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Thermodynamics of the $U_q(X_r^{(1)})$ Bethe ansatz system with q a root of unity

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The Bethe ansatz equations connected to the $U_q(X_r^{(1)})$ algebra are studied at $q = \exp(2\pi i/(l+g))$, with l being a level and g the dual Coxeter number. Based on a "string hypothesis" in the thermodynamic limit, the central charges relevant to the $X_r^{(1)}$ restricted solid-on-solid models and their fusion hierarchies are determined in two critical regimes. The calculation leads naturally to a generalized conjecture on the Rogers dilogarithmic function identity for an arbitrary pair $(X_r^{(1)}, l)$, which includes the earlier one for the simply-laced cases $X_r^{(1)} = A_r^{(1)}, D_r^{(1)}$ and $E_{6,7,8}^{(1)}$.

1. Introduction

Through the last few decades the Bethe ansatz has been recognized as a fairly universal structure in integrable systems in quantum field theory and statistical mechanics. A variety of models are known to share common Bethe equations and have been analyzed in a unified perspective. Among others, examples of particular interest are those models connected with the affine Lie algebra $X_r^{(1)}$ and its q-deformation $U_q(X_r^{(1)})$. In ref. [1] the Bethe equations for such systems are formulated in terms of the root system of the associated algebra.

The purpose of this paper is to study the $U_q(X_r^{(1)})$ Bethe equations at $q = \exp(2\pi i/(l+g))$, where l is a positive integer and g denotes the dual Coxeter number. The results are relevant to the critical level-l $X_r^{(1)}$ restricted solid-on-solid (RSOS) models and their fusion hierarchies. More specifically, we shall do a Yang and Yang type thermodynamic calculation [2] and extract the central charges [3] in two critical regimes from low-temperature asymptotics of the specific heat capacity

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for an infinite system. As an evaluation of the central charges, this is yet another but in principle equivalent approach to calculating the finite-size corrections to the ground-state energies [4]. The former route, however, has been taken in relatively fewer works compared with the finite-size correction approach for which one finds an immense literature (see for example ref. [5]).

In refs. [6,7] Bazhanov and Reshetikhin executed such a program for the simply-laced algebras $A_r^{(1)}$, $D_r^{(1)}$ and $E_{6,7,8}^{(1)}$. Their results were relevant to the fusion $A_r^{(1)}$ RSOS models built in ref. [8] and went even further beyond the existing list of solvable RSOS models. The present paper generalizes their result into arbitrary $X_r^{(1)}$. Besides the $A_r^{(1)}$ RSOS, the models themselves to which the results apply are yet to be constructed (remark 2.5). Some models of such sort are available in the "non-fusion" case for $B_r^{(1)}$, $C_r^{(1)}$, $D_r^{(1)}$ [9] and $G_2^{(1)}$ [10]. (See also ref. [11] for $A_{2r}^{(2)}$ and $A_{2r-1}^{(2)}$.)

A fascinating feature of Bazhanov and Reshetikhin's calculation [6,7] is that the central charge is naturally expressed by the Rogers dilogarithmic function $L(x)$ through combinations of the following quantity:

$$\frac{6}{\pi^2}\sum_{a=1}^{r}\sum_{m=1}^{l-1} L\left(f_m^{(a)}\right), \tag{1.1a}$$

$$L(x) = -\frac{1}{2}\int_0^x\left(\frac{\log(1-y)}{y}+\frac{\log y}{1-y}\right)\mathrm{d}y, \quad 0\leqslant x\leqslant 1, \tag{1.1b}$$

where the entry $0\leqslant f_m^{(a)}\leqslant 1$ $(1\leqslant a\leqslant r,\ 1\leqslant m\leqslant l-1)$ is determined purely from the Lie algebraic data. The authors of ref. [7] have used the conjecture [12] that the quantity (1.1a) is equal to

$$\frac{l\dim X_r}{l+g}-r \quad \text{for } X_r^{(1)} = A_r^{(1)},\ D_r^{(1)} \text{ and } E_{6,7,8}^{(1)},$$

which is the well-known Wess–Zumino–Witten (WZW) value (−rank) [13]. In fact, the proof had been essentially given for $X_r^{(1)} = A_1^{(1)}$ in refs. [14,15] and for $X_r^{(1)} = A_r^{(1)}$ by Kirillov [12]. Here we generalize the conjecture to arbitrary $X_r^{(1)}$ as follows:

$$\frac{6}{\pi^2}\sum_{a=1}^{r}\sum_{m=1}^{t_a l-1} L\left(f_m^{(a)}\right) = \frac{l\dim X_r}{l+g}-r \quad \forall l\geqslant 1, \tag{1.2}$$

where the vital factor t_a $(1\leqslant a\leqslant r)$ is the integer defined by $t_a = 2/|\alpha_a|^2$ with α_a being the ath simple root in the normalization $|\text{long root}|^2 = 2$. Again the entry $f_m^{(a)}$ is specified purely from the pair $(X_r^{(1)}, l)$ (see (A.1c)). The simply-laced case (1.1a) corresponds to the situation $t_a = 1$ for all $1\leqslant a\leqslant r$. The conjectural formula

(1.2), which is remarkable in its own right, is the key to our calculation of the central charges by the Bethe ansatz thermodynamics. It has been supported by numerical experiments for any $(X_r^{(1)}, l)$ with small levels l and ranks r. (See the remark after (A.1d) for the case $l = 1$.)

The outline of the paper is as follows. In sect. 2, we formulate the $U_q(X_r^{(1)})$ Bethe equations in terms of the root system following ref. [1] and thereby fix the notations. Thermodynamic quantities are evaluated in terms of "string" and "hole" densities. As a special feature for the level-l RSOS models at criticality, we employ an hypothesis (2.7) that for "color" a, only those strings with length $\leqslant t_a l$ contribute to the thermodynamic quantities. The central charge c is expressed through the Rogers dilogarithmic function with arguments $g_m^{(a)}(\pm\infty)$. We also include an interesting result on high-temperature asymptotics of the entropy. In sect. 3, the quantities $g_m^{(a)}(\pm\infty)$ are identified with those $f_m^{(a)}$'s attached to various subalgebras of $X_r^{(1)}$ at various levels depending on the regime. Then the resulting values of c are compared with several known cases with perfect agreement. The comparison is mainly based on the one-point function results of the RSOS models in refs. [16–19]. Sect. 4 contains a summary and discussion. In particular, we indicate further possible generalizations and present the extended results on the central charges for such situations. Appendix A gives a formulation of the dilogarithmic function identity (yet to be proved), each defined for the pair $(X_r^{(1)}, l)$ where $l \in \mathbb{Z}_{>0}$ corresponds to the level. The description of the important quantity $f_m^{(a)}$ is partly based on Kirillov and Reshetikhin's theory [20] on finite-dimensional representations of yangians. We include a few results and conjectures on the explicit form of the $f_m^{(a)}$ for exceptional Lie algebras. The conjectured generalization of the dilogarithm identity in ref. [12] appears to be invalid for the non-simply-laced algebras due to the lack of the factor t_a. (See note added.)

2. Thermodynamics of the $U_q(X_r^{(1)})$ Bethe ansatz system

We shall study the Bethe equations relevant to the level-l $X_r^{(1)}$ RSOS models at criticality. Besides the algebra $X_r^{(1)}$ and the level l (and the system size N tending to infinity), the equations contain two more integer parameters s and p subject to (2.4). They are supposed to specify the type of fusion $s\omega_p$ of the corresponding RSOS model in the sense explained in remark 2.5.

2.1. THE $U_q(X_r^{(1)})$ BETHE EQUATIONS

Let $X_r^{(1)}$ be one of the rank-r non-twisted affine Lie algebras [21] $A_r^{(1)}$ $(r \geqslant 1)$, $B_r^{(1)}$ $(r \geqslant 2)$, $C_r^{(1)}$ $(r \geqslant 1)$, $D_r^{(1)}$ $(r \geqslant 3)$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$, and $U_q(X_r^{(1)})$ be its q-deformation of the universal enveloping algebra [22]. We let α_a, ω_a $(1 \leqslant a \leqslant r)$ denote the simple roots and the fundamental weights of the classical part X_r, respectively. Introduce the bilinear form $(\,|\,)$ on the dual space of the Cartan subalgebra of X_r as follows:

$$\frac{2(\alpha_a|\omega_b)}{(\alpha_a|\alpha_a)}=\delta_{ab},\qquad C_{ab}=\frac{2(\alpha_a|\alpha_b)}{(\alpha_a|\alpha_a)}=(\text{Cartan matrix})_{ab}\quad\text{of } X_r. \tag{2.1}$$

We normalize the roots so that $|\text{long root}|^2=2$ and set

$$t_a=\frac{2}{(\alpha_a|\alpha_a)},\quad 1\leqslant a\leqslant r. \tag{2.2}$$

By definition $t_a=1$ if α_a is a long root and otherwise $t_a=2$ for B_r, C_r, F_4 and $t_a=3$ for G_2. One may equivalently define t_i as $t_i=a_i/a_i^\vee$, where a_i and $a_i^\vee$'s are the so-called Kac and dual Kac labels, respectively (cf. ref. [21]). For later convenience we also introduce the notation

$$B_{ab}=\frac{t_b}{t_{ab}}C_{ab},\qquad t_{ab}=\max(t_a,t_b),\quad 1\leqslant a,b\leqslant r. \tag{2.3}$$

It then follows that $B_{ab}=B_{ba}$ is the symmetrized Cartan matrix. In table 1, we list the Dynkin diagram with the numeration of its nodes, the dimension and the dual Coxeter number g for each X_r along with those $t_a\neq 1$.

Fix the integers l, p and s satisfying

$$l\geqslant 1,\quad 1\leqslant p\leqslant 1\quad\text{and}\quad 1\leqslant s\leqslant t_p l-1. \tag{2.4}$$

Choose further any positive integer N such that

$$N_a=Ns\frac{t_a}{t_p}(C^{-1})_{pa}=Ns(C^{-1})_{ap}\in\mathbb{Z}\quad\text{for all } 1\leqslant a\leqslant r. \tag{2.5}$$

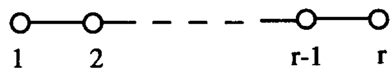
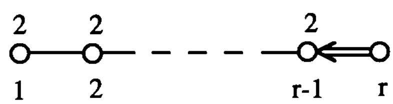
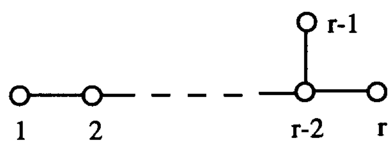
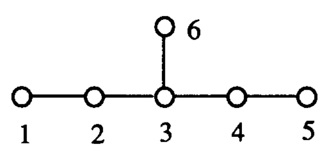
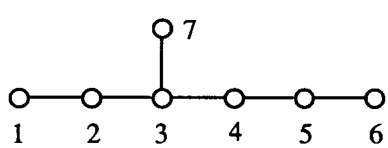
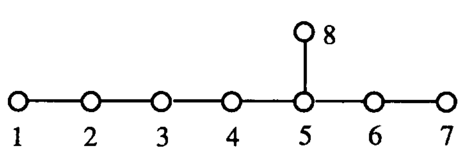
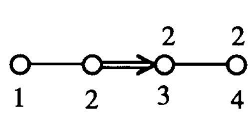
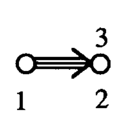
Note then that the N_a are in fact non-negative. Now we are going to write down the $U_q(X_r^{(1)})$ Bethe equations at $q=\exp[2\pi i/(l+g)]$, which is, in the sense explained below, related to "level-l $X_r^{(1)}$ RSOS model with type $s\omega_p$ fusion". They are the following system of simultaneous equations for the complex variables $\{u_j^{(a)}\}_{j=1}^{N_a}$, $1\leqslant a\leqslant r$:

$$\left(\frac{\mathrm{sh}\left(\frac{\pi}{2L}\left(u_j^{(a)}+i(s\omega_p|\alpha_a)\right)\right)}{\mathrm{sh}\left(\frac{\pi}{2L}\left(u_j^{(a)}-i(s\omega_p|\alpha_a)\right)\right)}\right)^N=\Omega_j^{(a)}\prod_{b=1}^{r}\prod_{k=1}^{N_b}\frac{\mathrm{sh}\left(\frac{\pi}{2L}\left(u_j^{(a)}-u_k^{(b)}+i(\alpha_a|\alpha_b)\right)\right)}{\mathrm{sh}\left(\frac{\pi}{2L}\left(u_j^{(a)}-u_k^{(b)}-i(\alpha_a|\alpha_b)\right)\right)}$$

$$\text{for}\quad 1\leqslant j\leqslant N_a,\quad 1\leqslant a\leqslant r, \tag{2.6}$$

where $L=l+g$ and $\Omega_j^{(a)}$ is some phase factor. Several remarks are in order.

