Physical Mathematics of Bethe Ansatz*

Atsuo Kuniba (Univ. Tokyo)

STATPHYS25 26 July 2013

(* MANY technical cautions omitted in the slide for simplicity.)

Spin 1/2 Heisenberg chain on a ring

$$H = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)$$

Diagonalization of H [Bethe 1931]

$$H|u_1, \ldots, u_N\rangle = E|u_1, \ldots, u_N\rangle, \quad E = \sum_{j=1}^{N} \frac{-8}{u_j^2 + 1},$$

if $u_1, \ldots, u_N$ satisfy Bethe equation

$$\left(\frac{u_j + i}{u_j - i}\right)^L = \prod_{k=1\ (k\neq j)}^{N} \frac{u_j - u_k + 2i}{u_j - u_k - 2i} \qquad (j = 1, \ldots, N).$$

$|u_1, \ldots, u_N\rangle$: Bethe vector, $\quad N = \sharp$ of down spins

Example. Length $L = 6$, $\sharp$ of down spins $N = 3$.

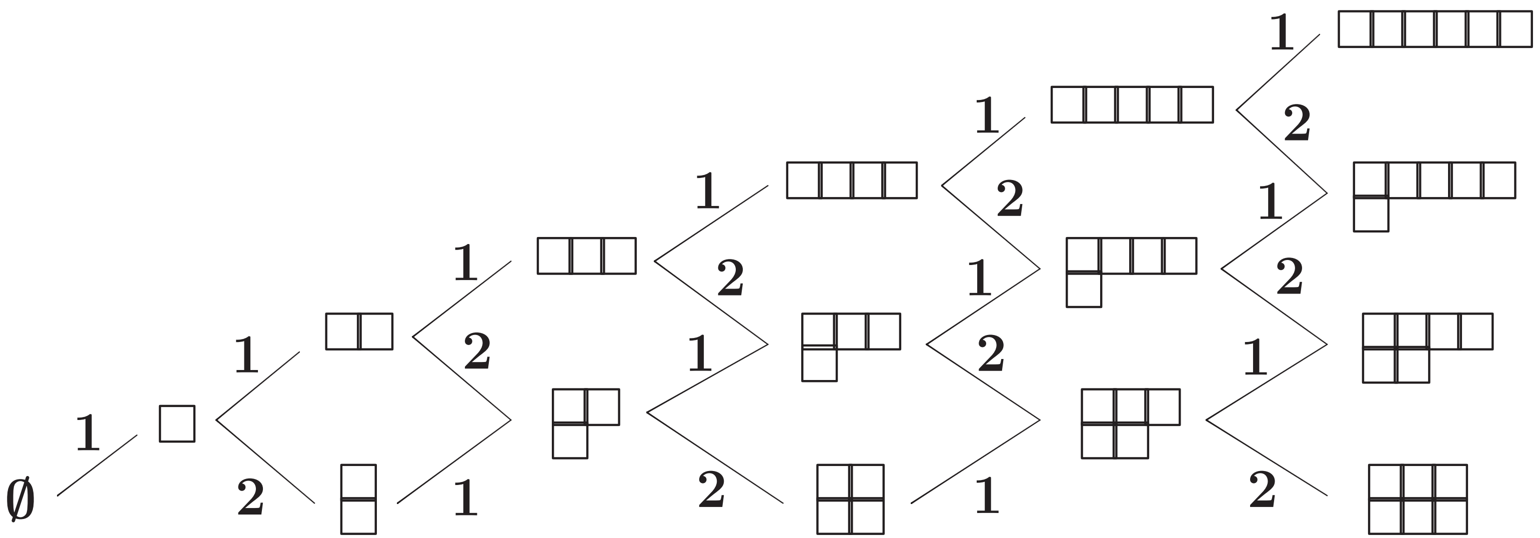
$$\left(\frac{u_1 + i}{u_1 - i}\right)^6 = \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)},$$
$$\left(\frac{u_2 + i}{u_2 - i}\right)^6 = \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)},$$
$$\left(\frac{u_3 + i}{u_3 - i}\right)^6 = \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.$$

Q1: How many solutions ?
Q2: How do they look like ?
Q3: How can they be labeled ?

A1: Kostka number
A2: String hypothesis
A3: Rigged configurations

Q&A1 : There should be 5 vectors for completeness because;

- Bethe vector is highest weight vector of sl_2.
- Irreducible decomposition of $(\text{spin } 1/2)^{\otimes 6}$:

