312100000000000 $\cdots$
$\cdots$ 00000000000032001032100000000 $\cdots$
$\cdots$ 00000000000000320100032100000 $\cdots$
$\cdots$ 00000000000000003021000032100 $\cdots$

$\cdots$ 00321000031002000000000000000 $\cdots$
$\cdots$ 00000321000310200000000000000 $\cdots$
$\cdots$ 00000000321003120000000000000 $\cdots$
$\cdots$ 00000000000321013200000000000 $\cdots$
$\cdots$ 00000000000000302132100000000 $\cdots$
$\cdots$ 00000000000000030021032100000 $\cdots$
$\cdots$ 00000000000000003000210032100 $\cdots$

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

Double (classical and quantum) origin of integrability

(1) Ultra-Discretization (UD) of soliton equations

- Key formula

$$\lim_{\varepsilon\to+0}\varepsilon\log\left(\exp(\frac{a}{\varepsilon})+\exp(\frac{b}{\varepsilon})\right)=\max(a,b)$$

$$\lim_{\varepsilon\to+0}\varepsilon\log\left(\exp(\frac{a}{\varepsilon})\times\exp(\frac{b}{\varepsilon})\right)=a+b$$

$$(+,\times)\longrightarrow(\max,+)$$

keeps distributive law:

$$AB+AC=A(B+C)\to\max(a+b,a+c)=a+\max(b,c)$$

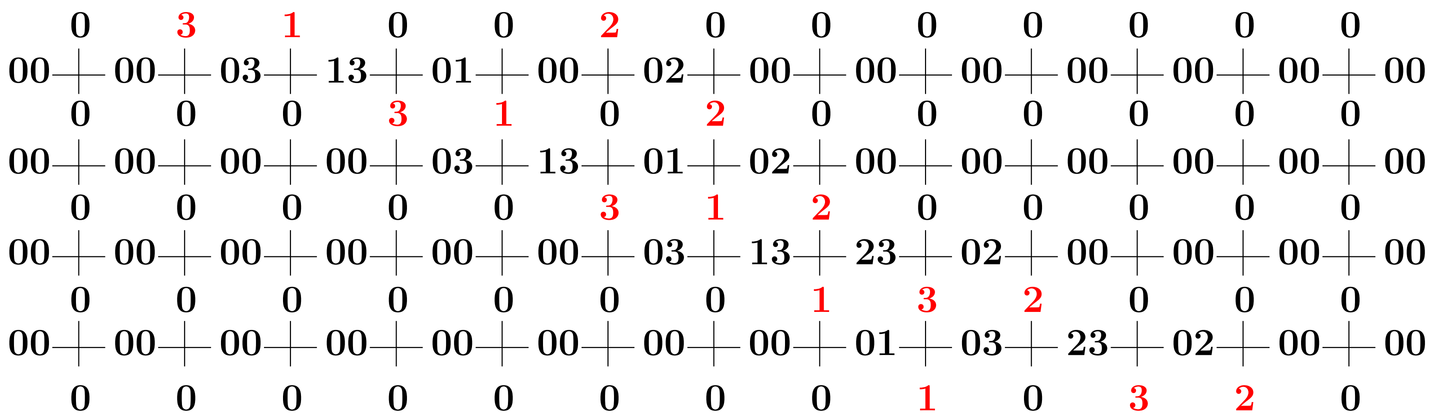
- UD of a discrete KdV equation gives an evolution equation of the $n=1$ BBS (1996).

(2) Solvable lattice model at “ Temperature 0 ”

Time evolution pattern

$$\cdots 0310020000000 \cdots$$
$$\cdots 0003102000000 \cdots$$
$$\cdots 0000031200000 \cdots$$
$$\cdots 0000000132000 \cdots$$
$$\cdots 0000000010320 \cdots$$

emerges from a configuration of a 2D lattice model in statistical mechanics



by forgetting the hidden variables on the horizontal edges.

- n-color box-ball system

  = 2D solvable vertex model associated with quantum group

  $U_q(\widehat{sl}_{n+1})$ at $q = 0$ ($q \sim$ temperature)

- Row transfer matrix at $q = 0$

  = deterministic map (defined by the unique configuration surviving at $q = 0$)

  = time evolution of box-ball system (forming a commuting family $T_1, T_2, \ldots T_\infty$)

- Proper formulation uses *crystal base theory* (theory of quantum group at $q = 0$).