TABLE 1

X_r	Dynkin Diagrams	dim	g
A_r		r^2+2r	$r+1$
B_r		$2r^2+r$	$2r-1$
C_r		$2r^2+r$	$r+1$
D_r		$2r^2-r$	$2r-2$
E_6		78	12
E_7		133	18
E_8		248	30
F_4		52	9
G_2		14	4

The classical simple Lie algebra X_r, the Dynkin diagram, the dimension of X_r and the dual Coxeter number g. In each Dynkin diagram, the nodes are numerated from 1 to r. The parameter t_a (2.2) has been given above the node a only when $t_a \neq 1$.

Remark 2.1. The Bethe equation (2.6) with all $\Omega_j^{(a)} = 1$ corresponds to a special case of ref. [1], where $\mathrm{U}_q(\mathrm{X}_r^{(k)})$ $(k = 1, 2, 3)$ are considered with more general choices of N_a under the presence of inhomogeneity for the spectral parameter as well as the representation parameter $s\omega_p$. Their equations, in which L is a generic parameter, are relevant to the spectrum of row-to-row transfer matrices for $\mathrm{U}_q(\mathrm{X}_r^{(1)})$ vertex models.

Remark 2.2. It was shown for the $\mathrm{A}_r^{(1)}$ case [6,7] that the Bethe equations for the RSOS models acquire some non-trivial phase factor $\Omega_j^{(a)}$ as above. Although its explicit form is not known in general, we expect, as in the $\mathrm{D}_r^{(1)}$ and $\mathrm{E}_{6,7,8}^{(1)}$ cases [7], that its effect would be properly taken into account in the thermodynamic limit by the forthcoming hypothesis (2.7) on the solution of (2.6).

Remark 2.3. The integer N stands for the length of a row of a square lattice, therefore the thermodynamic limit is achieved when $N \to \infty$. The quantities $u_j^{(a)}$ and N_a are called "pseudo momenta" and the "completion numbers of the color a", respectively.

Remark 2.4. The completion numbers (2.5) have been chosen so that $\omega_{\mathrm{tot}} \overset{\mathrm{def}}{=} Ns\omega_p - \sum_{a=1}^{r} N_a \alpha_a = 0$, which is relevant to the ground states of our RSOS models. From the viewpoint of spin chains and vertex models, the ω_{tot} corresponds to the "total magnetization" and the choice (2.5) implies the "completely antiferromagnetic sector". See e.g. ref. [23] for an argument concerning the Heisenberg chain.

Remark 2.5. By "level-l $\mathrm{X}_r^{(1)}$ RSOS model with type $s\omega_p$ fusion" we mean a "model" for which we can in general only provide the following very obscure and conjectural description.

Suppose q is generic and let V_ω denote the finite-dimensional irreducible $\mathrm{U}_q(\mathrm{X}_r)$ module with highest weight $\omega \in \oplus_{a=1}^{r} \mathbb{Z}_{\geq 0}\omega_a$. Consider $\mathrm{U}_q(X_r^{(1)})$ and the associated spectral parameter dependent quantum R-matrix $R(v) \in \mathrm{Hom}(\mathrm{V} \otimes \mathrm{V}', \mathrm{V}' \otimes \mathrm{V})$, where the V and V′ are chosen for each algebra as follows:

$$
\mathrm{V} = \mathrm{V}' = \begin{cases} V_{\omega_1} & \text{for } \mathrm{U}_q(\mathrm{A}_r^{(1)}) \text{ and } \mathrm{U}_q(\mathrm{C}_r^{(1)}), \\ V_{\omega_r} & \text{for } \mathrm{U}_q(\mathrm{B}_r^{(1)}), \\ V_{\omega_6} & \text{for } \mathrm{U}_q(\mathrm{E}_7^{(1)}), \\ V_{\omega_1} \oplus V_0 & \text{for } \mathrm{U}_q(\mathrm{E}_8^{(1)}), \\ V_{\omega_4} & \text{for } \mathrm{U}_q(\mathrm{F}_4^{(1)}), \\ V_{\omega_2} & \text{for } \mathrm{U}_q(\mathrm{G}_2^{(1)}), \end{cases}
$$

$$
\mathrm{V}, \mathrm{V}' \in \begin{cases} \{V_{\omega_{r-1}}, V_{\omega_r}\} & \text{for } \mathrm{U}_q(\mathrm{D}_r^{(1)}), \\ \{V_{\omega_1}, V_{\omega_5}\} & \text{for } \mathrm{U}_q(\mathrm{E}_6^{(1)}). \end{cases}
$$

Explicit forms of such $R(v)$'s may be found for $U_q(X_r^{(1)}) = U_q(A_r^{(1)})$, $U_q(C_r^{(1)})$ [24], $U_q(G_2^{(1)})$ [25] and $U_q(B_r^{(1)})$, $U_q(D_r^{(1)})$ [26]. Now starting from these fundamental $R(v)$'s, do the fusion procedure (cf. ref. [27]) using every possible degeneracy point $\det(R(v = v_0)) = 0$ to get as many inequivalent solutions as possible to the Yang–Baxter equation. The resulting R-matrices in general act on the tensor product of reducible $U_q(X_r)$ modules. The first conjecture is, the totality of such generically reducible $U_q(X_r)$ modules will contain the one isomorphic to a natural q-analogue $W_q(p, s)$ of $W(p, s)$ (A.4) for each s and p (cf. ref. [20]). Assuming this, one can in principle find $R^{s\omega_p}(v) \in \mathrm{End}(W_q(p, s) \otimes W_q(p, s))$ to be called a type $s\omega_p$ fusion R-matrix. Apply next the so-called vertex-SOS correspondence transformation [28] for the $R^{s\omega_p}(v)$ to produce a solvable face model equipped in general with edge and site variables (cf. ref. [8]). Finally, one specializes to $q = \exp[2\pi i/(l + g)]$ and restricts the site variables to the level-l dominant integral weights of $X_r^{(1)}$. Then the second conjecture is, for each choice of l, p, s as in (2.4), there exists an admissibility condition on adjacent site and edge variables under which all the allowed Boltzmann weight functions become well defined and fulfill the (generalized) star–triangle relations among themselves. If this is valid, the resulting Boltzmann weight functions define an object to be called the level-l $X_r^{(1)}$ RSOS model with type $s\omega_p$ fusion.

The above "construction" is indeed known to work essentially for $A_r^{(1)}$ [8] and $G_2^{(1)}$ with $s = 1$, $p = 2$ [10]. It is also natural to expect that those models in [9] have this origin as well. In fact, they are known even to admit an elliptic function generalization. (For $G_2^{(1)}$ [10], this is still a conjecture.) Some background responsible for this may be found in ref. [29].

2.2. THE THERMODYNAMICAL CALCULATION

We shall employ the following hypothesis on the solution of (2.6).

Hypothesis ("String Hypothesis"). Let $\mathcal{N}_m^{(a)}$ $(1 \leqslant a \leqslant r,\ m \geqslant 1)$ *be the number of the* $u_j^{(a)}$ *that tend to a pattern* $\{u_{a,m} + it_a^{-1}(m + 1 - 2n) \mid 1 \leqslant n \leqslant m\}$ *for some* $u_{a,m} \in \mathbb{R}$ *in the limit* $N \to \infty$. *Then*

$$\lim_{N \to \infty} \frac{\sum_{m=1}^{t_a l} \mathcal{N}_m^{(a)}}{N_a} = 1 \quad \text{for all } 1 \leqslant a \leqslant r. \tag{2.7}$$

This has been actually observed to be the case for a few simplest cases of $A_1^{(1)}$ [6], which we believe generally true. We call the pattern $\{u_{a,m} + it_a^{-1}(m + 1 - 2n) \mid 1 \leqslant n \leqslant m\}$ a string of color a, length m at real center $u_{a,m}$ or just a color-a m-string for short. The hypothesis means that for color a, only those strings with length $\leqslant t_a l$ contribute to the thermodynamic quantities. This is a natural extension of the one employed in ref. [7]. The real centers of the strings will form continuous distributions. In the working below, we will mainly use the letters a, b, c, d as the color indices and j, k, m, n as the length indices. In case a confusion might arise, the algebra dependence is also shown by a superscript, e.g. the Cartan matrix $C^{(X_r)}$. Fourier transformation of a function $h(u)$ will be denoted by $\hat{h}(x)$ and normalized as

$$h(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(x)\, \mathrm{e}^{iux}\, \mathrm{d}x, \qquad \hat{h}(x) = \int_{-\infty}^{\infty} h(u)\, \mathrm{e}^{-iux}\, \mathrm{d}u.$$

We find it convenient mainly to work in the Fourier transformed picture and often suppress the arguments of functions as $h = h(u)$, $\hat{h} = \hat{h}(x)$.