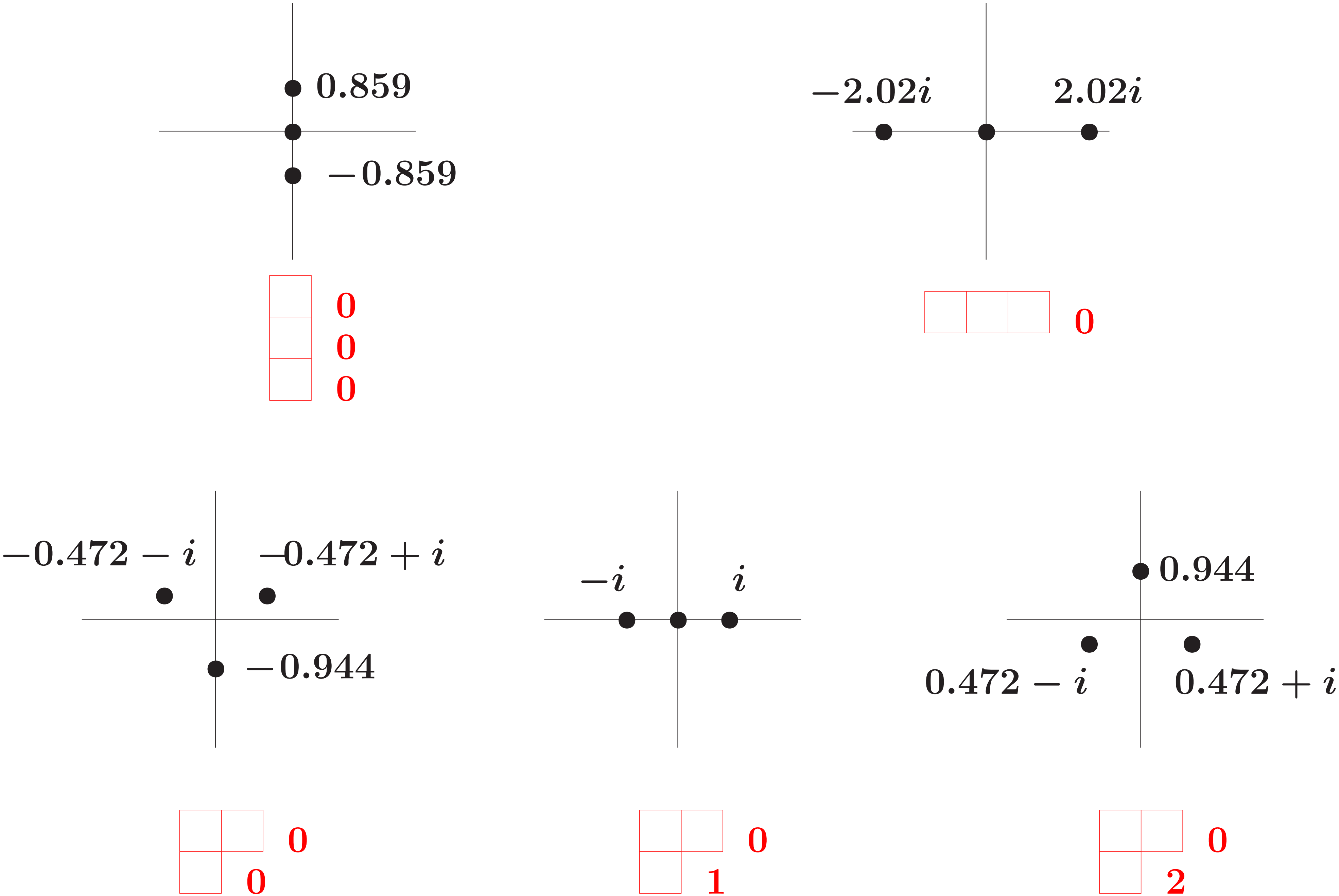


[5= multiplicity of ⊞] corresponds to the 5 paths from ∅:

121212, 111222, 121122, 112122, 112212

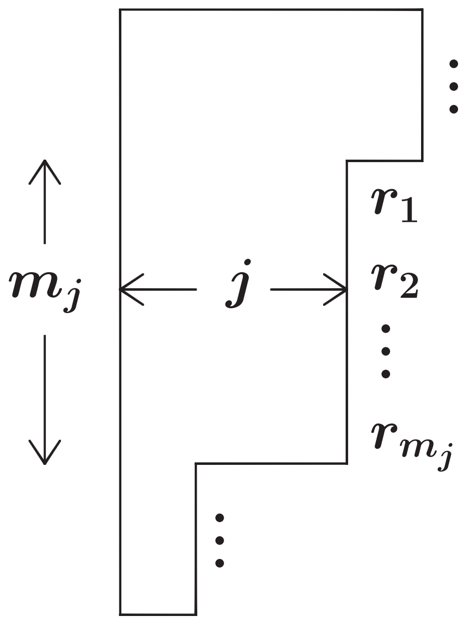
General case: Kostka number $K_{(L-N,N),(1^L)} = \binom{L}{N} - \binom{L}{N-1}$

Q&A2, 3 : Strings and Rigged Configurations



Rigged configuration for sl_2 (spin $\frac{1}{2}$)$^{\otimes L}$

Young diagram = configuration, $\{r_i\}$ = rigging



$$0 \le r_1 \le \cdots \le r_{m_j} \le p_j$$

$\cdots$ Fermionic selection rule

$$p_j = L - 2\sum_{k\ge1} \min(j,k) m_k$$

$\cdots$ vacancy number

$$\sharp \text{ of rigged configurations} = \sum_{\{m_j\}} \prod_{j\ge1} \binom{p_j + m_j}{m_j}$$

The RHS is called Fermionic form and deduced from physicist's (=Bethe's) counting of the string solutions.

{ Bethe roots } ⟷ 1 : 1? ⟷ { irreducible sl_2 components }

$\updownarrow$ 1 : 1 rigorously !

{ Bethe roots } ⟷ 1 : 1? ⟷ { rigged configurations }

Theorem. ("alle Lösungen": Bethe's Fermionic formula 1931)

$$\text{Kostka number} \quad K_{(L-N,N),(1^L)} = \sum_{\{m_i\}} \prod_{i\geq 1} \binom{p_i + m_i}{m_i}$$

Ultimate generalization

$sl_2 \longrightarrow$ arbitrary simple Lie algebra $\mathfrak{g}$

Spin 1/2 rep. $\longrightarrow$ Kirillov-Reshetikhin module of $U_q(\hat{\mathfrak{g}})$

Kostka number $\longrightarrow$ q-analogue of branching coefficient.