Some outcomes from such insight

- $\exists$ Integrable cellular automata with quantum group symmetry.

Example: $\widehat{\mathrm{so}}_{10}$-automaton

$$\cdots 000\bar{2}\bar{4}211 0000\bar{1}\bar{1}\bar{4}00000000000000000000000000000000 \cdots$$
$$\cdots 00000000\bar{2}\bar{4}21100\bar{1}\bar{1}\bar{4}00000000000000000000000000000 \cdots$$
$$\cdots 0000000000000\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4}00000000000000000000000000 \cdots$$
$$\cdots 000000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4}00000000000000000000000 \cdots$$
$$\cdots 000000000000000000000003\bar{0}\bar{0}\bar{3}\bar{4}\bar{4}000000000000000000 \cdots$$
$$\cdots 00000000000000000000000000311\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}0000000000000 \cdots$$
$$\cdots 0000000000000000000000000000031100\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}00000000 \cdots$$
$$\cdots 000000000000000000000000000000003110000\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}000 \cdots$$

- Particles and antiparticles undergo pair-creations/annihilations.

- n-color BBS $= \widehat{sl}_{n+1}$-automaton $= \widehat{\mathrm{so}}_{2n+2}$-automaton in antipaticle-free sector.

- Soliton & scattering most naturally captured in quantum group framework.

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	$\overset{\text{UD}}{\longrightarrow}$	Cellular automata Box-ball systems	$\overset{0 \leftarrow q}{\longleftarrow}$	Lattice statistical models Solvable vertex models
Inverse scattering method		KKR bijection		Bethe ansatz

- Kerov-Kirillov-Reshetikhin (KKR) bijection (1986) asserts "formal completeness" of the hypothetical string solutions to the Bethe equation at combinatorial level.

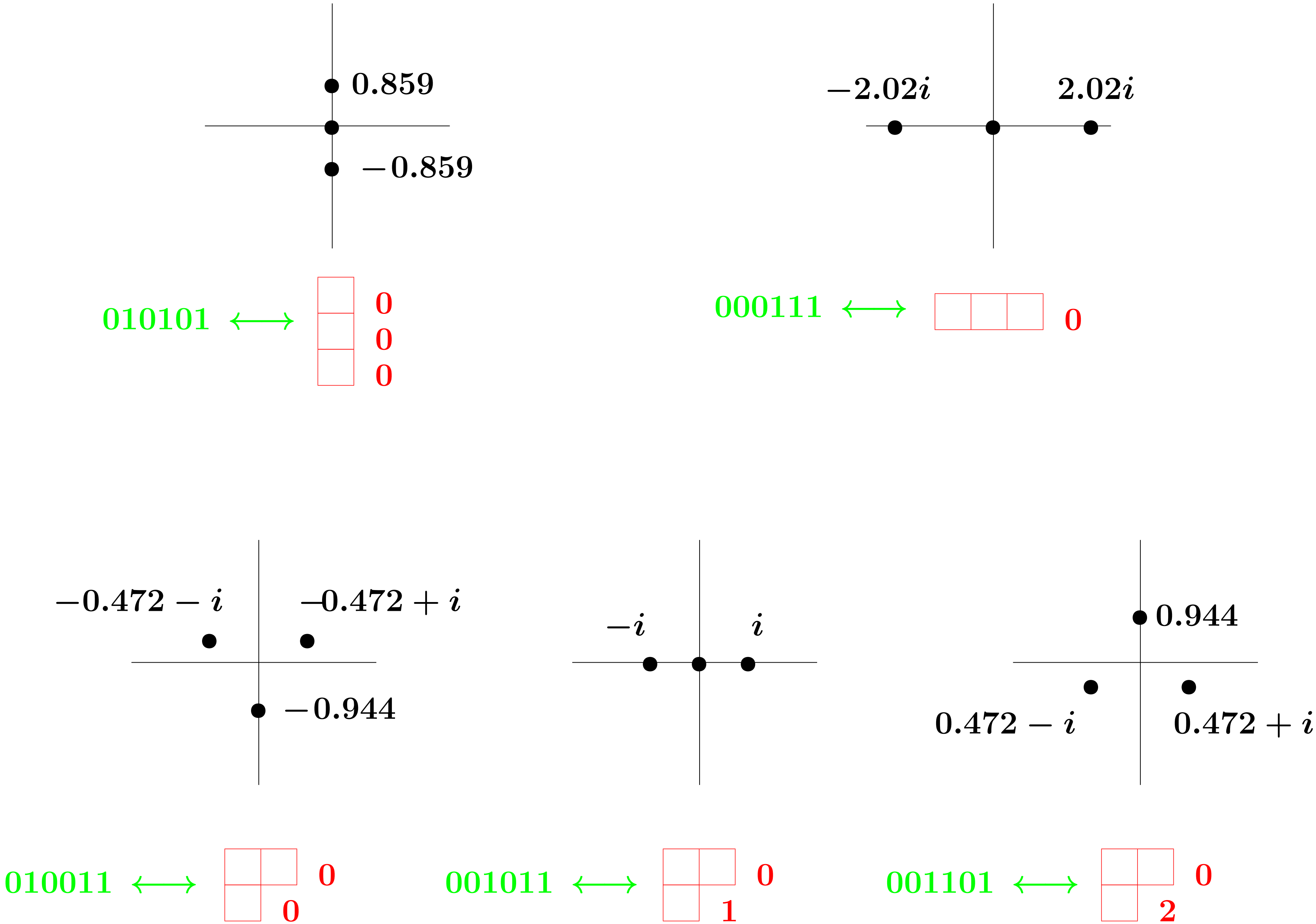
- Its unexpected connection to BBS was discovered in 2002.