Now the thermodynamical calculation goes as follows (cf. ref. [6,7]). Consider the logarithm of both sides of (2.6) and sum them up over those $u_j^{(a)}$ belonging to a color-a m-string at real center $u(1 \leqslant a \leqslant r,\ 1 \leqslant m \leqslant t_a l)$. If one introduces the string and hole densities [2] $\rho_m^{(a)}(u)$, $\sigma_m^{(a)}(u)$ for each color a and length m, they are shown to obey the equation

$$\begin{aligned}
&\delta_{pa}\phi_{a,m}(u, s/t_p)\\
&\quad = \rho_m^{(a)}(u) + \sigma_m^{(a)}(u)\\
&\qquad + \sum_{b=1}^{r}\sum_{k=1}^{t_b l}\sum_{n=1}^{k}\int_{-\infty}^{\infty} \mathrm{d}v\ \phi_{a,m}\big(u - v - it_b^{-1}(k+1-2n),\, C_{ab}/t_a\big)\rho_k^{(b)}(v),
\end{aligned} \tag{2.8a}$$

$$\phi_{a,m}(u, \Delta) = \frac{iL}{\pi^2}\frac{\partial}{\partial u}\sum_{j=1}^{m}\log\left(\frac{\mathrm{sh}\left(\frac{\pi}{2L}\left(u + it_a^{-1}(m+1-2j) + i\Delta\right)\right)}{\mathrm{sh}\left(\frac{\pi}{2L}\left(u + it_a^{-1}(m+1-2j) - i\Delta\right)\right)}\right). \tag{2.8b}$$

Passing to the Fourier component, this is equivalent to

$$\delta_{pa}\hat{A}_{pa}^{(L)sm} = \hat{\sigma}_m^{(a)} + \sum_{b=1}^{r}\sum_{k=1}^{t_b l}\hat{M}_{ab}\hat{A}_{ab}^{(L)mk}\hat{\rho}_k^{(b)} \quad \text{for } 1 \leqslant a \leqslant r, \quad 1 \leqslant m \leqslant t_a l. \tag{2.9}$$

Here the $\hat{A}_{ab}^{(L)mk}$ and $\hat{M}_{ab}$, $(1 \leqslant a,\ b \leqslant r,\ 1 \leqslant m \leqslant t_a l,\ 1 \leqslant k \leqslant t_b l)$ are defined as

$$\hat{M}_{ab} = \hat{M}_{ba} = B_{ab} + 2\delta_{ab}\big(\mathrm{ch}(x/t_a) - 1\big), \tag{2.10a}$$

$$\hat{A}_{ab}^{(L)mk} = \hat{A}_{ba}^{(L)km} = \frac{\mathrm{sh}(\min(m/t_a,\, k/t_b)x)\,\mathrm{sh}((L - \max(m/t_a,\, k/t_b))x)}{\mathrm{sh}(x/t_{ab})\,\mathrm{sh}(Lx)}, \tag{2.10b}$$

where B_{ab} and t_{ab} are given in (2.3). We will also use the following function $\hat{K}_a^{mn}$ $(1 \leqslant a \leqslant r,\ 1 \leqslant m,\ n \leqslant t_a l - 1)$:

$$\hat{K}_a^{mn} = \hat{K}_a^{nm} = \delta_{mn} + \frac{1}{2\mathrm{ch}(x/t_a)}\left(C_{mn}^{(A_{t_a l-1})} - 2\delta_{mn}\right), \tag{2.11}$$

which involves the Cartan matrix of $A_{t_a l-1}$. Given these definitions, one can readily check the following identities:

$$\hat{A}_{ab}^{(L)mk} - \frac{\mathrm{sh}(mx/t_a)}{\mathrm{sh}(lx)}\hat{A}_{a\ \ b}^{(L)t_a l\,k} = \hat{A}_{ab}^{(l)mk}, \tag{2.12a}$$

$$2\mathrm{ch}(x/t_a)\sum_{m=1}^{t_a l-1}\hat{A}_{aa}^{(l)km}\hat{K}_a^{mn} = \delta_{kn} \quad \text{for } 1 \leqslant a \leqslant r, \quad 1 \leqslant k,\ n \leqslant t_a l - 1. \tag{2.12b}$$

Set $m = t_a l$, $x \to 0$ in (2.9) and simplify the right-hand side by the relation $\sum_{k=1}^{t_b l} k\hat{\rho}_k^{(b)}(0) = s t_b/t_p (C^{-1})_{pb}$ derivable from (2.5) and the hypothesis (2.7). The result turns out to be $\hat{\sigma}_{t_a l}^{(a)}(0) = 0$, which means that

$$\sigma_{t_a l}^{(a)}(u) = \hat{\sigma}_{t_a l}^{(a)}(x) = 0 \quad \text{for all } u,\ x \in \mathbb{R} \quad \text{and} \quad 1 \leqslant a \leqslant r, \tag{2.13}$$

since the density must satisfy $\sigma_{t_a l}^{(a)}(u) \geqslant 0$, $\forall u \in \mathbb{R}$. From the same equation one can then express $\hat{\rho}_{t_d l}^{(d)}$ as

$$\hat{\rho}_{t_d l}^{(d)} = \hat{Y}_{dp}\hat{A}_{p\ \ p}^{(L)s\,t_p l} - \sum_{a=1}^{r}\sum_{b=1}^{r}\sum_{k=1}^{t_b l-1}\hat{Y}_{da}\hat{M}_{ab}\hat{A}_{a\ \ b}^{(L)t_a l\,k}\hat{\rho}_k^{(b)} \quad \text{for } 1 \leqslant d \leqslant r, \tag{2.14}$$

where $\hat{Y}$ is the inverse of the $r \times r$ symmetric matrix $(\hat{M}_{ab}\hat{A}_{a\ \ b}^{(L)t_a l\,t_b l})_{1 \leqslant a,b \leqslant r}$. In fact, by the definition $\hat{Y}$ satisfies the following relations for $1 \leqslant a,\ b \leqslant r$:

$$\sum_{b=1}^{r}\hat{M}_{ab}\hat{A}_{a\ \ b}^{(L)m\,t_b l}\hat{Y}_{bc} = \delta_{ac}\frac{\mathrm{sh}(mx/t_a)}{\mathrm{sh}(lx)},$$

$$\sum_{b=1}^{r}\hat{Y}_{ab}\hat{M}_{bc}\hat{A}_{b\ \ c}^{(L)t_b l\,k} = \delta_{ac}\frac{\mathrm{sh}(kx/t_c)}{\mathrm{sh}(lx)}. \tag{2.15}$$

Eliminating $\hat{\rho}_{t_a l}^{(a)}$ by (2.14), one finds, using (2.12a) and (2.15), that (2.9) reduces to

$$\delta_{pa}\hat{A}_{pa}^{(l)sm} = \hat{\sigma}_m^{(a)} + \sum_{b=1}^{r}\sum_{k=1}^{t_b l-1}\hat{M}_{ab}\hat{A}_{ab}^{(l)mk}\hat{\rho}_k^{(b)} \quad \text{for } 1 \leqslant a \leqslant r, \quad 1 \leqslant m \leqslant t_a l - 1. \tag{2.16}$$

We further rewrite this by multiplying $\hat{K}_a^{nm}$ and summing over $m=1,\dots,t_a l-1$ with the help of (2.12b) to arrive at

$$\frac{\delta_{pa}\delta_{sn}}{2\mathrm{ch}(x/t_p)} = \sum_{m=1}^{t_a l-1} \hat{K}_a^{nm}\hat{\sigma}_m^{(a)} + \sum_{b=1}^{r}\sum_{k=1}^{t_b l-1} \hat{J}_{ab}^{nk}\hat{\rho}_k^{(b)} \quad \text{for } 1 \leqslant a \leqslant r, \quad 1 \leqslant n \leqslant t_a l-1. \tag{2.17}$$

Here, $\hat{J}_{ab}^{nk} = \hat{M}_{ab}\sum_{m=1}^{t_a l-1}\hat{K}_a^{nm}\hat{A}_{ab}^{(l)mk}$ is defined in the range $1 \leqslant a,\ b \leqslant r$, $1 \leqslant n \leqslant t_a l-1$, $1 \leqslant k \leqslant t_b l-1$ and has the form

$$\hat{J}_{ab}^{nk} = \frac{\hat{M}_{ab}}{2\mathrm{ch}(x/t_a)}\left(\frac{\mathrm{sh}(x/t_a)}{\mathrm{sh}(x/t_{ab})}\delta_{t_b n, t_a k} + \sum_{j=1}^{t_b/t_a-1}\frac{\mathrm{sh}(jx/t_b)}{\mathrm{sh}(x/t_{ab})}\left(\delta_{t_b(n+1)-t_a j, t_a k} + \delta_{t_b(n-1)+t_a j, t_a k}\right)\right), \tag{2.18}$$

wherein the sum $\sum_{j=1}^{t_b/t_a-1}$ is to be understood as zero if $t_a \geqslant t_b$.

Now we turn to the calculation of the thermal energy density $\mathscr{E}$. So far, the explicit formula for the eigenvalues of the row-to-row transfer matrix is known only for $A_r^{(1)}$ fusion models (cf. ref. [7]) and those examples in the latter or ref. [1] concerning the "non-fusion" case in the direction of the auxially space. Therefore we proceed by postulating the following form that seems to be a natural extension of the eqs. (4.5) and (4.5) in ref. [7] for the $A_r^{(1)}$ case:

$$\mathscr{E} = \epsilon \sum_{m=1}^{t_p l}\int_{-\infty}^{\infty} \mathrm{d}u\ \phi_{p,m}(u,\ s/t_p)\rho_m^{(p)}(u)$$
$$= \frac{\epsilon}{2\pi}\sum_{m=1}^{t_p l}\int_{-\infty}^{\infty}\mathrm{d}x\ \hat{A}_{pp}^{(L)sm}(x)\hat{\rho}_m^{(p)}(x). \tag{2.19}$$

Here $\epsilon = \pm$ is the sign factor that specifies the regime in consideration (cf. ref. [6,7]). See (2.8b) for the definition of $\phi_{p,m}(u,\ s/t_p)$. Again one can simplify this expression by means of (2.12a), (2.14) and (2.15) as

$$\mathscr{E} = \epsilon\mathscr{E}_0 + \epsilon\sum_{m=1}^{t_p l-1}\int_{-\infty}^{\infty}\mathrm{d}u\ A_{pp}^{(l)sm}(u)\rho_m^{(p)}(u),$$
$$\mathscr{E}_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\ \hat{A}_p{}^{(L)s}{}_p{}^{t_p l}(x)\hat{Y}_{pp}(x)\hat{A}_p{}^{(L)t_p l}{}_p{}^{s}(x). \tag{2.20}$$

Next we evaluate the entropy density $\mathscr{S}$ responsible for the combinatorial degrees of freedom to arrange the strings and holes. Taking (2.13) into account, it is given by

$$\mathscr{S}=\sum_{a=1}^{r}\sum_{m=1}^{t_a l-1}\int_{-\infty}^{\infty}\mathrm{d}u\Big(\big(\rho_m^{(a)}(u)+\sigma_m^{(a)}(u)\big)\log\big(\rho_m^{(a)}(u)+\sigma_m^{(a)}(u)\big)$$

$$-\rho_m^{(a)}(u)\log\rho_m^{(a)}(u)-\sigma_m^{(a)}(u)\log\sigma_m^{(a)}(u)\Big). \tag{2.21}$$

Having the thermal energy and the entropy at hand, we are now in a position to seek the equilibrium state. Denoting the temperature by $T=\beta^{-1}$, the condition $\delta\mathscr{F}/\delta\rho_m^{(a)}(u)=0$ for the free energy density $\mathscr{F}=\mathscr{E}-T\mathscr{S}$ with the constraint (2.16) leads to

$$\frac{\epsilon t_a\delta_{pa}\delta_{sn}}{4\mathrm{ch}(t_a\pi u/2)}=T\sum_{m=1}^{t_a l-1}\int_{-\infty}^{\infty}\mathrm{d}v\,K_a^{nm}(u-v)\log\big(1+\exp\big(\beta\epsilon_m^{(a)}(v)\big)\big)$$

$$-T\sum_{b=1}^{r}\sum_{k=1}^{t_b l-1}\int_{-\infty}^{\infty}\mathrm{d}v\,J_{ab}^{nk}(u-v)\log\big(1+\exp\big(-\beta\epsilon_k^{(b)}(v)\big)\big), \tag{2.22a}$$

$$\epsilon_m^{(a)}=T\log\big(\sigma_m^{(a)}/\rho_m^{(a)}\big), \tag{2.22b}$$

for $1\leqslant a\leqslant r$, $1\leqslant n\leqslant t_a l-1$.