Bethe ansatz vs Corner transfer matrix for $U_q(\hat{\mathfrak{g}})$-symmetric spin chain indicates

Conjecture [Hatayama-K-Okado-Takagi-Yamada 1999]

$$\sum_{p:\text{ paths}} q^{E(p)} = \sum_{\{m_i^{(a)}\}:\text{ rigged conf.}} q^{c(\{m_i^{(a)}\})} \prod_{a} \prod_{i\geq 1} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}_q$$

Phys:

LHS= "trace" of Baxter's corner transfer matrix

RHS= generating function of "Bethe momentum" based on string hypothesis

Math:

LHS= q-analogue of branching coeff. reformulated by representation theory of $U_q(\hat{\mathfrak{g}})$

RHS= (q-)Fermionic form of $\mathfrak{g}$, where a runs over vertices of the Dynkin diagram

Theorem

Conjecture is true for $\mathfrak{g} = sl_n$ [Kirillov-Schilling-Shimozono 2002]

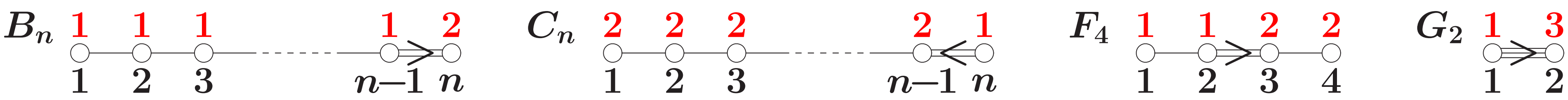
Conjecture is true for any $\mathfrak{g}$ at $q = 1$ [by many results by 2011]

T-systems and Y-systems

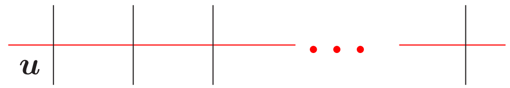
Difference equations related to Fermionic formula

$\{T_m^{(a)}(u)\}$, $\{Y_m^{(a)}(u)\}$; $a \in \{$vertices of Dynkin diagram of $\mathfrak{g}\}$, $m \in \mathbb{Z}_{\geq 1}$, $u \in \mathbb{C}$

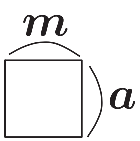
$t_a = 1$ for $\mathfrak{g} =$ ADE. For $\mathfrak{g} =$ BCFG, $\{t_a\}$ are specified as



$T_m^{(a)}(u) =$ Commuting transfer matrices in $U_q(\hat{\mathfrak{g}})$ vertex models

$= \mathrm{Tr}_{W_m^{(a)}(u)}\Big($  $\Big)$

$W_m^{(a)}(u)$: Kirillov-Reshetikhin (KR) module of $U_q(\hat{\mathfrak{g}})$ (Beloved One)

$\simeq V(\lambda)$ with $\lambda =$  for $\mathfrak{g} = sl_n$

$\simeq \bigoplus_\lambda$(Fermionic form of 1-site chain)$_\lambda V(\lambda)$ for general $\mathfrak{g}$

ADE

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + \prod_{b\sim a} T^{(b)}_m(u).$$

B_n

$$\begin{aligned}
T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad (1\le a\le n-2),\\
T^{(n-1)}_m(u-1)T^{(n-1)}_m(u+1) &= T^{(n-1)}_{m-1}(u)T^{(n-1)}_{m+1}(u) + T^{(n-2)}_m(u)T^{(n)}_{2m}(u),\\
T^{(n)}_{2m}(u-\tfrac{1}{2})T^{(n)}_{2m}(u+\tfrac{1}{2}) &= T^{(n)}_{2m-1}(u)T^{(n)}_{2m+1}(u) + T^{(n-1)}_m(u-\tfrac{1}{2})T^{(n-1)}_m(u+\tfrac{1}{2}),\\
T^{(n)}_{2m+1}(u-\tfrac{1}{2})T^{(n)}_{2m+1}(u+\tfrac{1}{2}) &= T^{(n)}_{2m}(u)T^{(n)}_{2m+2}(u) + T^{(n-1)}_m(u)T^{(n-1)}_{m+1}(u).
\end{aligned}$$

C_n

$$\begin{aligned}
T^{(a)}_m(u-\tfrac{1}{2})T^{(a)}_m(u+\tfrac{1}{2}) &= T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad (1\le a\le n-2),\\
T^{(n-1)}_{2m}(u-\tfrac{1}{2})T^{(n-1)}_{2m}(u+\tfrac{1}{2}) &= T^{(n-1)}_{2m-1}(u)T^{(n-1)}_{2m+1}(u) + T^{(n-2)}_{2m}(u)T^{(n)}_m(u-\tfrac{1}{2})T^{(n)}_m(u+\tfrac{1}{2}),\\
T^{(n-1)}_{2m+1}(u-\tfrac{1}{2})T^{(n-1)}_{2m+1}(u+\tfrac{1}{2}) &= T^{(n-1)}_{2m}(u)T^{(n-1)}_{2m+2}(u) + T^{(n-2)}_{2m+1}(u)T^{(n)}_m(u)T^{(n)}_{m+1}(u),\\
T^{(n)}_m(u-1)T^{(n)}_m(u+1) &= T^{(n)}_{m-1}(u)T^{(n)}_{m+1}(u) + T^{(n-1)}_{2m}(u).
\end{aligned}$$