- **Example. Spin $\frac{1}{2}$ periodic Heisenberg chain**

$$H = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)$$

For $L = 6$ sites in 3 down-spin sector, the Bethe equation reads

$$\left(\frac{u_1 + i}{u_1 - i}\right)^6 = \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)},$$
$$\left(\frac{u_2 + i}{u_2 - i}\right)^6 = \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)},$$
$$\left(\frac{u_3 + i}{u_3 - i}\right)^6 = \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.$$



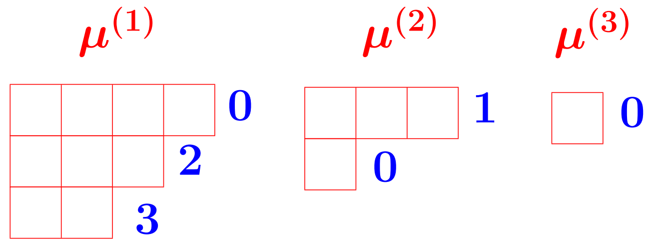
0.859
−0.859
010101 ⟷
0
0
0
−2.02i
2.02i
000111 ⟷
0
−0.472 − i
−0.472 + i
−0.944
010011 ⟷
0
0
−i
i
001011 ⟷
0
1
0.944
0.472 − i
0.472 + i
001101 ⟷
0
2

KKR bijection for sl_{n+1}

$$\{\text{highest states}\} \overset{1:1}{\longleftrightarrow} \{\text{rigged configurations}\}$$

$n = 3$ example

0000111021132200000 $\longleftrightarrow$



"Bethe vectors" — "Bethe roots"

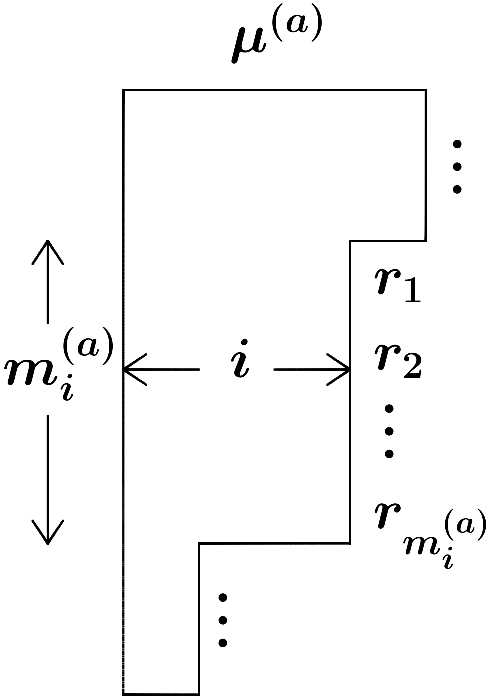
Solitons — Strings (bound states of magnons)