With the aid of (2.22) one can extract the low-temperature asymptotics of the entropy as follows [30,31]. Shift the variable $u\to u-(2/t_p\pi)\log T$ in (2.22). For $T\ll 1$, the result reduces to the following equation for the function $\varphi_m^{(a)}(u)\overset{\mathrm{def}}{=}\epsilon_m^{(a)}(u-(2/t_p\pi)\log T)/T$:

$$\tfrac{1}{2}\epsilon t_a\delta_{pa}\delta_{sn}\,\mathrm{e}^{-t_p\pi u/2}=\sum_{m=1}^{t_a l-1}\int_{-\infty}^{\infty}\mathrm{d}v\,K_a^{nm}(v)\log\big(1+\exp\big(\varphi_m^{(a)}(u-v)\big)\big)$$

$$-\sum_{b=1}^{r}\sum_{k=1}^{t_b l-1}\int_{-\infty}^{\infty}\mathrm{d}v\,J_{ab}^{nk}(v)\log\big(1+\exp\big(-\varphi_k^{(b)}(u-v)\big)\big), \tag{2.23}$$

which implies that $\varphi_m^{(a)}(u)$ remains finite in the low-temperature limit $T\to 0$. Considering the same shift for the Fourier transformation of (2.17) and comparing the result with the u-derivative of (2.23), one deduces the following behavior valid at $u \to \infty$ (cf. refs. [30,31]):

$$\rho_m^{(a)}(u) \simeq -\frac{2\epsilon}{t_p \pi} f\big(\beta\epsilon_m^{(a)}(u)\big) \frac{\mathrm{d}\epsilon_m^{(a)}(u)}{\mathrm{d}u},$$

$$\sigma_m^{(a)}(u) \simeq -\frac{2\epsilon}{t_p \pi} \big(1 - f\big(\beta\epsilon_m^{(a)}(u)\big)\big) \frac{\mathrm{d}\epsilon_m^{(a)}(u)}{\mathrm{d}u}, \tag{2.24}$$

where $f(\varphi) = (1 + e^{\varphi})^{-1}$ is the Fermi distribution function. By using (2.22b) and (2.24) in (2.21), we find the asymptotics of the entropy for $T \to 0$,

$$\mathscr{S} = \frac{4\epsilon T}{t_p \pi} \sum_{a=1}^{r} \sum_{m=1}^{t_a l - 1} \int_{\varphi_m^{(a)}(-\infty)}^{\varphi_m^{(a)}(\infty)} \mathrm{d}\varphi \big(f(\varphi) \log f(\varphi) + (1 - f(\varphi)) \log(1 - f(\varphi)) \big). \tag{2.25a}$$

Here we have attached an extra factor 2 to take into account the contribution from the negative large u discarded by firstly passing to the limit $u \to u - (2/t_p \pi) \log T$, $T \to 0$. Introducing $g_m^{(a)}(u) = f(\exp(\varphi_m^{(a)}(u)))$ further, the entropy (2.25a) is expressed in terms of the Rogers dilogarithmic function $L(x)$ (1.1b) as

$$\mathscr{S} = \frac{8\epsilon T}{t_p \pi} \sum_{a=1}^{r} \sum_{m=1}^{t_a l - 1} \big(L\big(g_m^{(a)}(\infty)\big) - L\big(g_m^{(a)}(-\infty)\big) \big). \tag{2.25b}$$

Finally we relate this asymptotic expression to the central charge c. The argument of ref. [4] implies that $\partial^2 \mathscr{F} / \partial T^2 = -\pi c / 3 v_{\mathrm{F}}$ in the low-temperature limit, where the Fermi velocity is chosen to be $v_{\mathrm{F}} = t_p / 4$ in our case. Combining this with the relation $\partial \mathscr{S} / \partial T = -\partial^2 \mathscr{F} / \partial T^2$, we arrive at the expression for the central charge,

$$c = \epsilon \frac{6}{\pi^2} \sum_{a=1}^{r} \sum_{m=1}^{t_a l - 1} \big(L\big(g_m^{(a)}(\infty)\big) - L\big(g_m^{(a)}(-\infty)\big) \big). \tag{2.26}$$

The quantities $g_m^{(a)}(\pm\infty)$ in general depend on the regimes $\epsilon = \pm 1$ and are to be determined from (2.23). In the next section we will see that they are given by the purely Lie algebraic data $f_m^{(a)}$ described in appendix A. We remark that entropy of a form similar to (2.25b) has been obtained in ref. [30] for the higher spin Heisenberg chain.

2.3. HIGH-TEMPERATURE LIMIT

Here we study the high-temperature behavior of the equilibrium free energy

$$\mathscr{F} = \epsilon \mathscr{E}_0 - T \sum_{m=1}^{t_p l - 1} \int_{-\infty}^{\infty} \mathrm{d}u \, A_{pp}^{(l)sm}(u) \log\big(1 + \exp\big(-\beta\epsilon_m^{(p)}(u)\big)\big). \tag{2.27}$$

When $T \to \infty$, the leading part of the asymptotics of the $\epsilon_m^{(a)}(u)$ is expected to become a constant. The constant solution to (2.22) in this limit can be extracted from (A.1c′) as

$$f\left(\exp\left(\beta\epsilon_m^{(a)}(u)\right)\right) = f_m^{(a)} \quad \text{as } T \to \infty, \tag{2.28}$$

where f on the l.h.s. is the Fermi function specified after (2.24) and the quantity $f_m^{(a)}$ has been described in the appendix. Combining (2.27) and (2.28), we have

$$\lim_{T\to\infty} \frac{\mathscr{F}}{T} = \sum_{m=1}^{t_p l - 1} \hat{A}_{pp}^{(l)sm}(x=0) \log\left(1 - f_m^{(p)}\right). \tag{2.29}$$

Substituting (A.7) into this and using (2.10b), we obtain an interesting result,

$$\lim_{T\to\infty} \mathscr{S} = -\lim_{T\to\infty} \frac{\mathscr{F}}{T} = \log Q_s^{(p)}. \tag{2.30}$$

Here the quantity $Q_s^{(p)}$ is the "yangian character" at $q = \exp(2\pi i/(l+g))$ detailed in appendix A. When $T \to \infty$, the entropy density $\mathscr{S}$ is generally expected to become $\lim_{N\to\infty}(1/N)\log(\dim \mathscr{H}(N))$, where $\mathscr{H}(N)$ denotes the space on which the length-N row-to-row transfer matrix acts. Thus the result (2.30) implies that $\lim_{N\to\infty}(\dim \mathscr{H}(N))^{1/N} = Q_s^{(p)}$, which is consistent to our picture for the "RSOS model with fusion type $s\omega_p$" in remark 2.5. For $X_r^{(1)} = A_1^{(1)}$, this has been observed in ref. [6].

Remark 2.6 (by V.V. Bazhanov). In ref. [39] the thermodynamic Bethe ansatz equations for non-critical RSOS models of ADE type have been conjectured. The corresponding lattice models are not known in general and it would be interesting to find them. These equations enable one to calculate the central charges c, the dimension of the perturbation Δ_t (thermal exponent), the spectrum and the S-matrix of the excitations of the corresponding scaling field theory near criticality. In particular, for $X_r^{(1)} = E_8^{(1)}$, $l = 2$, $p = s = 1$, one gets [39] in this way $c = 1/2$, $\Delta_t = 1/8$ and the S-matrix identical to Zamolodchikov's one for the magnetic Ising model [36]. [Eq. (5.4) in ref. [39] has been missprinted. The correct formula reads as $c = 2\,\text{rank}(G)/(g+2)$ and the words "minimal unitary" before the equation and "by operator $\phi_{(1,3)}$" after it should be omitted.] In addition, eqs. (2.30) and (A.12a) of the present paper in this case gives

$$e^{\mathscr{S}} = 1 + \sqrt{2}, \tag{2.31}$$

in high-temperature limit. This means that the corresponding lattice model (if any) should have the largest eigenvalue of the incidence matrix equal to (2.31).

3. List of the central charges

The remaining task in the central charge calculation is to specify the $g_m^{(a)}(\pm\infty) = f(\exp(\varphi_m^{(a)}(\pm\infty)))$ in (2.26) by using (2.23). We do this in subsect. 3.1 and thereby list the values of c. As it turns out, $g_m^{(a)}(\pm\infty)$ will be identified with the Lie algebraic data $f_m^{(a)}$ described in appendix A attached to various $(X_{r'}^{\prime(1)}, l')$ with $X_{r'}^{\prime(1)}$ being a subalgebra of $X_r^{(1)}$. Subsect. 3.2 is devoted to comparisons of the results for c with known cases.

3.1. DETERMINATION OF $g_m^{(a)}(\pm\infty)$

Consider the limit $u \to \pm\infty$ in eq. (2.23). Since $K_a^{nm}(u)$ and $J_{ab}^{nk}(u)$ decay exponentially, the quantity $g_m^{(a)}(\pm\infty) = (1 + \exp(\varphi_m^{(a)}(\pm\infty)))^{-1}$ satisfies

$$\tfrac{1}{2}\epsilon t_a \delta_{pa} \delta_{sn}\, e^{\mp t_p \pi\infty/2} = -\sum_{m=1}^{t_a l-1} \hat{K}_a^{nm}(0) \log g_m^{(a)}(\pm\infty)$$

$$+\sum_{b=1}^{r} \sum_{k=1}^{t_b l-1} \hat{J}_{ab}^{nk}(0) \log\left(1 - g_k^{(b)}(\pm\infty)\right), \qquad (3.1)$$

if, and as we assume, the $\varphi_m^{(a)}(u)$ approaches to $\varphi_m^{(a)}(\pm\infty)$ quickly enough. In the appendix, we have defined, for each pair $(X_r^{(1)}, l)$, the quantity $f_m^{(a)}(1 \leqslant a \leqslant r,\ 1 \leqslant m \leqslant t_a l - 1)$ that solve (3.1) with zero on its left-hand side. See (A.1c′). Thus we obtain

$$g_m^{(a)}(+\infty) = f_m^{(a)} \quad \text{for all } 1 \leqslant a \leqslant r, \quad 1 \leqslant m \leqslant t_a l - 1. \qquad (3.2)$$

On the other hand, we have a divergent left-hand side in (3.1) for $g_m^{(a)}(-\infty)$ whose sign also depends on the choice $\epsilon = \pm 1$. Such an equation is fulfilled by the behavior

$$\begin{aligned}
\epsilon = +1: &\quad g_m^{(a)}(u) \xrightarrow{u\to-\infty} +0 &&\text{for } (a, m) \in G_+, \\
\epsilon = -1: &\quad g_m^{(a)}(u) \xrightarrow{u\to-\infty} 1-0 &&\text{for } (a, m) \in G_-, \qquad (3.3)\\
\epsilon = \pm 1: &\quad 0 < g_m^{(a)}(-\infty) < 1 &&\text{for } (a, m) \notin G_\epsilon,
\end{aligned}$$

for some subset $G_\pm$ of $G \overset{\text{def}}{=} \{(a, m) \mid 1 \leqslant a \leqslant r,\ 1 \leqslant m \leqslant t_a l - 1\}$. In order to find the $G_\pm$, we consider the following linear equation for the unknown constants

$h_m^{(a)}$, $(a, m) \in \mathrm{G}$, which corresponds to the proper cancellation of the divergent terms in (3.1):

$$\delta_{pa}\delta_{sn} = \begin{cases} \sum_{m=1}^{t_a l-1} \hat{K}_a^{nm}(0) h_m^{(a)} & \text{for } \epsilon = +1 \\ \sum_{b=1}^{r} \sum_{k=1}^{t_b l-1} \hat{J}_{ab}^{nk}(0) h_k^{(b)} & \text{for } \epsilon = -1 \end{cases} \qquad (a, n) \in \mathrm{G}. \tag{3.4}$$