F_4

$$\begin{aligned}
T^{(1)}_m(u-1)T^{(1)}_m(u+1) &= T^{(1)}_{m-1}(u)T^{(1)}_{m+1}(u) + T^{(2)}_m(u),\\
T^{(2)}_m(u-1)T^{(2)}_m(u+1) &= T^{(2)}_{m-1}(u)T^{(2)}_{m+1}(u) + T^{(1)}_m(u)T^{(3)}_{2m}(u),\\
T^{(3)}_{2m}(u-\tfrac{1}{2})T^{(3)}_{2m}(u+\tfrac{1}{2}) &= T^{(3)}_{2m-1}(u)T^{(3)}_{2m+1}(u) + T^{(2)}_m(u-\tfrac{1}{2})T^{(2)}_m(u+\tfrac{1}{2})T^{(4)}_{2m}(u),\\
T^{(3)}_{2m+1}(u-\tfrac{1}{2})T^{(3)}_{2m+1}(u+\tfrac{1}{2}) &= T^{(3)}_{2m}(u)T^{(3)}_{2m+2}(u) + T^{(2)}_m(u)T^{(2)}_{m+1}(u)T^{(4)}_{2m+1}(u),\\
T^{(4)}_m(u-\tfrac{1}{2})T^{(4)}_m(u+\tfrac{1}{2}) &= T^{(4)}_{m-1}(u)T^{(4)}_{m+1}(u) + T^{(3)}_m(u).
\end{aligned}$$

G_2

$$\begin{aligned}
T^{(1)}_m(u-1)T^{(1)}_m(u+1) &= T^{(1)}_{m-1}(u)T^{(1)}_{m+1}(u) + T^{(2)}_{3m}(u),\\
T^{(2)}_{3m}(u-\tfrac{1}{3})T^{(2)}_{3m}(u+\tfrac{1}{3}) &= T^{(2)}_{3m-1}(u)T^{(2)}_{3m+1}(u) + T^{(1)}_m(u-\tfrac{2}{3})T^{(1)}_m(u)T^{(1)}_m(u+\tfrac{2}{3}),\\
T^{(2)}_{3m+1}(u-\tfrac{1}{3})T^{(2)}_{3m+1}(u+\tfrac{1}{3}) &= T^{(2)}_{3m}(u)T^{(2)}_{3m+2}(u) + T^{(1)}_m(u-\tfrac{1}{3})T^{(1)}_m(u+\tfrac{1}{3})T^{(1)}_{m+1}(u),\\
T^{(2)}_{3m+2}(u-\tfrac{1}{3})T^{(2)}_{3m+2}(u+\tfrac{1}{3}) &= T^{(2)}_{3m+1}(u)T^{(2)}_{3m+3}(u) + T^{(1)}_m(u)T^{(1)}_{m+1}(u-\tfrac{1}{3})T^{(1)}_{m+1}(u+\tfrac{1}{3}).
\end{aligned}$$

$\boxed{ADE}$ $$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1)=\frac{\prod_{b\sim a}(1+Y_m^{(b)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}.$$

$\boxed{C_n}$ $$Y_m^{(a)}(u-\tfrac{1}{2})Y_m^{(a)}(u+\tfrac{1}{2})=\frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}\quad(1\le a\le n-2),$$

$$Y_{2m}^{(n-1)}(u-\tfrac{1}{2})Y_{2m}^{(n-1)}(u+\tfrac{1}{2})=\frac{(1+Y_{2m}^{(n-2)}(u))(1+Y_m^{(n)}(u))}{(1+Y_{2m-1}^{(n-1)}(u)^{-1})(1+Y_{2m+1}^{(n-1)}(u)^{-1})},$$

$$Y_{2m+1}^{(n-1)}(u-\tfrac{1}{2})Y_{2m+1}^{(n-1)}(u+\tfrac{1}{2})=\frac{1+Y_{2m+1}^{(n-2)}(u)}{(1+Y_{2m}^{(n)}(u)^{-1})(1+Y_{2m+2}^{(n)}(u)^{-1})},$$

$$Y_m^{(n)}(u-1)Y_m^{(n)}(u+1)=\frac{(1+Y_{2m}^{(n-1)}(u+\tfrac{1}{2}))(1+Y_{2m}^{(n-1)}(u-\tfrac{1}{2}))(1+Y_{2m-1}^{(n-1)}(u))(1+Y_{2m+1}^{(n-1)}(u))}{(1+Y_{m-1}^{(n)}(u)^{-1})(1+Y_{m+1}^{(n)}(u)^{-1})}.$$