- highest states $= i_1 i_2 \dots i_L$ $(0 \le i_k \le n)$ satisfying the highest condition:

$$\#_0\{i_1, \dots, i_k\} \ge \#_1\{i_1, \dots, i_k\} \ge \cdots \ge \#_n\{i_1, \dots, i_k\} \quad (\forall k)$$

- rigged configuration: $((\mu^{(1)}, r^{(1)}), \dots, (\mu^{(n)}, r^{(n)}))$

$$\left.\begin{array}{l} \mu^{(1)}, \dots, \mu^{(n)} : \text{configuration} = n\text{-tuple of Young diagrams} \\ r^{(1)}, \dots, r^{(n)} : \text{rigging} = \text{integers assigned to each row} \end{array}\right\} + \text{selection rule (next page)}$$



$$m_i^{(a)} = \#(\text{length } i \text{ rows in } \mu^{(a)}), \ \textstyle\sum_{i\geq 1} i m_i^{(a)} = |\mu^{(a)}|$$

$$0 \leq r_1 \leq \cdots \leq r_{m_i^{(a)}} \leq h_i^{(a)}$$

$\cdots$ "Fermionic" selection rule

$$h_i^{(a)} = L\delta_{a,1} - \textstyle\sum_{b=1}^{n} C_{ab} \sum_{j\geq 1} \min(i,j) m_j^{(b)}$$

$\cdots$ vacancy for *holes*

$$C_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1}$$

(C_{ab}) $\cdots$ Cartan matrix of sl_{n+1}

$$\#\text{ of rigging choices for a fixed configuration} = \prod_{a=1}^{n} \prod_{i\geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}$$

This is an sl_{n+1} generalization of Bethe's formula for # of string solutions (1931).

hat also eine Möglichkeit weniger, die des letzten Komplexes von n Wellen, λ_{q_n}, kann schließlich nur noch

$$Q_n' - (q_n - 1) = Q_n + 1$$

verschiedene Werte annehmen, wo

$$Q_n(N, q_1 q_2 \ldots) = N - 2 \sum_{p<n} p\, q_p - 2 \sum_{p \geqslant n} n\, q_p. \tag{44}$$

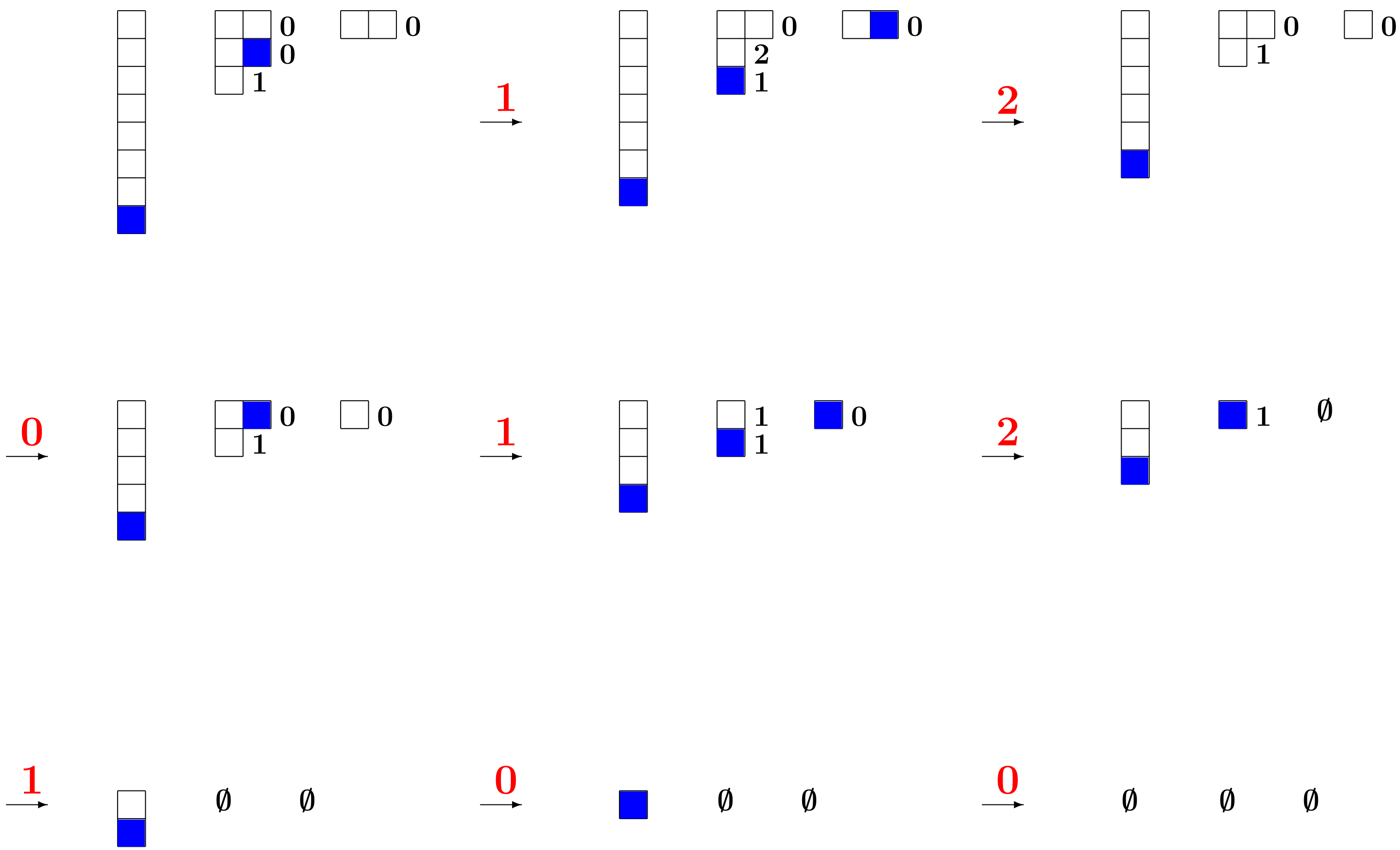
Schließlich ist zu berücksichtigen, daß Vertauschung der λ der verschiedenen Wellenkomplexe mit gleicher Anzahl n von Wellen nicht zu neuen Lösungen führt. Die gesamte Zahl unserer Lösungen wird somit