The solution consistently satisfies $h_m^{(a)} \geqslant 0$ for all $(a, m) \in \mathrm{G}$ and determines the set G_ϵ via the rule $h_m^{(a)} > 0 \leftrightarrow (a, m) \in \mathrm{G}_\epsilon$ as

$$\mathrm{G}_+ = \{(p, m) \mid 1 \leqslant m \leqslant t_p l - 1\}, \tag{3.5a}$$

$$\mathrm{G}_- = \begin{cases} \{(a, st_a/t_p) \mid 1 \leqslant a \leqslant r\} & \text{if } s/t_p \in \mathbb{Z}, \\ \mathrm{G} \cap \mathrm{G}(s, p) & \text{if } s/t_p \notin \mathbb{Z}, \end{cases} \tag{3.5b}$$

$$\mathrm{G}(s, p) = \left\{ \left(a, \frac{s-s_0}{t_p}\right), \left(a, \frac{s-s_0}{t_p} + 1\right) \middle| 1 \leqslant a \leqslant r,\ t_a = 1 \right\} \tag{3.5c}$$

$$\cup \left\{ (a, s-s_0), (a, s), (a, s-s_0+t_p) \mid 1 \leqslant a \leqslant r,\ t_a = t_p \right\},$$

$$s_0 \equiv s \bmod t_p, \quad 0 \leqslant s_0 \leqslant t_p - 1. \tag{3.5d}$$

These patterns are conveniently visualized in a width-r tableau whose ath column $(1 \leqslant a \leqslant r)$ consists of the $t_a l - 1$ rectangles each having the depth $1/t_a$ and the width 1. The mth rectangle from the top of the ath column corresponds to the

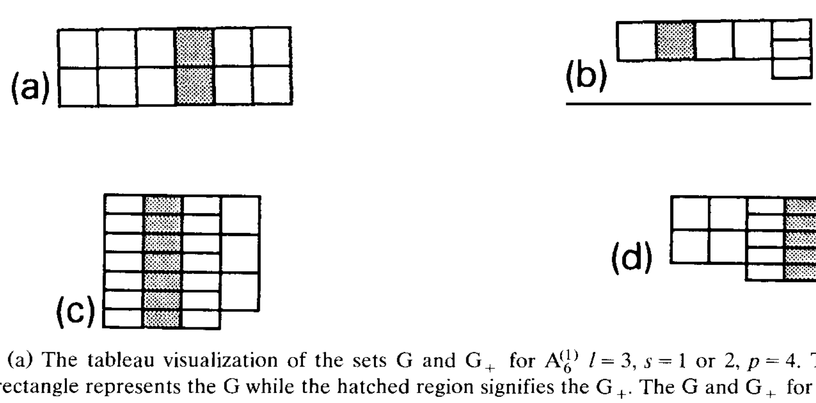


Fig. 1. (a) The tableau visualization of the sets G and G_+ for $\mathrm{A}_6^{(1)}$ $l = 3$, $s = 1$ or 2, $p = 4$. The 2×6 outer rectangle represents the G while the hatched region signifies the G_+. The G and G_+ for $\mathrm{D}_6^{(1)}$ and $\mathrm{E}_6^{(1)}$ with the same l, s, p are also represented by this figure. (b) The sets G and G_+ (hatched) for $\mathrm{B}_5^{(1)}$, $l = 2$, $s = 1$, $p = 2$. (c) The sets G and G_+ (hatched) for $\mathrm{C}_4^{(1)}$, $l = 4$, $1 \leqslant s \leqslant 7$, $p = 2$. (d) The sets G and G_+ (hatched) for $\mathrm{F}_4^{(1)}$, $l = 3$, $1 \leqslant s \leqslant 5$, $p = 4$.

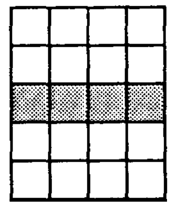


Fig. 2. (a) The sets G and G_- (hatched) for $A_4^{(1)}$, $l = 6$, $s = 3$, $t_p = 1$.

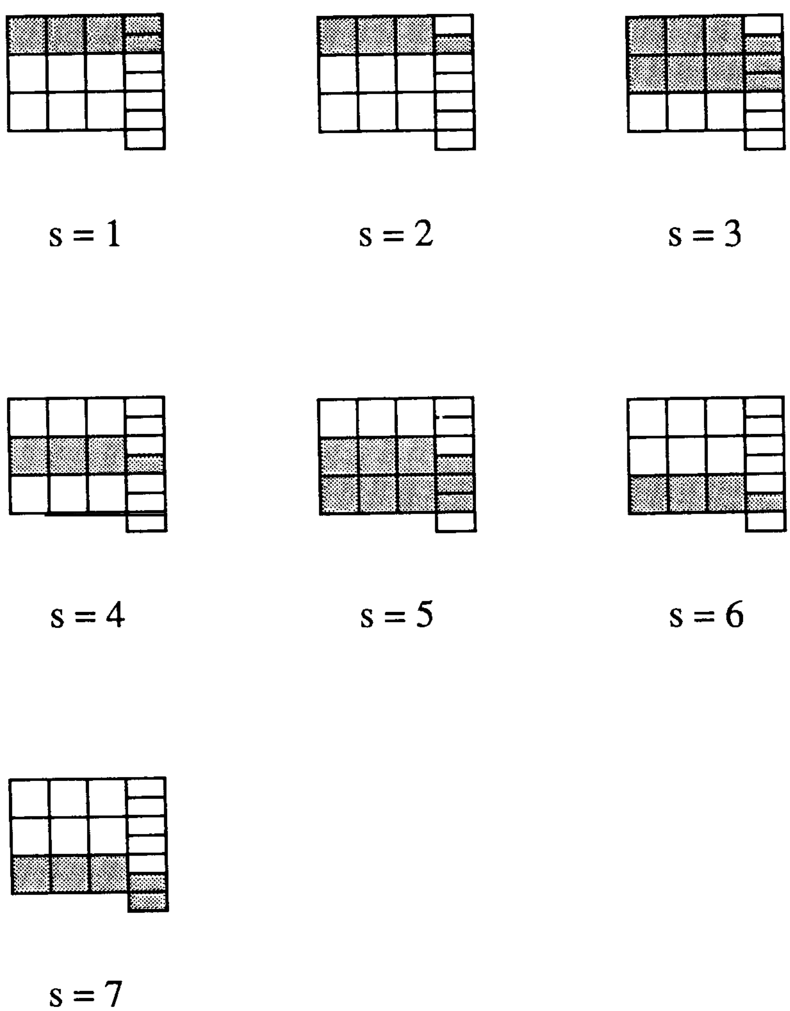


Fig. 2. (b) The sets G and G_- (hatched) for $B_4^{(1)}$, $l = 4$, $1 \leqslant s \leqslant 7$, $t_p = 2$. When s is even, the tableau also corresponds to the G and G_- for $B_4^{(1)}$, $l = 4$, $s/2$, $t_p = 1$.

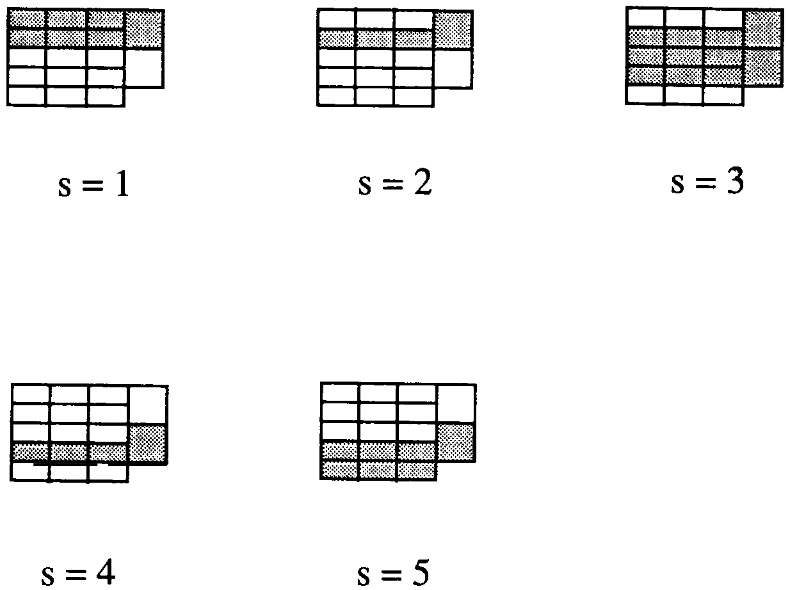


Fig. 2. (c) The sets G and G_- (hatched) for $C_4^{(1)}$, $l = 3$, $1 \leqslant s \leqslant 5$, $t_p = 2$. When s is even, the tableau also corresponds to the G and G_- for $C_4^{(1)}$, $l = 3$, $s/2$, $t_p = 1$.

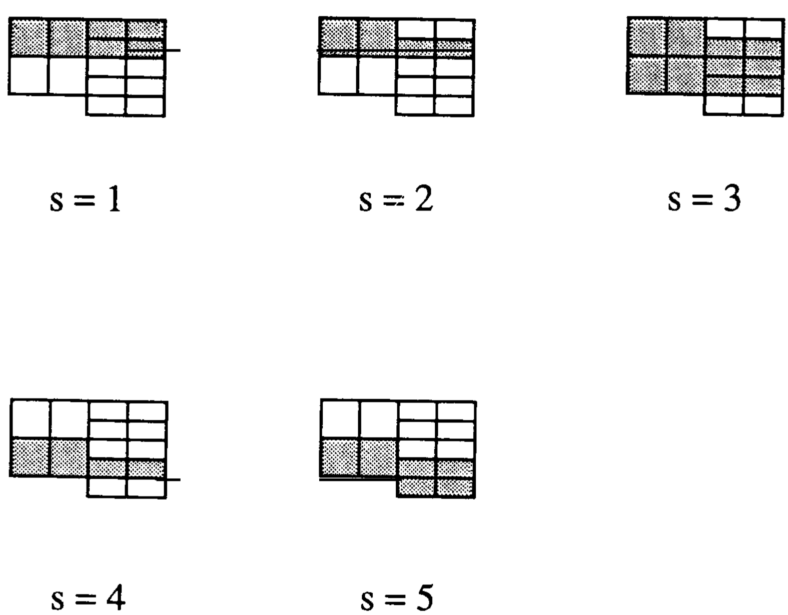


Fig. 2. (d) The sets G and G_- (hatched) for $F_4^{(1)}$, $l = 3$, $1 \leqslant s \leqslant 5$, $t_p = 2$. When s is even, the tableau also corresponds to the G and G_- for $F_4^{(1)}$, $l = 3$, $s/2$, $t_p = 1$.

