Matrix product solutions to the reflection equation from three dimensional integrability

Atsuo Kuniba (Inst. Phys. Komaba, Univ. Tokyo)
Vincent Pasquier (Univ. Paris Saclay)

Reference: J. Phys. A **51** 255204 (2018)

Mathematical Society of Japan Autumn Meeting
Okayama University, 24 Sep. 2018

Introduction (TE → YBE)

Yang Baxter equation (YBE):

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x)$$

Tetrahedron equation (TE) is a 3D generalization of YBE:

$$R_{124}\, R_{135}\, R_{236}\, R_{456} = R_{456}\, R_{236}\, R_{135}\, R_{124}$$

Introduction (TE → YBE)

Yang Baxter equation (YBE):

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x)$$

Tetrahedron equation (TE) is a 3D generalization of YBE:

$$R_{124}\, R_{135}\, R_{236}\, R_{456} = R_{456}\, R_{236}\, R_{135}\, R_{124}$$

Regarding the spaces 4,5,6 as auxiliary and writing it as

$$R_{124}\, R_{135}\, R_{236} = R_{456}\, (R_{236}\, R_{135}\, R_{124})(R_{456})^{-1}$$

one finds TE = YBE up to conjugation in the auxiliary space 4,5,6

= a quantization of YBE

Introduction (TE → YBE)

Yang Baxter equation (YBE):

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x)$$

Tetrahedron equation (TE) is a 3D generalization of YBE:

$$R_{124}\, R_{135}\, R_{236}\, R_{456} = R_{456}\, R_{236}\, R_{135}\, R_{124}$$

Regarding the spaces 4,5,6 as auxiliary and writing it as

$$R_{124}\, R_{135}\, R_{236} = R_{456}\, (R_{236}\, R_{135}\, R_{124})(R_{456})^{-1}$$

one finds TE = YBE up to conjugation in the auxiliary space 4,5,6
= a quantization of YBE

This almost trivial observation is known to yield infinitely many solutions to YBE in a **matrix product form** having applications to statistical mechanics etc.

Aim: Extend this 3D approach to the **Reflection equation**.

Quantized reflection equation

(Ordinary) reflection equation (RE):

$$L_{12}(x/y)K_2(x)\ L_{21}(xy)K_1(y) = K_1(y)\ L_{12}(xy)K_2(x)\ L_{21}(x/y)$$

Quantized reflection equation

(Ordinary) reflection equation (RE):

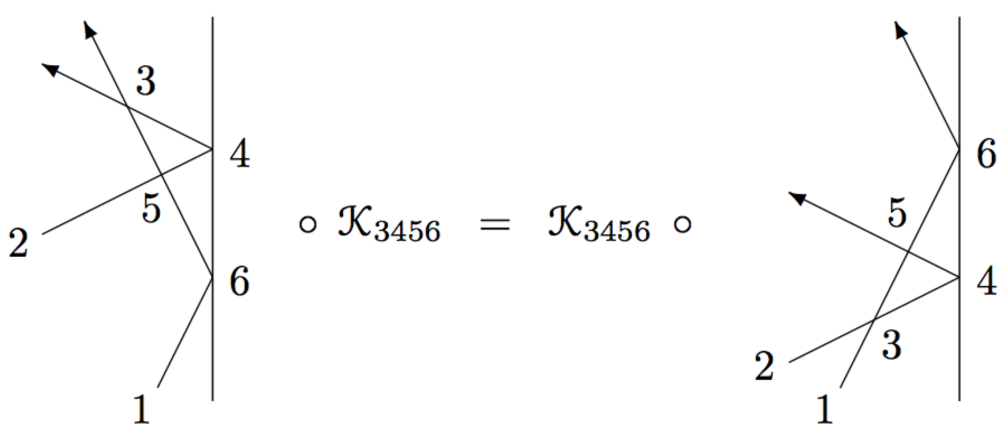
$$L_{12}(x/y)K_2(x)\ L_{21}(xy)K_1(y) = K_1(y)\ L_{12}(xy)K_2(x)\ L_{21}(x/y)$$

Quantized reflection equation := RE up to conjugation

$$(L_{12}K_2L_{21}K_1)\mathcal{K} = \mathcal{K}(K_1L_{12}K_2L_{21})$$

where L and K also act on the **auxiliary space** and $\mathcal{K}$ is the conjugation.
If all the space indices are written out explicitly, it reads

$$L_{123}K_{24}L_{215}K_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}K_{16}L_{125}K_{24}L_{213}$$



A solution to the quantized RE (1/2)

q-bosons and their Fock spaces (q: generic)

$$F_q = \bigoplus_{m\geq 0} \mathbb{C}|m\rangle, \quad F_q^* = \bigoplus_{m\geq 0} \mathbb{C}\langle m|, \quad \langle m|m'\rangle = (q^2)_m \delta_{m,m'}, \quad (q)_m = \prod_{i=1}^m (1-q^i)$$

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{h}|m\rangle = m|m\rangle, \quad \mathbf{k} = q^{\mathbf{h}+\frac{1}{2}}$$

$F_{q^2}, F_{q^2}^*, \mathbf{A}^+, \mathbf{A}^-, \mathbf{K} :=$ the same objects with q replaced by q^2.