$$z(N, q_1 q_2 \ldots) = \prod_{n=1}^{\infty} \frac{(Q_n + q_n) \ldots (Q_n + 1)}{q_n!} = \prod_n \binom{Q_n + q_n}{q_n}, \tag{45}$$

wo die Q_n durch (44) definiert sind.

8. Wir werden nun nachweisen, daß wir die richtige Anzahl Lösungen erhalten haben.

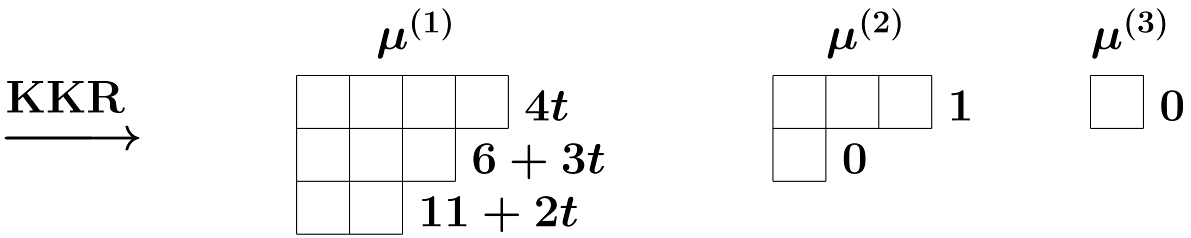
Example of KKR algorithm



Top left rigged configuration $\xrightarrow{\text{KKR}}$ **00121021**

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000
$t = 1$: 000000001111000022103200000000000000000000000000
$t = 2$: 000000000000111100022132000000000000000000000000
$t = 3$: 000000000000000011110021322000000000000000000000
$t = 4$: 000000000000000000001110211322000000000000000000
$t = 5$: 000000000000000000000001100211322100000000000000
$t = 6$: 000000000000000000000000011000211032210000000000
$t = 7$: 000000000000000000000000000110000211003221000000



- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

Rigged configuration = action angle variable of BBS!

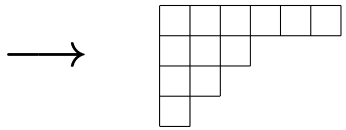
In particular, $\mu^{(1)}$ = list of amplitude of solitons.

$(\mu^{(1)}, \ldots, \mu^{(n)})$ will be called a soliton content (= generalized amplitude).