$\boxed{F_4}$ $$Y_m^{(1)}(u-1)Y_m^{(1)}(u+1)=\frac{1+Y_m^{(2)}(u)}{(1+Y_{m-1}^{(1)}(u)^{-1})(1+Y_{m+1}^{(1)}(u)^{-1})},$$

$$Y_m^{(2)}(u-1)Y_m^{(2)}(u+1)=\frac{(1+Y_m^{(1)}(u))(1+Y_{2m}^{(3)}(u-\tfrac{1}{2}))(1+Y_{2m}^{(3)}(u+\tfrac{1}{2}))(1+Y_{2m-1}^{(3)}(u))(1+Y_{2m+1}^{(3)}(u))}{(1+Y_{m-1}^{(2)}(u)^{-1})(1+Y_{m+1}^{(2)}(u)^{-1})},$$

$$Y_{2m}^{(3)}(u-\tfrac{1}{2})Y_{2m}^{(3)}(u+\tfrac{1}{2})=\frac{(1+Y_m^{(2)}(u))(1+Y_{2m}^{(4)}(u))}{(1+Y_{2m-1}^{(3)}(u)^{-1})(1+Y_{2m+1}^{(3)}(u)^{-1})},$$

$$Y_{2m+1}^{(3)}(u-\tfrac{1}{2})Y_{2m+1}^{(3)}(u+\tfrac{1}{2})=\frac{1+Y_{2m+1}^{(4)}(u)}{(1+Y_{2m}^{(3)}(u)^{-1})(1+Y_{2m+2}^{(3)}(u)^{-1})},$$

$$Y_m^{(4)}(u-\tfrac{1}{2})Y_m^{(4)}(u+\tfrac{1}{2})=\frac{1+Y_m^{(3)}(u)}{(1+Y_{m-1}^{(4)}(u)^{-1})(1+Y_{m+1}^{(4)}(u)^{-1})}.$$

(B_n, G_2 : omitted due to space shortage)

T-system: proposed for transfer matrices [K-Nakanishi-Suzuki 1994]

Proved for q-character of KR module [Nakajima, ADE 2003], [Hernandez, $\forall\mathfrak{g}$ 2006]

Y-system is an algebraic form of thermodynamic Bethe ansatz equation of type $\mathfrak{g}$ under string hypothesis.

$Y^{(a)}_m(u) \sim$ Boltzmann factor Exp(dressed energy) of string/hole excitation with color a, length m, rapidity u.

sl_2 example:

$$\log Y_m(u) = \text{known fcn.} + \int_{-\infty}^{\infty} \frac{\log(1+Y_{m-1}(v))(1+Y_{m+1}(v))}{4\cosh\frac{\pi(u-v)}{2}} dv$$

$$\rightsquigarrow Y_m(u-i)Y_m(u+i) = (1+Y_{m-1}(u))(1+Y_{m+1}(u)).$$

Y-system was proposed by

ADE: Al. Zamolodchikov (1991), Ravanini-Tateo-Valleriani (1993).

$\forall\mathfrak{g}$: K-Nakanishi (1992).

Level restricted T-system and Y-system

Introduce $\ell \in \mathbb{Z}_{\geq 2}$ called level.

Level ℓ restricted T and Y-system are those closing among

$$\{T_m^{(a)}(u)\} \text{ and } \{Y_m^{(a)}(u)\} \text{ with } 1 \leq m \leq t_a\ell - 1,$$

obtained by imposing $T_{t_a\ell}^{(a)}(u) = 1$ and $Y_{t_a\ell}^{(a)}(u)^{-1} = 0$.

Example: $(\mathfrak{g}, \ell) = (C_2,\ 2)$.

$$\begin{aligned}
T_1^{(1)}(u-\tfrac{1}{2})T_1^{(1)}(u+\tfrac{1}{2}) &= T_2^{(1)}(u) + T_1^{(2)}(u),\\
T_2^{(1)}(u-\tfrac{1}{2})T_2^{(1)}(u+\tfrac{1}{2}) &= T_1^{(1)}(u)T_3^{(1)}(u) + T_1^{(1)}(u-\tfrac{1}{2})T_1^{(1)}(u+\tfrac{1}{2}),\\
T_3^{(1)}(u-\tfrac{1}{2})T_3^{(1)}(u+\tfrac{1}{2}) &= T_2^{(1)}(u) + T_1^{(2)}(u),\\
T_1^{(2)}(u-1)T_1^{(2)}(u+1) &= 1 + T_2^{(1)}(u).
\end{aligned}$$

This closes among $T_1^{(1)}(u)$, $T_2^{(1)}(u)$, $T_3^{(1)}(u)$, $T_1^{(2)}(u)$.

Restricted T-system $\simeq$ algebra of transfer matrices for level ℓ RSOS models

Restricted Y-system $\simeq$ functional equation of TBA dressed energy

They are evolution equations of finitely many functions in the u direction.

Periodicity conjecture

Level ℓ restricted T-system and Y-system obey

$$T_m^{(a)}(u+2(h^\vee+\ell)) = T_m^{(a)}(u), \qquad Y_m^{(a)}(u+2(h^\vee+\ell)) = Y_m^{(a)}(u).$$

($h^\vee$ = dual Coxeter number of $\mathfrak{g}$)

Example: $(A_2, 2)$. Writing $T^{(a)}(u) = T_1^{(a)}(u)$,

$$T^{(1)}(u-1)T^{(1)}(u+1) = 1 + T^{(2)}(u), \quad T^{(2)}(u-1)T^{(2)}(u+1) = 1 + T^{(1)}(u).$$

Periodicity holds due to $T^{(1)}(u+5) = T^{(2)}(u), \quad T^{(2)}(u+5) = T^{(1)}(u)$.