Set $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \simeq \mathbb{C}^2$ and define L and K as follows:

A solution to the quantized RE (1/2)

q-bosons and their Fock spaces (q: generic)

$$F_q = \bigoplus_{m\geq 0}\mathbb{C}|m\rangle,\quad F_q^* = \bigoplus_{m\geq 0}\mathbb{C}\langle m|,\quad \langle m|m'\rangle = (q^2)_m\delta_{m,m'},\quad (q)_m = \prod_{i=1}^m (1-q^i)$$

$$\mathbf{a}^+|m\rangle = |m+1\rangle,\quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle,\quad \mathbf{h}|m\rangle = m|m\rangle,\quad \mathbf{k} = q^{\mathbf{h}+\frac{1}{2}}$$

$F_{q^2}, F^*_{q^2}, \mathbf{A}^+, \mathbf{A}^-, \mathbf{K} :=$ the same objects with q replaced by q^2.

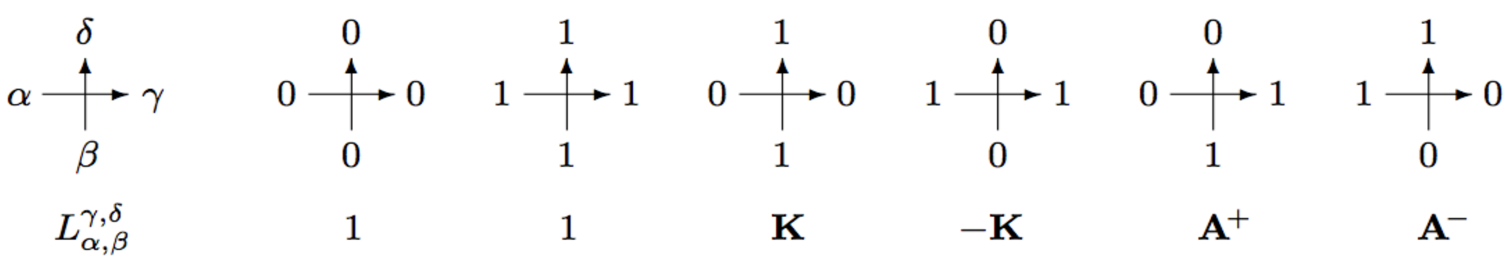
Set $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \simeq \mathbb{C}^2$ and define L and K as follows:

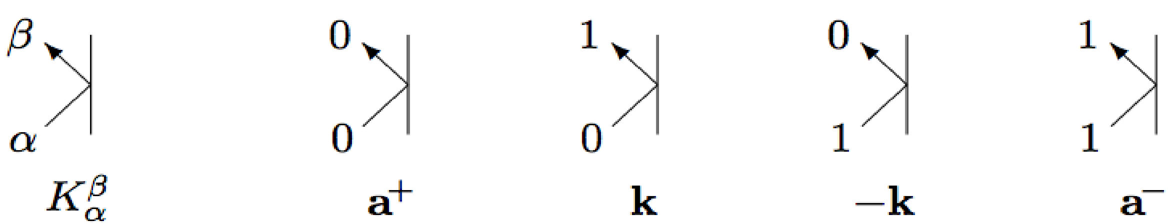
$$L \in \mathrm{End}(V\otimes V\otimes F_{q^2})$$

$$L(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum v_\gamma \otimes v_\delta \otimes L^{\gamma,\delta}_{\alpha,\beta}|m\rangle$$

$$K \in \mathrm{End}(V\otimes F_q)$$

$$K(v_\alpha \otimes |m\rangle) = \sum v_\beta \otimes K^\beta_\alpha |m\rangle$$





A solution to the quantized RE (2/2)

For this L and K, the quantized RE is 16 linear equations on $\mathcal{K} \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$

Example.

$$
\begin{aligned}
&[1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{a}^- - 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{k},\ \mathcal{K}] = 0,\\
&(1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+)\mathcal{K}\\
&= \mathcal{K}(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^- - \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k}),\\
&(1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-)\mathcal{K} = \mathcal{K}(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^-),\\
&[1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k}, \mathcal{K}] = 0.
\end{aligned}
$$

Fix the normalization by $\mathcal{K}(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$.