Quiz: What are the soliton contents or at least their amplitude?

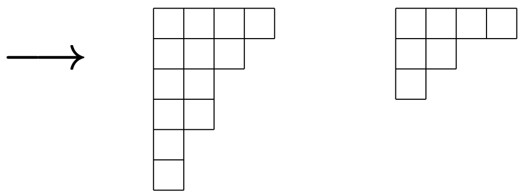
110111100111000111000000000000000000000000000000
001000011000111000111111000000000000000000000000
000100000110000111000000111111000000000000000000
000010000001100000111000000000111111000000000000

$\boldsymbol{\mu}^{(1)}$



210221001220012211000000000000000000000000000000
002100220112201000221100000000000000000000000000
000021002001120221000022110000000000000000000000
000000210200012100221000002211000000000000000000
000000002120001021000221000000221100000000000000
000000000012200100210000221000000022110000000000
000000000001022010002100000221000000002211000000
000000000000100202100021000000221000000000221100

$\boldsymbol{\mu}^{(1)}$ $\boldsymbol{\mu}^{(2)}$



Randomized box-ball system

$$\begin{array}{ccc} \textbf{BBS state} & & \textbf{Soliton content} \\ i_1 i_2 \ldots i_L 00000 \ldots & \overset{\text{KKR}}{\longmapsto} & (\mu^{(1)}, \ldots, \mu^{(n)}) \end{array}$$

Randomize $i_1 i_2 \ldots i_L$ by introducing the i.i.d. measure on the set of states:

$$\{0, 1, \ldots, n\} \to (0,1); \qquad i \ \mapsto \ p_i \qquad (p_0 + \cdots + p_n = 1).$$

Problems

(1) Find the induced measure $\mathrm{Prob}(\mu^{(1)}, \ldots, \mu^{(n)})$ on soliton contents.

(2) Determine the scaling form of the most probable $(\mu^{(1)}, \ldots, \mu^{(n)})$ when $L \to \infty$.

Technical Lemma

$$\mathrm{Prob}(\mu^{(1)}, \ldots, \mu^{(n)}) \simeq \mathrm{Prob}_+(\mu^{(1)}, \ldots, \mu^{(n)}) \quad \text{if} \ \ p_0 \geq p_1 \geq \cdots \geq p_n$$

as far as the leading scaling behavior is concerned, where

$$\mathrm{Prob}_+(\mu^{(1)}, \ldots, \mu^{(n)}) := \text{induced measure for the } \textit{highest} \text{ BBS states.}$$

Theorem (“Fermionic” measure for the soliton contents)

$$\mathrm{Prob}_+(\mu^{(1)},\ldots,\mu^{(n)}) = Z_L^{-1}\mathrm{e}^{-\beta_1|\mu^{(1)}|-\cdots-\beta_n|\mu^{(n)}|}\prod_{a=1}^{n}\prod_{i\geq 1}\binom{h_i^{(a)}+m_i^{(a)}}{m_i^{(a)}},$$

$$\mathrm{e}^{\beta_a} := \frac{p_{a-1}}{p_a}, \qquad Z_L = \text{normalization const. (partition function)}.$$

The above Lemma and Theorem yield a solution to Problem (1) for $p_0 \geq \cdots \geq p_n$.

Problem (2) is handled by Thermodynamic Bethe Ansatz (TBA).

Introduce the scaled string and hole densities $\rho_i^{(a)}, \sigma_i^{(a)}$ by

$$m_i^{(a)} \simeq L\rho_i^{(a)}, \quad h_i^{(a)} \simeq L\sigma_i^{(a)}, \quad \sigma_i^{(a)} = \delta_{a,1} - \sum_{b=1}^{n} C_{ab}\sum_{j\geq 1}\min(i,j)\rho_j^{(b)},$$

The most probable $(\mu^{(1)},\ldots,\mu^{(n)})$ is given by $\{\rho_i^{(a)}\}$ achieving the minimum of

$$F = \sum_{a=1}^{n}\sum_{i\geq 1}\Bigl(\beta_a i\rho_i^{(a)} - (\rho_i^{(a)}+\sigma_i^{(a)})\log(\rho_i^{(a)}+\sigma_i^{(a)}) - \rho_i^{(a)}\log\rho_i^{(a)} - \sigma_i^{(a)}\log\sigma_i^{(a)}\Bigr)$$

The condition $\frac{\delta F}{\delta \rho_i^{(a)}} = 0$ leads to the TBA equation

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \log(1 + (Y_j^{(b)})^{-1})$$

in terms of $Y_i^{(a)} = \dfrac{\sigma_i^{(a)}}{\rho_i^{(a)}}$ with the boundary condition $\lim_{i \to \infty} \dfrac{1 + Y_{i+1}^{(a)}}{1 + Y_i^{(a)}} = e^{\beta_a}$.