$$(T^{(1)}(0), T^{(2)}(1)) = (a, b),$$

$$(T^{(1)}(2), T^{(2)}(3)) = \left(\frac{1+T^{(2)}(1)}{T^{(1)}(0)}, \frac{1+T^{(1)}(2)}{T^{(2)}(1)}\right) = \left(\frac{1+b}{a}, \frac{1+\frac{1+b}{a}}{b}\right) = \left(\frac{1+b}{a}, \frac{1+a+b}{ab}\right),$$

$$(T^{(1)}(4), T^{(2)}(5)) = \left(\frac{1+T^{(2)}(3)}{T^{(1)}(2)}, \frac{1+T^{(1)}(4)}{T^{(2)}(3)}\right) = \left(\frac{1+\frac{1+a+b}{ab}}{\frac{1+b}{a}}, \frac{1+\frac{1+a}{b}}{\frac{1+a+b}{ab}}\right) = \left(\frac{1+a}{b}, a\right)$$

$$T^{(1)}(6) = \frac{1+T^{(2)}(5)}{T^{(1)}(4)} = \frac{1+a}{\frac{1+a}{b}} = b = T^{(2)}(1)$$

$$0\{10,30,50,70\} \qquad (E_8,2):\quad \{T_1^{(1)}(u),T_1^{(3)}(u),T_1^{(5)}(u),T_1^{(7)}(u)\}_{u=0}^{32}$$

$$2\left\{\frac{11}{5},\frac{431}{15},\frac{101291}{25},\frac{31}{35}\right\}$$

$$4\left\{\frac{83}{45},\frac{69696833}{230625},\frac{45718438593497}{22157296875},\frac{103041}{1525}\right\}$$

$$6\left\{\frac{102041}{1025},\frac{8821291833971}{66471890625},\frac{3603424631077797294639}{3463462467773 4375},\frac{14562107}{415125}\right\}$$

$$8\left\{\frac{1061807}{1441125},\frac{5276210022879156 53}{153931665234375},\frac{144652414821069001465529527}{6161870815433349609375},\frac{2176297573}{492384375}\right\}$$

$$10\left\{\frac{15241182}{312625},\frac{1741858802351675418 4}{133590695185546875},\frac{65852952390687824418240896525206}{1926354863674850921630859375},\frac{32206227374}{211021875}\right\}$$

$$12\left\{\frac{23381761}{6226875},\frac{443940578926110770904 1}{912869750434570312 5},\frac{2553966816510832754529086992801664 48}{881307350131244296646118164062 5},\frac{6587423634821}{129778453125}\right\}$$

$$14\left\{\frac{289412993}{98476875},\frac{240117200327845738829501 9}{2875539713868896484375},\frac{1134215951212517251168445050240215777 13}{542004020330715242437362670898437 5},\frac{8472179120234}{2252658515625}\right\}$$

$$16\left\{\frac{391949128}{4689375},\frac{7397263161797774132227049}{584693075153342285156 25},\frac{1290705517162033306270461619591091257193}{569104221347251004559230804443359375},\frac{14335608965944}{129778453125}\right\}$$

$$18\left\{\frac{66998956}{126613125},\frac{2107149795677823486002 41}{928084246275146484375},\frac{1724707384403205750584318844948339136 63}{113820844269450200911846160888671875},\frac{74693044181731}{13626737578125}\right\}$$

$$20\left\{\frac{11232037}{1563125},\frac{15762599424019577436 47}{246474832617333984375},\frac{2104768617341673326572332456823959011}{25293520948766711313743591308593 75},\frac{1211696207719}{450531703125}\right\}$$

$$22\left\{\frac{3077201}{1245375},\frac{1154015828669881829 27}{426005883536132812 5},\frac{23354104411061828987973549647671}{2467660580367484030609130859375},\frac{175786811543}{2883965625}\right\}$$

$$24\left\{\frac{4476646}{2188375},\frac{471838863103501 93}{124684648839843 75},\frac{359394552467269919534330 03}{17123154343776452636718 75},\frac{991662341}{13294378125}\right\}$$

$$26\left\{\frac{7058}{4575},\frac{1842216632119}{879609515625},\frac{1312893318319321061 59}{1150215885547558593 75},\frac{7222892}{312625}\right\}$$

$$28\left\{\frac{1181}{615},\frac{61893029}{42204375},\frac{156275914764469}{18471799828125},\frac{46522}{27675}\right\}$$

$$30\left\{\frac{23}{15},\frac{32333}{1845},\frac{4966808}{187575},\frac{1781}{2135}\right\}$$

$$32\{10,30,50,70\}$$

Periodicity of Y-system for $(\mathfrak{g}, \ell)$ was proposed:

(ADE, 2): Al. Zamolodchikov (1991),

(ADE, ℓ): Ravanini-Tateo-Valleriani (1993),

$(\mathfrak{g}, \ell)$: K-Nakanishi-Suzuki (1994).

Many cases proved in

$(A_n, 2)$: Frenkel-Szenes (1995), Gliozzi-Tateo (1996),

(A_n, ℓ): Volkov, Henriques (2007),

(ADE, 2): Fomin-Zelevinsky (2003) Cluster algebra,

(ADE, ℓ): Keller (arXiv:0807.1960) Cluster algebra/category.

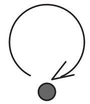
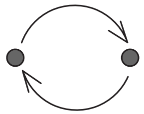
Periodicity of T-system:

Proposed and partially proved in Inoue-Iyama-K-Nakanishi-Suzuki (2008),

(A_n, ℓ) case: proof also contained in Henriques (2007).

Theorem. [Inoue-Iyama-Keller-K-Nakanishi 2013]
Periodicity conjecture of T and Y-systems is true for $\forall(\mathfrak{g}, \ell)$.