A solution to the quantized RE (2/2)

For this L and K, the quantized RE is 16 linear equations on $\mathcal{K} \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q)$

Example.

$$[1\otimes \mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- - 1\otimes \mathbf{k} \otimes \mathbf{A}^- \otimes \mathbf{k}, \mathcal{K}] = 0,$$
$$(1\otimes \mathbf{a}^- \otimes 1 \otimes \mathbf{k} + 1 \otimes \mathbf{k} \otimes \mathbf{A}^- \otimes \mathbf{a}^+)\mathcal{K}$$
$$= \mathcal{K}(\mathbf{A}^- \otimes \mathbf{a}^+ \otimes \mathbf{A}^- \otimes \mathbf{k} + \mathbf{A}^- \otimes \mathbf{k} \otimes 1 \otimes \mathbf{a}^- - \mathbf{K} \otimes \mathbf{a}^- \otimes \mathbf{K} \otimes \mathbf{k}),$$
$$(1\otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{a}^-)\mathcal{K} = \mathcal{K}(\mathbf{A}^+ \otimes \mathbf{a}^- \otimes \mathbf{K} \otimes \mathbf{k} + \mathbf{K} \otimes \mathbf{a}^+ \otimes \mathbf{A}^- \otimes \mathbf{k} + \mathbf{K} \otimes \mathbf{k} \otimes 1 \otimes \mathbf{a}^-),$$
$$[1\otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{k}, \mathcal{K}] = 0.$$

Fix the normalization by $\mathcal{K}(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle.$

Proposition. The solution to the quantized RE is given by

$\mathcal{K}$ = the intertwiner of the Soibelman representation of the quantized coordinate ring Aq(Sp4) labeled by the longest element of its Weyl group.

Remark. The intertwiner has been obtained explicitly in [K-Okado 2012], which yielded the first solution to the 3D reflection equation proposed by [Isaev-Kulish 1997] .

Matrix product construction of S(z) and K(z)

Introduce the *boundary vectors* $|\chi_s\rangle = \sum_{m\geq 0} \frac{|sm\rangle}{(q^{2s^2})_m} \in F_{q^2} \qquad |\eta_s\rangle = \sum_{m\geq 0} \frac{|sm\rangle}{(q^{s^2})_m} \in F_q \qquad (s=1,2)$

Conjecture: $\mathcal{K}(|\chi_s\rangle \otimes |\eta_k\rangle \otimes |\chi_s\rangle \otimes |\eta_k\rangle) = |\chi_s\rangle \otimes |\eta_k\rangle \otimes |\chi_s\rangle \otimes |\eta_k\rangle \qquad (1 \leq s \leq k \leq 2)$

Matrix product construction of S(z) and K(z)

Introduce the *boundary vectors* $|\chi_s\rangle = \sum_{m\geq 0} \frac{|sm\rangle}{(q^{2s^2})_m} \in F_{q^2}$ $\quad |\eta_s\rangle = \sum_{m\geq 0} \frac{|sm\rangle}{(q^{s^2})_m} \in F_q$ $\quad (s=1,2)$

Conjecture: $\mathcal{K}(|\chi_s\rangle \otimes |\eta_k\rangle \otimes |\chi_s\rangle \otimes |\eta_k\rangle) = |\chi_s\rangle \otimes |\eta_k\rangle \otimes |\chi_s\rangle \otimes |\eta_k\rangle \qquad (1 \leq s \leq k \leq 2)$

For any $n \geq 1$ use the vector spaces with labels like $\mathbf{V} = V^{\otimes n}$ and $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$

Construct the operators in *matrix product forms* as

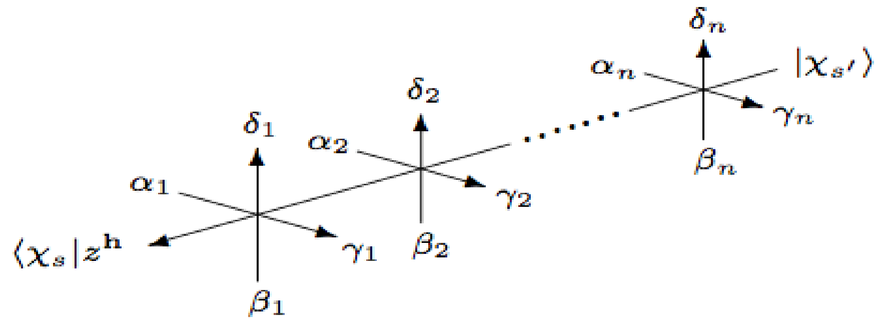
$$S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(z), S^{s,s'}_{\mathbf{1},\mathbf{2}}(z) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{2}}{\mathbf{V}}) \ (s,s'=1,2) \quad K^{\mathrm{tr}}_{\mathbf{1}}(z), K^{k,k'}_{\mathbf{1}}(z) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}) \ (k,k'=1,2)$$