This is equivalent to the (constant) Y-system

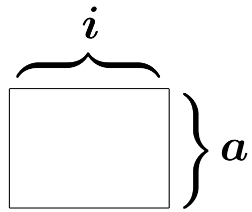
$$\left(Y_i^{(a)}\right)^2 = \frac{(1 + Y_{i-1}^{(a)})(1 + Y_{i+1}^{(a)})}{\left(1 + (Y_i^{(a-1)})^{-1}\right)\left(1 + (Y_i^{(a+1)})^{-1}\right)}$$

Solution (Rare case for which an exact formula can be given)

$$Y_i^{(a)} = \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{Q_i^{(a-1)} Q_i^{(a+1)}},$$

$$Q_i^{(a)} = Q_i^{(a)}(p_0, \ldots, p_n) = \frac{\det(p_k^{\lambda_j + n - j})_{j,k=0}^{n}}{\det(p_k^{n-j})_{j,k=0}^{n}} \qquad \Big((\lambda_0, \ldots, \lambda_n) = (\overbrace{i \ldots i}^{a}\, \overbrace{0, \ldots, 0}^{n+1-a}) \Big)$$

= Schur function for $a \times i$ rectangular Young diagram

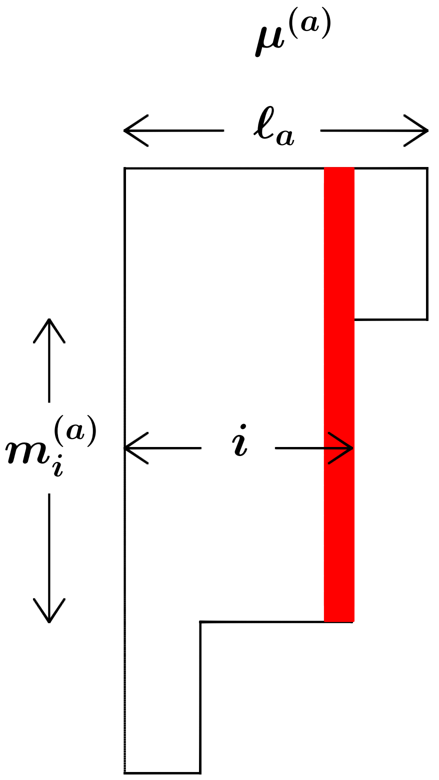


Result. The limit shape of soliton content $(\mu^{(1)}, \ldots, \mu^{(n)})$ is given by

$$\eta_i^{(a)} := \lim_{L\to\infty} \frac{1}{L}(\text{Length of the } i \text{ th column of } \mu^{(a)}) = \frac{Q_{i-1}^{(a-1)} Q_i^{(a+1)}}{Q_i^{(a)} Q_{i-1}^{(a)} Q_1^{(1)}}$$

$$\text{width } \ell_a \text{ of } \mu^{(a)} \simeq \frac{\log L}{\log \frac{p_{a-1}}{p_a}} \quad (L \to \infty \text{ if } p_0 > \cdots > p_n)$$

Special case $p_a = \frac{q^a}{1+q+\cdots+q^n} \quad (0 < q \le 1).$



Scaled column length of $\mu^{(a)}$

$$\eta_i^{(a)} = \frac{q^{i+a-1}(1-q)(1-q^a)(1-q^{n+1-a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})}$$