Quivers and Cluster algebra formulation

Q: quiver (finite oriented graph without loop  and 2-cycle )

$I = \{1, \ldots, N\}$: vertex set, $x = (x_1, \ldots, x_N)$: I-tuple of variables

x_i: cluster variable, (Q, x): seed.

Cluster algebra $\mathcal{A}_Q$ is defined by (i)–(iv). [Fomin-Zelevinsky 2002]

(i) Start from the initial seed (Q, x) as above.

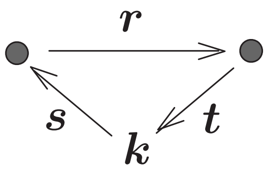
(ii) For each $k \in I$, define another seed (R, y)
by $(R, y) = \mu_k(Q, x)$ (mutation at k, defined on the next page).

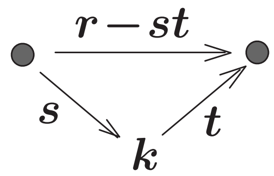
(iii) Iterate mutations for every new seed at every k,
and collect all (possibly infinite) seeds.

(iv) $\mathcal{A}_Q = \mathbb{Z}$-subalgebra of $\mathbb{Q}(x_1, \ldots, x_N)$ generated by $\forall$cluster variables.

Mutation at k : $\mu_k(Q,x) = (R,y)$

- A new quiver R is obtained from Q by reversing ∀arrows incident with k and

$Q =$ 

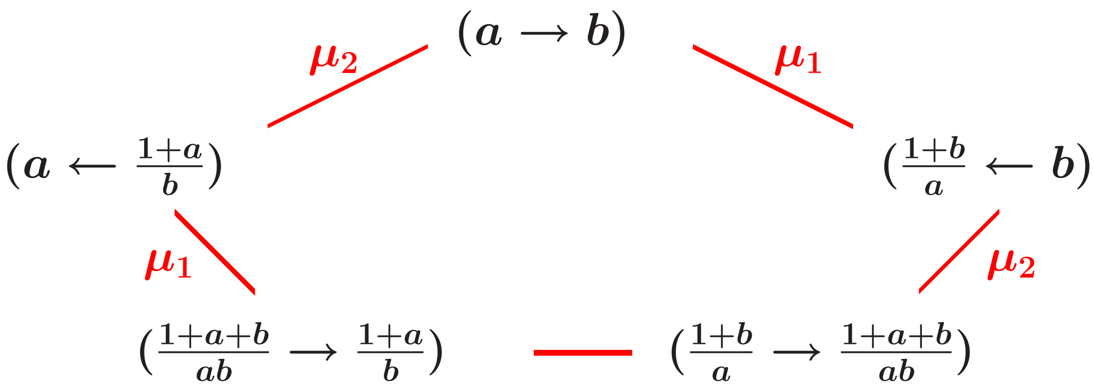
$R =$ 

$(s,t \geq 0)$

- New cluster variables $y = (y_1, \ldots, y_n)$ are given by

$$y_i = x_i \ (i \neq k), \qquad y_k = \frac{1}{x_k}\left(\prod_{\text{arrows } j\to k \text{ of } Q} x_j + \prod_{\text{arrows } k\to j \text{ of } Q} x_j\right)$$

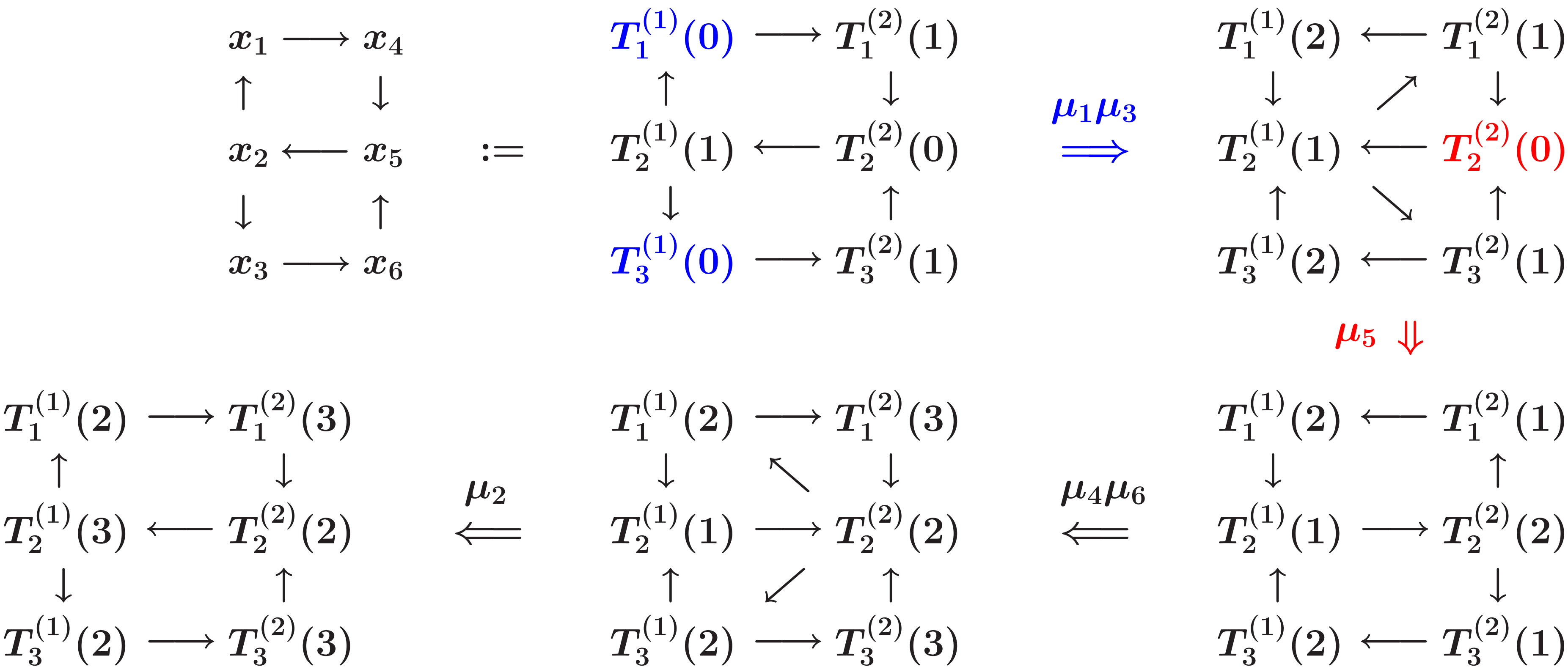
Example. $I = \{1,2\}$. Seed $(Q,x) = (1 \to 2, \{a,b\})$ denoted by $(a \to b)$.