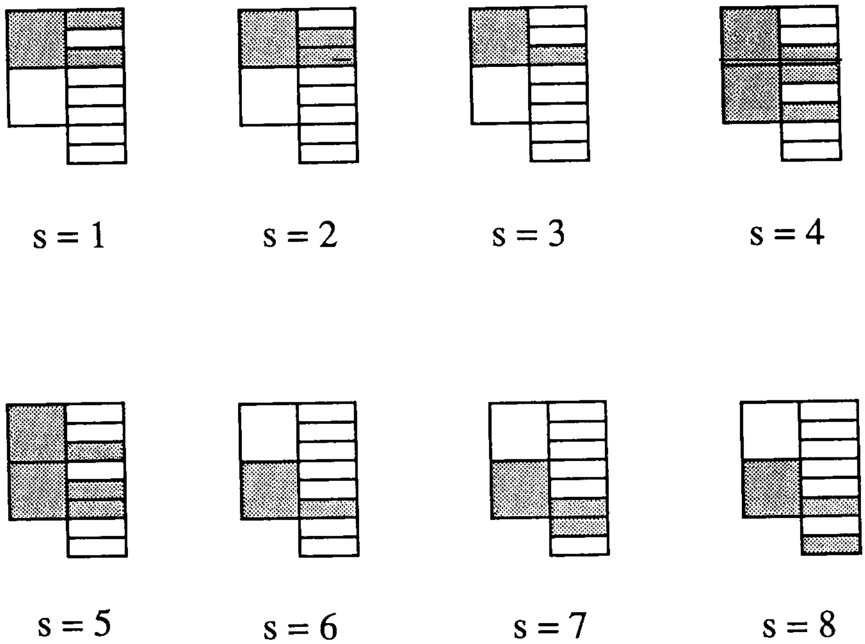


Fig. 2. (e) The sets G and G_- (hatched) for $G_2^{(1)}$, $l=3$, $1 \leqslant s \leqslant 8$, $t_p=3$. When $s=0 \bmod 3$, the tableau also corresponds to the G and G_- for $G_2^{(1)}$, $l=3$, $s/3$, $t_p=1$.

element $(a, m) \in G$. Then we see from (3.5) that the simply-laced algebras $X_r^{(1)} = A_r^{(1)}$, $D_r^{(1)}$ and $E_{6,7,8}^{(1)}$ possess especially simple patterns: $G_+ = p$th column, and $G_- = s$th row. In fact, G_+ (3.5a) is always just the pth column for any $X_r^{(1)}$ irrespective of s. On the other hand, G_- (3.5b) for non-simply-laced algebras exhibits slightly complicated patterns depending on both s and p. See figs. 1 and 2 for some examples of G_+ and G_-, respectively. To summarize so far, besides (3.2), we have seen in (3.3) that $g_m^{(a)}(-\infty) = (1-\epsilon)/2$ for $(a, m) \in G_\epsilon$ and the $G_\pm$ is specified by (3.5). This completely removes the divergence in (3.1) and leaves us with the following equation for the remaining $g_m^{(a)}(-\infty)$:

$$0 = \sum_{\substack{m=1 \\ (a,m) \notin G_\epsilon}}^{t_a l - 1} \hat{K}_a^{nm}(0) \log g_m^{(a)}(-\infty) - \sum_{b=1}^{r} \sum_{\substack{k=1 \\ (b,k) \notin G_\epsilon}}^{t_b l - 1} \hat{J}_{ab}^{nk}(0) \log\big(1 - g_k^{(b)}(-\infty)\big), \tag{3.6}$$

which should hold for any $(a, n) \in G \setminus G_\epsilon$. A little inspection using (2.11) and (2.18) then shows that (3.6) in fact decouples into a few independent set of equations. Moreover, each of such equation sets coincides with (A.1c,c′) for some $(X_{r'}^{\prime(1)}, l')$ and therefore enables us to identify the $g_m^{(a)}(-\infty)$'s with those $f_m^{(a)}$'s $\{f_m^{(a)} | (X_{r'}^{\prime(1)}, l')\}$ attached to the data $(X_{r'}^{\prime(1)}, l')$. Here we shall only illustrate this decoupling with two examples, from which the other cases will be easily inferred.

Example 1: $(X_r^{(1)}, l) = (F_4^{(1)}, 3)$; $\epsilon = +1$. This case has been depicted in fig. 1d with the G_+ for $p = 4$ hatched. As p varies from $p = 1$ to $p = 4$, the "hatched region" moves from left to right and the 14 $g_m^{(a)}(-\infty)$'s, $(a, m) \in G$ are determined as follows. (The union of the sets here and the following example means the one retaining the multiplicity of the elements.)

$$\begin{aligned} p=1:&\quad \{0, 0\} \cup \{f_m^{(a)} \,|\, (C_3^{(1)}, 3)\},\\ p=2:&\quad \{0, 0\} \cup \{f_m^{(a)} \,|\, (A_1^{(1)}, 3)\} \cup \{f_m^{(a)} \,|\, (A_2^{(1)}, 6)\},\\ p=3:&\quad \{0, 0, 0, 0, 0\} \cup \{f_m^{(a)} \,|\, (A_2^{(1)}, 3)\} \cup \{f_m^{(a)} \,|\, (A_1^{(1)}, 6)\},\\ p=4:&\quad \{0, 0, 0, 0, 0\} \cup \{f_m^{(a)} \,|\, (B_3^{(1)}, 3)\}. \end{aligned} \tag{3.7a}$$

These are essentially derivable by identifying the shape of the "non-hatched region" $G \setminus G_+$ with the G's for some $(X_{r'}^{\prime(1)}, l')$. Notice that the decouplings are independent of the choice of s. Substituting $g_m^{(a)}(-\infty)$ (3.7a), $g_m^{(a)}(\infty)$ (3.2) into (2.26) and using (3.8) and (A.1a), one gets (3.9i) with $l = 3$.

Example 2: $(X_r^{(1)}, l) = (G_2^{(1)}, 3)$; $p = 2$; $\epsilon = -1$. The G_- for each $1 \leqslant s \leqslant 8 (= lt_2 - 1, (2.4))$ has been depicted in fig. 2e. Now the "hatched region" corresponds to $g_m^{(a)}(-\infty) = 1$ and the 10 $g_m^{(a)}(-\infty)$'s, $(a, m) \in G$ are determined as

$$\begin{aligned} &s = 1, 2, 7, 8: \{1, 1, 1\} \cup \{f_m^{(a)} \,|\, (G_2^{(1)}, 2)\} \cup \{f_m^{(a)} \,|\, (A_1^{(1)}, 2)\},\\ &s = 3, 6: \{1, 1\} \cup \{f_m^{(a)} \,|\, (G_2^{(1)}, 2)\} \cup \{f_m^{(a)} \,|\, (A_1^{(1)}, 3)\},\\ &s = 4, 5: \{1, 1, 1, 1, 1\} \cup \{f_m^{(a)} \,|\, (A_1^{(1)}, 3)\} \cup \{f_m^{(a)} \,|\, (A_1^{(1)}, 2)\} \cup \{f_m^{(a)} \,|\, (A_1^{(1)}, 3)\}. \end{aligned} \tag{3.7b}$$

As these examples show, the $g_m^{(a)}(-\infty)$'s are evaluated in terms of those $f_m^{(a)}$'s associated to various $(X_{r'}^{\prime(1)}, l')$ with $X_{r'}^{\prime(1)} \subset X_r^{(1)}$.

Having identified the $g_m^{(a)}(\pm\infty)$'s with the $f_m^{(a)}$'s, the central charges are now computable from (2.26) by means of the (conjectured) dilogarithmic function identity (A.1a). Below we list the result of c for any choice of the regime $\epsilon = \pm 1$, the algebra $X_r^{(1)}$, the level l and the integer parameters s, p obeying (2.4). We use the notation

$$\mathscr{L}(X_r^{(1)}, l) = \frac{l \dim X_r}{l + g} - r \quad \text{for } l \geqslant 0. \tag{3.8}$$

In particular, we have $\mathscr{L}(X_r^{(1)}\ 0) = -r$ for any $X_r^{(1)}$ and employ a natural convention $\mathscr{L}(A_0^{(1)}, l) = \mathscr{L}(C_0^{(1)}, l) = 0$ for any l. See (3.11) and (3.12) for further properties. The data $\dim X_r$ and g are available in table 1. The results for $X_r^{(1)} = A_r^{(1)}$, $D_r^{(1)}$ and $E_{6,7,8}^{(1)}$ recover those in refs. [6,7].

Regime $\epsilon = +1$. The results are independent of s. To present them neatly, we shall write e.g. $a, b \in \mathscr{D}(X_r)$ to mean that the Dynkin diagram $\mathscr{D}(X_r)$ of X_r contains the nodes a and $b (1 \leqslant a, b \leqslant r)$. Then, using the t_a (2.2) attached to X_r (not $X'_{r'}$), we have

$$c = \mathscr{L}\left(X_r^{(1)}, l\right) - \sum_{X'_{r'} \in \mathscr{H}_p} \mathscr{L}\left(X_{r'}^{\prime(1)}, l'\right),$$

$$l'/l = \min\{t_a \mid a \in \mathscr{D}(X'_{r'})\}. \tag{3.9a}$$

Here $\mathscr{H}_p$ denotes the set of the subalgebras $X'_{r'}$ of X_r whose $\mathscr{D}(X'_{r'})$ are obtained by removing the pth node from the $\mathscr{D}(X_r)$. Explicitly, they are given by

$$A_r^{(1)} (r \geqslant 1); \tag{3.9b}$$

$$c = \mathscr{L}\left(A_r^{(1)}, l\right) - \mathscr{L}\left(A_{p-1}^{(1)}, l\right) - \mathscr{L}\left(A_{r-p}^{(1)}, l\right) \quad 1 \leqslant p \leqslant r.$$

$$B_r^{(1)} (r \geqslant 2); \tag{3.9c}$$

$$\begin{aligned} c &= \mathscr{L}\left(B_r^{(1)}, l\right) - \mathscr{L}\left(A_{p-1}^{(1)}, l\right) - \mathscr{L}\left(B_{r-p}^{(1)}, l\right) \quad 1 \leqslant p \leqslant r-2, \\ &= \mathscr{L}\left(B_r^{(1)}, l\right) - \mathscr{L}\left(A_{p-1}^{(1)}, l\right) - \mathscr{L}\left(A_{r-p}^{(1)}, 2l\right) \quad p = r-1, r \end{aligned}$$

$$C_r^{(1)} (r \geqslant 1); \tag{3.9d}$$

$$c = \mathscr{L}\left(C_r^{(1)}, l\right) - \mathscr{L}\left(A_{p-1}^{(1)}, 2l\right) - \mathscr{L}\left(C_{r-p}^{(1)}, l\right) \quad 1 \leqslant p \leqslant r,$$

$$D_r^{(1)} (r \geqslant 3); \tag{3.9e}$$

$$\begin{aligned} c &= \mathscr{L}\left(D_r^{(1)}, l\right) - \mathscr{L}\left(A_{p-1}^{(1)}, l\right) - \mathscr{L}\left(D_{r-p}^{(1)}, l\right) \quad 1 \leqslant p \leqslant r-2, \\ &= \mathscr{L}\left(D_r^{(1)}, l\right) - \mathscr{L}\left(A_{r-1}^{(1)}, l\right) \quad p = r-1, r. \end{aligned}$$

$$E_6^{(1)}; \tag{3.9f}$$

$$\begin{aligned} c &= \mathscr{L}\left(E_6^{(1)}, l\right) - \mathscr{L}\left(D_5^{(1)}, l\right) \quad p = 1, 5, \\ &= \mathscr{L}\left(E_6^{(1)}, l\right) - \mathscr{L}\left(A_1^{(1)}, l\right) - \mathscr{L}\left(A_4^{(1)}, l\right) \quad p = 2, 4, \\ &= \mathscr{L}\left(E_6^{(1)}, l\right) - 2\mathscr{L}\left(A_2^{(1)}, l\right) - \mathscr{L}\left(A_1^{(1)}, l\right) \quad p = 3, \\ &= \mathscr{L}\left(E_6^{(1)}, l\right) - \mathscr{L}\left(A_5^{(1)}, l\right) \quad p = 6. \end{aligned}$$

$$E_7^{(1)}; \tag{3.9g}$$

$$
\begin{aligned}
c &= \mathscr{L}(E_7^{(1)}, l) - \mathscr{L}(D_6^{(1)}, l) && p=1,\\
&= \mathscr{L}(E_7^{(1)}, l) - \mathscr{L}(A_1^{(1)}, l) - \mathscr{L}(A_5^{(1)}, l) && p=2,\\
&= \mathscr{L}(E_7^{(1)}, l) - \mathscr{L}(A_1^{(1)}, l) - \mathscr{L}(A_2^{(1)}, l) - \mathscr{L}(A_3^{(1)}, l) && p=3,\\
&= \mathscr{L}(E_7^{(1)}, l) - \mathscr{L}(A_4^{(1)}, l) - \mathscr{L}(A_2^{(1)}, l) && p=4,\\
&= \mathscr{L}(E_7^{(1)}, l) - \mathscr{L}(A_1^{(1)}, l) - \mathscr{L}(D_5^{(1)}, l) && p=5,\\
&= \mathscr{L}(E_7^{(1)}, l) - \mathscr{L}(E_6^{(1)}, l) && p=6,\\
&= \mathscr{L}(E_7^{(1)}, l) - \mathscr{L}(A_6^{(1)}, l) && p=7.
\end{aligned}
$$