$$S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(z) = \mathrm{Tr}_a(z^{\mathbf{h}_a} L_{1_1 2_1 a} \cdots L_{1_n 2_n a}), \qquad K^{\mathrm{tr}}_{\mathbf{1}}(z) = \mathrm{Tr}_a\left(z^{\mathbf{h}_a} K_{1_1 a} \cdots K_{1_n a}\right),$$

$$S^{s,s'}_{\mathbf{1},\mathbf{2}}(z) = \langle \chi_s | z^{\mathbf{h}_a} L_{1_1 2_1 a} \cdots L_{1_n 2_n a} | \chi_{s'} \rangle \qquad K^{k,k'}_{\mathbf{1}}(z) = \langle \eta_k | z^{\mathbf{h}_a} K_{1_1 a} \cdots K_{1_n a} | \eta_{k'} \rangle.$$

Here *a* denotes the auxiliary space. The matrix elements are expressed in 3D diagrams, e.g,

$$S^{s,s'}(z)^{\boldsymbol{\gamma},\boldsymbol{\delta}}_{\boldsymbol{\alpha},\boldsymbol{\beta}} =$$



Main result

A pair (S(z), K(z)) satisfying

$$S_{12}(x/y)K_2(x)\, S_{21}(xy)K_1(y) = K_1(y)\, S_{12}(xy)K_2(x)\, S_{21}(x/y)$$

with S(z) further satisfying YBE among itself is called a solution of RE.

Main result

A pair (S(z), K(z)) satisfying

$$S_{12}(x/y)K_2(x)\, S_{21}(xy)K_1(y) = K_1(y)\, S_{12}(xy)K_2(x)\, S_{21}(x/y)$$

with S(z) further satisfying YBE among itself is called a solution of RE.

Theorem:

The type A case of the table is a solution of RE. Admitting the conjecture on the previous page, all the pairs for the other **g** are also solutions of RE.

$\mathfrak{g}$	R matrix	K matrix
$A^{(1)}_{n-1}$	$S^{\mathrm{tr}}(z)$	$K^{\mathrm{tr}}(z)$,
$D^{(2)}_{n+1}$	$S^{1,1}(z)$	$K^{1,1}(z)$, $K^{1,2}(z)$, $K^{2,1}(z)$, $K^{2,2}(z)$
$B^{(1)}_{n}$	$S^{2,1}(z)$	$K^{2,1}(z)$, $K^{2,2}(z)$
$\tilde{B}^{(1)}_{n}$	$S^{1,2}(z)$	$K^{1,2}(z)$, $K^{2,2}(z)$
$D^{(1)}_{n}$	$S^{2,2}(z)$	$K^{2,2}(z)$

Main result

A pair (S(z), K(z)) satisfying

$$S_{12}(x/y)K_2(x)\, S_{21}(xy)K_1(y) = K_1(y)\, S_{12}(xy)K_2(x)\, S_{21}(x/y)$$

with S(z) further satisfying YBE among itself is called a solution of RE.

Theorem:

The type A case of the table is a solution of RE. Admitting the conjecture on the previous page, all the pairs for the other **g** are also solutions of RE.

$\mathfrak{g}$	R matrix	K matrix
$A^{(1)}_{n-1}$	$S^{\mathrm{tr}}(z)$	$K^{\mathrm{tr}}(z)$,
$D^{(2)}_{n+1}$	$S^{1,1}(z)$	$K^{1,1}(z)$, $K^{1,2}(z)$, $K^{2,1}(z)$, $K^{2,2}(z)$
$B^{(1)}_{n}$	$S^{2,1}(z)$	$K^{2,1}(z)$, $K^{2,2}(z)$
$\tilde{B}^{(1)}_{n}$	$S^{1,2}(z)$	$K^{1,2}(z)$, $K^{2,2}(z)$
$D^{(1)}_{n}$	$S^{2,2}(z)$	$K^{2,2}(z)$

Remarks.

S^{tr}, $S^{s,s'}$ and their YBE were proved in [Bazhanov-Sergeev 2006] and [K-Sergeev 2013].

They are the $U_p(\mathfrak{g})$ quantum R matrices of the fundamental representation (for S^{tr}) and the spin representations (for $S^{s,s'}$) for some p.

A similar result on **G_2 Reflection equation** is known [K 2018].