“Exchange graph”

Cluster algebra formulation of T-system. $(A_2, 4)$ example

$$T_1^{(1)}(u-1)T_1^{(1)}(u+1) = T_2^{(1)}(u) + T_1^{(2)}(u),$$
$$T_2^{(2)}(u-1)T_2^{(2)}(u+1) = T_1^{(2)}(u)T_3^{(2)}(u) + T_2^{(1)}(u),$$
$$T_3^{(1)}(u-1)T_3^{(1)}(u+1) = T_2^{(1)}(u) + T_3^{(2)}(u).$$



$$(Q, x(u+2)) = \mu_2\mu_4\mu_6\mu_5\mu_3\mu_1(Q, x(u)) \quad \text{for} \quad x = \{T_m^{(a)}\}.$$

Similarly, T-system for any $(\mathfrak{g}, \ell)$ is formulated as

$(Q, x(u+2)) = \mu(Q, x(u))$ by an appropriate choice of

$$\begin{cases} Q & : \text{quiver}, \\ x(u) = \{x_i(u)\} & : \text{cluster variables suitably identified with } T_m^{(a)}(u)\text{'s}, \\ \mu = \mu_{i_1} \cdots \mu_{i_s} & : \text{composite mutation.} \end{cases}$$

Periodicity for any $(\mathfrak{g}, \ell)$ is formulated as

$$(Q, x(u)) = \mu^{h^\vee + \ell}(Q, x(u)).$$

Y-system can be built in cluster algebra with coefficients.

Periodicity of T and Y-systems follows from periodicity in (generalized) cluster category.

Dilogarithm identity

$$L(x) = -\frac{1}{2}\int_0^x \left(\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right), \qquad L(x) + L(1-x) = L(1) = \frac{\pi^2}{6}.$$

Let $Y_m^{(a)}(u) = Y_m^{(a)} > 0$ be the unique positive solution to the level ℓ restricted constant (u-independent) Y-system.

Conjecture [A.N. Kirillov 1989]

$$\frac{6}{\pi^2}\sum_{a=1}^{\mathrm{rank}\,\mathfrak{g}}\sum_{m=1}^{t_a\ell-1} L\left(\frac{Y_m^{(a)}}{1+Y_m^{(a)}}\right) = \frac{\ell\dim\mathfrak{g}}{\ell+h^\vee} - \mathrm{rank}\,\mathfrak{g} \qquad (\ell \geq 1).$$

Example: $\mathfrak{g} = sl_2$.

$$\frac{6}{\pi^2}\sum_{m=1}^{\ell-1} L\left(\frac{\sin^2\frac{\pi}{\ell+2}}{\sin^2\frac{\pi(m+1)}{\ell+2}}\right) = \frac{3\ell}{\ell+2} - 1.$$

- Utilized in so many calculations of central charges in

  Integrable perturbations of CFT,

  Low temperature specific heat & finite size corrections of spin chains,
  Asymptotics of affine Lie algebra characters.

- Proved in several cases individually by

Lewin, Watson, Kirillov, Richmond, Szekeres, Gliozzi, Tateo, Chapoton,...

> **Theorem.** [Inoue-Iyama-Keller-K-Nakanishi 2013]
>
> Dilogarithm conjecture is true for $\forall(\mathfrak{g}, \ell)$.

Essence of proof : Reduction to a couting problem on tropical Y-system
by periodicity and cluster algebra

- Easier to prove more general functional dilogarithm identity

$$\frac{6}{\pi^2} \sum_{u \text{ mod period}} \sum_{a=1}^{\mathrm{rank}\mathfrak{g}} \sum_{m=1}^{t_a \ell - 1} L\left(\frac{Y_m^{(a)}(u)}{1+Y_m^{(a)}(u)}\right) = \ell \dim \mathfrak{g} - (\ell + h^\vee)\mathrm{rank}\mathfrak{g}.$$

- LHS is invariant by varying $Y_m^{(a)}(u) > 0$ as long as Y-system is satisfied.

cf. [J. Phys. A: special issue 2014] "Cluster algebras in mathematical physics"