$E_8^{(1)}$; (3.9h)

$$
\begin{aligned}
c &= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(E_7^{(1)}, l) && p=1,\\
&= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(A_1^{(1)}, l) - \mathscr{L}(E_6^{(1)}, l) && p=2,\\
&= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(A_2^{(1)}, l) - \mathscr{L}(D_5^{(1)}, l) && p=3,\\
&= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(A_3^{(1)}, l) - \mathscr{L}(A_4^{(1)}, l) && p=4,\\
&= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(A_4^{(1)}, l) - \mathscr{L}(A_2^{(1)}, l) - \mathscr{L}(A_1^{(1)}, l) && p=5,\\
&= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(A_6^{(1)}, l) - \mathscr{L}(A_1^{(1)}, l) && p=6,\\
&= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(D_7^{(1)}, l) && p=7,\\
&= \mathscr{L}(E_8^{(1)}, l) - \mathscr{L}(A_7^{(1)}, l) && p=8.
\end{aligned}
$$

$F_4^{(1)}$; (3.9i)

$$
\begin{aligned}
c &= \mathscr{L}(F_4^{(1)}, l) - \mathscr{L}(C_3^{(1)}, l) && p=1,\\
&= \mathscr{L}(F_4^{(1)}, l) - \mathscr{L}(A_{p-1}^{(1)}, l) - \mathscr{L}(A_{4-p}^{(1)}, 2l) && p=2, 3,\\
&= \mathscr{L}(F_4^{(1)}, l) - \mathscr{L}(B_3^{(1)}, l) && p=4,
\end{aligned}
$$

$G_2^{(1)}$; (3.9j)

$$
\begin{aligned}
c &= \mathscr{L}(G_2^{(1)}, l) - \mathscr{L}(A_1^{(1)}, 3l) && p=1,\\
&= \mathscr{L}(G_2^{(1)}, l) - \mathscr{L}(A_1^{(1)}, l) && p=2.
\end{aligned}
$$

Regime $\epsilon = -1$. For the simply-laced algebras $X_r^{(1)} = A_r^{(1)}$, $D_r^{(1)}$ and $E_{6,7,8}^{(1)}$, the results depend only on $X_r^{(1)}$, l and s [7], while the other cases exhibit a dependence on p as well. They are given as follows:

$X_r^{(1)}$;

$$c = \mathscr{L}\left(X_r^{(1)}, \frac{s}{t_p}\right) + \mathscr{L}\left(X_r^{(1)}, l - \frac{s}{t_p}\right) - \mathscr{L}\left(X_r^{(1)}, l\right) + r \quad \text{if } \frac{s}{t_p} \in \mathbb{Z}, \tag{3.10a}$$

$$= \mathscr{L}\left(X_r^{(1)}, \frac{s - s_0}{t_p}\right) + \mathscr{L}\left(X_r^{(1)}, l - \frac{s - s_0}{t_p} - 1\right) - \mathscr{L}\left(X_r^{(1)}, l\right)$$

$$+ \sum_{a=1}^{r} (t_a + 1) - \tfrac{1}{2}\delta_{t_p, 3} \quad \text{if } \frac{s}{t_p} \notin \mathbb{Z}, \tag{3.10b}$$

where s_0 is the integer uniquely specified by (3.5d).

Remark 3.1. The central charge (3.10a) corresponds to the coset pair [32]

$$\begin{array}{lccc} & X_r^{(1)} & \oplus\ X_r^{(1)} & \supset X_r^{(1)} \\ \text{level} & l - \frac{s}{t_p} & \frac{s}{t_p} & l. \end{array}$$

Similarly, the value (3.10b) arises from the decomposition of the three-fold tensor product

$$\begin{array}{cccc} X_r^{(1)} & \oplus\ X_r^{(1)} & \oplus\ Y_{r'}^{(1)} & \supset X_r^{(1)} \\ l - s' - 1 & s' & 1 & l \end{array}$$

where $s' = (s - s_0)/t_p$ and $Y_{r'}^{(1)} \supset X_r^{(1)}$ is specified for each non-simply-laced algebra as follows:

$$X_r^{(1)} \subset Y_{r'}^{(1)},$$

$$B_r^{(1)} \subset D_{r+1}^{(1)},$$

$$C_r^{(1)} \subset A_{2r-1}^{(1)},$$

$$F_4^{(1)} \subset E_6^{(1)},$$

$$G_2^{(1)} \subset B_3^{(1)}.$$

Remark 3.2. The following properties (3.11) and (3.12) have enabled us to present the results (3.9) and (3.10) in a few fairly unified formulas,

$$\begin{aligned} \mathscr{L}(\mathrm{A}_1^{(1)}, l) &= \mathscr{L}(\mathrm{C}_1^{(1)}, l), \\ \mathscr{L}(\mathrm{B}_2^{(1)}, l) &= \mathscr{L}(\mathrm{C}_2^{(1)}, l), \\ \mathscr{L}(\mathrm{A}_3^{(1)}, l) &= \mathscr{L}(\mathrm{D}_3^{(1)}, l), \\ 2\mathscr{L}(\mathrm{A}_1^{(1)}, l) &= \mathscr{L}(\mathrm{D}_2^{(1)}, l). \end{aligned} \tag{3.11}$$

These are just the consequences of the equivalence relations among the classical Lie algebras X_r. On the other hand, when $l = 1$, we have

$$\begin{aligned} &\mathscr{L}(\mathrm{X}_r^{(1)}, 1) = 0 \quad \text{for } \mathrm{X}_r^{(1)} = \mathrm{A}_r^{(1)}(r \geqslant 0), \quad \mathrm{D}_r^{(1)}(r \geqslant 2) \quad \text{and} \quad \mathrm{E}_{6,7,8}^{(1)}, \\ &\mathscr{L}(\mathrm{B}_r^{(1)}, 1) = \mathscr{L}(\mathrm{A}_1^{(1)}, 2) = \tfrac{1}{2}, \quad r \geqslant 2, \\ &\mathscr{L}(\mathrm{C}_r^{(1)}, 1) = \mathscr{L}(\mathrm{A}_{r-1}^{(1)}, 2) = \frac{r(r-1)}{r+2}, \quad r \geqslant 1, \\ &\mathscr{L}(\mathrm{F}_4^{(1)}, 1) = \mathscr{L}(\mathrm{A}_2^{(1)}, 2) = \tfrac{6}{5}, \\ &\mathscr{L}(\mathrm{G}_2^{(1)}, 1) = \mathscr{L}(\mathrm{A}_1^{(1)}, 3) = \tfrac{4}{5}. \end{aligned} \tag{3.12}$$

From (3.8) and (A.1a), both sides of these identities are expressed by the sum $\sum_{(a,m)\in \mathrm{G}} L(f_m^{(a)})$ in which the corresponding summation domain G's possess the identical tableau visualization.

3.2. AGREEMENT OF THE CENTRAL CHARGES WITH KNOWN CASES

Let us check the agreement of the results (3.9) and (3.10) with the known (and the conjecturally known) values of c which are mainly implied from the one-point function calculations [16–19,25,33]. From these works, we shall quote the central charges in a slightly loose sense in that the GKO–Virasoro character [32] emerges as the one-dimensional configuration sum (1d sum) (cf. refs. [16–19]). The so-called regime II to I (respectively III to IV) transition line therein is to be compared with the regime $\epsilon = +1$ (respectively $\epsilon = -1$) here and by abuse of terminologies, we will refer to them for some models without specifications.

The case $X_r^{(1)} = A_r^{(1)}$. The expressions (3.9b) and (3.10a) transform into each other by the interchanges $r + 1 \leftrightarrow p$, $s \leftrightarrow p$ [7]. Moreover, for all the known 1d sum results so far, the generalized level–rank duality is known to hold [34,35], which implies the above symmetry. Thus we shall exclusively consider the regime $\epsilon = -1$.

Unifying the prior established results, l general, $r=p=s=1$ [33], l, s general, $r=p=1$ [16] and l, r general, $s=p=1$ [17], the 1d sum for general l, r, s $(p=1)$ had been conjectured [8] as the branching function for the coset pair

$$\begin{array}{lccccc} & \mathrm{A}_r^{(1)} & \oplus & \mathrm{A}_r^{(1)} & \supset & \mathrm{A}_r^{(1)} \\ \text{level} & l-s & & s & & l \end{array} \tag{3.13}$$

in agreement with the central charge (3.10a) $(p=1)$. It was later verified [19] that the proposed 1d sum [8] indeed leads to the above pair.

The cases $X_r^{(1)}=B_r^{(1)}$ *and* $D_r^{(1)}$. The level-l $\mathrm{X}_r^{(1)}$ RSOS models corresponding to $s=p=1$ were studied [18] in regime III and the 1d sums have been shown to become the branching function for the pair

$$\begin{array}{lccccc} & \mathrm{X}_r^{(1)} & \oplus & \mathrm{X}_r^{(1)} & \supset & \mathrm{X}_r^{(1)} \\ \text{level} & l-1 & & 1 & & l \end{array} \tag{3.14}$$

The resulting central charges realize (3.10a) with $s=p=1$.

The case $X_r^{(1)}=C_r^{(1)}$. For the level-l $\mathrm{C}_r^{(1)}$ RSOS models with $s=p=1$, the 1d sum relevant to the one-point function in regime II is known [18] as the branching function for the pair

$$\begin{array}{lccccc} & \mathrm{C}_l^{(1)} & \oplus & \mathrm{A}_{2l-1}^{(1)} & \supset & \mathrm{C}_l^{(1)} \\ \text{level} & r-1 & & 1 & & r \end{array} \tag{3.15}$$

which yields $c=2l-1-l(l+1)(2l+1)/(r+l)(r+l+1)$. A little manipulation shows that this coincides with $\mathscr{L}(\mathrm{C}_r^{(1)},l)-\mathscr{L}(\mathrm{C}_{r-1}^{(1)},l)$ in agreement with (3.9d).