Strings

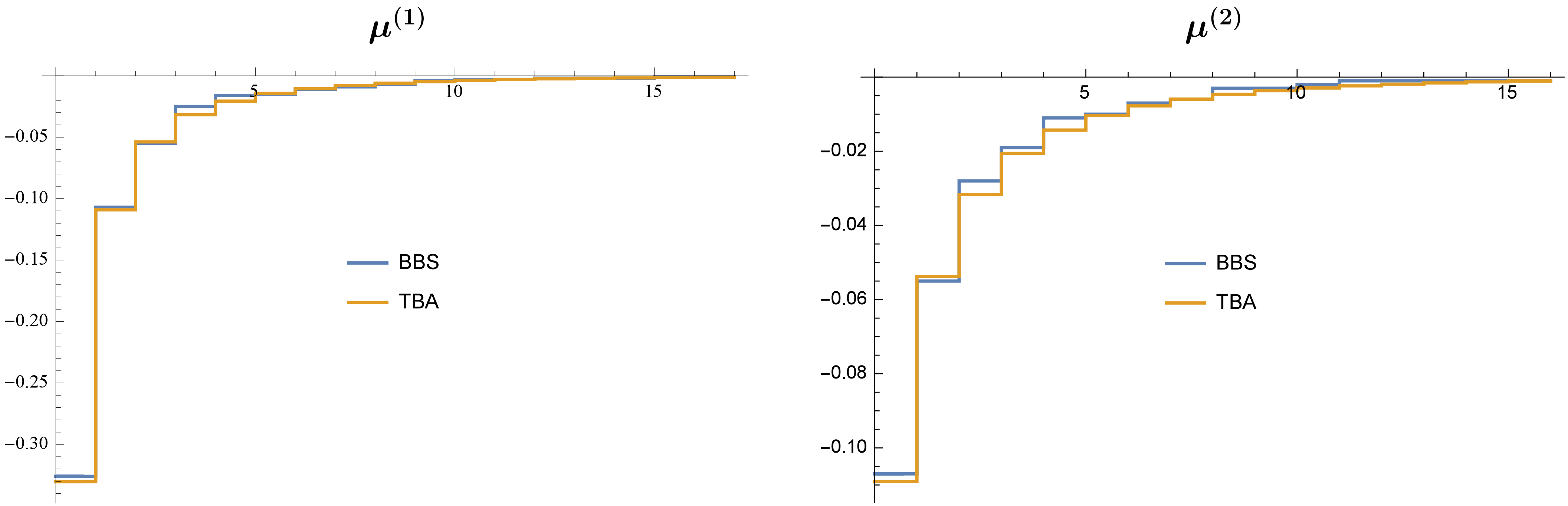
$$\rho_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} m_i^{(a)} = \frac{q^{i+a-1}(1-q)^2(1-q^a)(1-q^{n+1-a})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

Holes

$$\sigma_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} h_i^{(a)} = \frac{q^{a-1}(1-q)^2(1-q^i)(1-q^{n+i+1})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

2-color BBS with $L = 1000$ sites with distribution $(p_0, p_1, p_2) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$.

Vertically L^{-1} scaled soliton contents.



- **$0 < q < 1$: Subcritical**

  $\mu^{(a)}$ scales with depth $O(L)$, width $O(\log L)$

- **$q = 1$: Critical**

  $\mu^{(1)}$ scales with depth $O(L)$, width $O(\sqrt{L})$

Summary

1. **Rigged configurations are action-angle variables of n-color BBS.**
2. **In the context of BBS, a configuration $(\mu^{(1)}, \ldots, \mu^{(n)})$ is a soliton content generalizing the list of amplitude $\mu^{(1)}$.**
3. **Randomizing local states by the probabilities $p_0, \ldots, p_n$ induces the measure for $(\mu^{(1)}, \ldots, \mu^{(n)})$ expressed by the Bethe type "Fermionic" formula.**
4. **By TBA, exact leading scaling form of the soliton content is determined in terms of Schur functions involving $p_0, \ldots, p_n$.**
5. **Renormalized velocity and currents in BBS soliton gas (P. Dorey, V. Pasquier, AK, work in progress).**

Review part: R. Inoue, AK and T.Takagi
"Integrable structure of box-ball systems: crystal, Bethe ansatz, ultradiscretization and tropical geometry", JPA(2012), arXiv:1109.5349.

Randomized BBS:

AK, H. Lyu and M.Okado
"Randomized box-ball systems, limit shape of rigged configurations and thermodynamic Bethe ansatz", NPB(2018), arXiv:1808.02626.

AK and H. Lyu
"Large deviations and one-sided scaling limit of randomized multicolor box-ball system", JSP(to appear) arXiv:1808.08074.

Merci beaucoup !