The cases $X_r^{(1)}=E_{6,7,8}^{(1)}$. The central charge (3.10a) corresponds to the coset pair $\mathrm{E}_k^{(1)}\oplus\mathrm{E}_k^{(1)}\supset\mathrm{E}_k^{(1)}$ $(k=6,7,8)$ with the levels $(l-1)+1=l$, which yield the proposed values by Zamolodchikov [36].

The cases $X_r^{(1)}=G_2^{(1)}$. The 1d sum for the vertex model corresponding to $s=1$, $p=2$ is known to become the $\mathrm{G}_2^{(1)}$ string function on level-1 $\mathrm{B}_3^{(1)}$ modules viewed as the $\mathrm{G}_2^{(1)}$ modules via the embedding $\mathrm{G}_2^{(1)}\hookrightarrow\mathrm{B}_3^{(1)}$ [25]. The RSOS model for $s=1$, $p=2$ (trigonometric) has been built in ref. [10] with a conjectural elliptic extention. In view of these and the analogy from the $\mathrm{C}_r^{(1)}$ case, we conjecture here that the proposed elliptic version exists and its 1d sum in regime III becomes the branching function for the pair

$$\begin{array}{lccccc} & \mathrm{G}_2^{(1)} & \oplus & \mathrm{B}_3^{(1)} & \supset & \mathrm{G}_2^{(1)} \\ \text{level} & l-1 & & 1 & & l \end{array} \tag{3.16}$$

which yields $c=\frac{7}{2}(1-16/(l+3)(l+4))$. This is indeed equal to (3.10b) with $s=1$ and $p=2$. The first half of the conjecture is actually valid for $l=1$ when the $\mathrm{G}_2^{(1)}$

RSOS model reduces to the hard hexagon model [33]. In this case one also knows that $c = \frac{4}{5}$ in regime II, which again agrees with the $p = 2$ of (3.9j).

4. Discussions

4.1. SUMMARY

In this paper, we have studied the thermodynamics of the Bethe ansatz system (2.6) labelled by an arbitrary non-twisted affine Lie algebra $X_r^{(1)}$ and the three integer parameters l, s, p satisfying (2.4). The results are relevant to the critical level-l $X_r^{(1)}$ restricted solid-on-solid (RSOS) model with type $s\omega_p$ fusion whose basic features are roughly sketched in remark 2.5. Besides the known examples [8–10], such RSOS models are yet to be constructed. Our main result in the list of the central charges (3.9) and (3.10) in two critical regimes. They are compared with the several known values in subsect. 3.2 with perfect agreement. For the simply-laced algebras $X_r^{(1)} = A_r^{(1)}$, $D_r^{(1)}$ and $E_{6,7,8}^{(1)}$, these results reproduce those in refs. [6,7].

As a calculational device, we have found it especially clarifying to use the tableaux that visualize the domain $G = \{(a, m) \mid 1 \leqslant a \leqslant r,\ 1 \leqslant m \leqslant t_a l - 1\}$ corresponding to the summation $\sum_{(a,m) \in G} L(f_m^{(a)})$. With their aid, one can easily observe how the central charges become various combinations of the WZW values ($-$rank) $\mathscr{L}(X_{r'}^{\prime(1)}, l')$ (3.8) for several subalgebras and levels $(X_{r'}^{\prime(1)}, l')$.

In the course of the derivation, we have made three crucial assumptions. The first one is the hypothesis (2.7) on the solution of the Bethe equation (2.6) in the thermodynamic limit. The second one is the form (2.19) of the thermal energy, which is a natural extension of the one in ref. [7] for $A_r^{(1)}$. As the third one, we have formulated a generalized conjecture (A.1) on the Rogers dilogarithmic function for an arbitrary pair $(X_r^{(1)}, l)$. For the simply-laced cases $X_r^{(1)} = A_r^{(1)}$, $D_r^{(1)}$ and $E_{6,7,8}^{(1)}$, it coincides with the earlier one [12]. Besides being an indispensable tool in the central charge calculation, it appears to be of intrinsic mathematical interest. In the appendix, we have also raised a few results and conjectures (A.10)–(A.14) on the explicit form of the important entry $f_m^{(a)}$ (A.1c, c′) of the dilogarithmic function. They are more conveniently presented by passing to the quantity $Q_m^{(a)}$ as in (A.7). Then the $Q_m^{(a)}$, which may be viewed as a "yangian character", has been shown to appear in the high-temperature asymptotics of the entropy (2.30). The conjecture (A.1) is indeed valid for $X_r^{(1)} = A_r^{(1)}$ [12]. (See note added.)

4.2. FURTHER EXTENTIONS

Let us discuss possible generalizations of the analysis in the present paper.

The first natural direction would be to study the Bethe ansatz systems related with the twisted affine Lie algebras $X_r^{(k)} = A_{2r-1}^{(2)}$, $A_{2r}^{(2)}$, $D_{r+1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$ and the associated RSOS models. In ref. [11], such models have been actually constructed for $A^{(2)}_{2r}$ and $A^{(2)}_{2r-1}$. An intriguing feature therein is, the level-l $A^{(2)}_{2r-1}$ RSOS model [11] turns out to be the level–rank duality partner of the level-r $C^{(1)}_l$ RSOS model in ref. [9]. As observed for the duality pair $(A^{(1)}_{r-1}, l, s\omega_p, \epsilon) \leftrightarrow (A^{(1)}_{l-1}, r, p\omega_s, -\epsilon)$ [7,34], it may be interesting to study how the two apparently different Bethe equations lead to the same result for $(A^{(2)}_{2r-1}, l) \leftrightarrow (C^{(1)}_l, r)$.

The second possibility is to consider the Bethe equations related to the "non-unitary RSOS" models. Roughly speaking, they correspond to the specialization $q = \exp[2\pi i t/(l+g)]$ with $t > 1$ being an integer coprime with $l+g$. In ref. [37], possible universality classes of such $A^{(1)}_r$ RSOS models have been proposed in terms of the non-unitary coset models. In particular, their central charges are combinations of the WZW values at fractional levels. Thus, one might hope that the analysis of the corresponding Bethe equations may lead to a further generalization of the conjectured dilogarithm identity (A.1).

As the third direction of possible extentions, one may analyze the Bethe equation (2.6) with $s\omega_p$ replaced by $s_1\omega_1 + \dots + s_r\omega_r$, where s_a is an integer in the range $0 \leqslant s_a \leqslant t_a l - 1$. It would correspond to considering the more general type of fusion RSOS models. In fact, such examples have been built in ref. [8] for the $A^{(1)}_r$ case. To avoid a technical difficulty, let us exclusively consider the situation that for all non-zero s_a the (inverse square) root length t_a are equal. Then, leaving several justifications aside, one can proceed in a fairly parallel way as in sects. 2 and 3. In particular, the formula (2.26), the $g^{(a)}_m(\infty)$ (3.2) and the equation (3.6) formally remain unchanged. The crucial change is the left-hand side of (3.4) into $\delta_{s_a n}$. It governs the structure of the "hatched region" G_ϵ hence $g^{(a)}_m(-\infty)$ and ultimately the central charge c (2.26). One can roughly see how (3.4) changes so by tracing the following chain; the completion number (2.5); $N_a = N\sum_{p=1}^r (C^{-1})_{ap} s_p$, the left-hand side of (2.8a); $\phi_{a,m}(u, s_a/t_a)$, the left-hand side of (2.17); $\delta_{s_a n}/2\mathrm{ch}(x/t_a)$, the thermal energy (2.19); $\mathscr{E} = (\epsilon/2\pi)\sum_{p=1}^r \sum_{m=1}^{t_p l} \int_{-\infty}^{\infty} \mathrm{d}x\, \hat{A}^{(L)s_p m}_{pp}(x) \hat{\rho}^{(p)}_m(x)$, the left-hand side of (2.22a); $\epsilon t_a \delta_{s_a n}/4\mathrm{ch}(t_a \pi u/2)$, the left-hand side of (3.1); $\frac{1}{2}\epsilon t_a \delta_{s_a n}\, \mathrm{e}^{\mp t_a \pi\infty/2}$. Noting the assumption that the t_a are all equal for non-zero s_a, we are led to modify the left-hand side of (3.4) into $\delta_{s_a n}$. Let $G^{(s,p)}_\epsilon$ denote the G_ϵ specified in (3.5). (We warn the reader against the confusion of this with the $G(s, p)$ defined in (3.5c).) Determine the new G_ϵ from the modified (3.4) by the same rule $h^{(a)}_m > 0 \leftrightarrow (a, m) \in G_\epsilon$ as before. Then the result simply reads

$$G_\epsilon = \cup_{p=1}^r G^{(s_p, p)}_\epsilon. \tag{4.1}$$

Namely, the "hatched region" in the generalized situation $s_1\omega_1 + \dots + s_r\omega_r$ is just the union of "fundamental cases" $s_p\omega_p$. See figs. 3a, b for the examples. Having known the "hatched region", one can easily find the solution of (3.6) in terms of the $f^{(a)}_m$ for some $(X^{(1)}_r, l)$. The rest of the calculation is straightforward and we shall present the final result.

Fig. 3. (a) The set G and G_+ (hatched) for $C_7^{(1)}$, $l = 5$, $1 \leqslant s_2,\ s_5 \leqslant 9$, $s_i = 0$ for $i \neq 2, 5$. The three non-hatched regions respectively correspond to $(X_{r'}^{\prime(1)}, l') = (A_1^{(1)}, 10)$, $(A_2^{(1)}, 10)$ and $(C_2^{(1)}, 5)$ from left to right. (b) The sets G and G_- (hatched) for $C_7^{(1)}$, $l = 5$, $\{s_1, s_2, \dots, s_6\} = \{0, 0, 0, 0, 2, 7\}$ (modulo order), $s_7 = 0$. The three non-hatched regions respectively correspond to $(X_{r'}^{\prime(1)}, l') = (C_7^{(1)}, 1)$, $(C_7^{(1)}, 2)$ and $(C_7^{(1)}, 1)$ from top to bottom.

Regime $\epsilon = +1$. Using the notations in (3.9a), we have

$$c = \mathscr{L}\big(X_r^{(1)}, l\big) - \sum_{X'_{r'} \in \mathscr{H}_s} \mathscr{L}\big(X_{r'}^{\prime(1)}, l'\big),$$

$$l'/l = \min\{t_a \mid a \in \mathscr{D}(X'_{r'})\}. \tag{4.2}$$

Here, for $s = (s_1, \dots, s_r)$, the symbol $\mathscr{H}_s$ stands for the set of subalgebras $X'_{r'}$ of X_r whose $\mathscr{D}(X'_{r'})$ are obtained by removing the ath node from the $\mathscr{D}(X_r)$ for all the a's such that $s_a > 0$. The formula (3.9a) corresponds to the choice $s_a = s\delta_{pa}$.

Regime $\epsilon = -1$. The central charges are easily obtainable from (4.1). However, the presentation of the general result will require tedious notations to count the overlapping of the hatched regions, which we find not necessarily essential. Here we shall only provide the result for the case $s_a/t_a \in \mathbb{Z}$ for all $1 \leqslant a \leqslant r$. Given such s_a/t_a's, define the strictly increasing sequence of integers $0 = k_0 < 1 \leqslant k_1 < \dots < k_m < k_{m+1} = l$ for some $1 \leqslant m \leqslant r$ by saying that $k_i (1 \leqslant i \leqslant m)$ is the ith smallest non-zero integer in the set $\{s_a/t_a \mid 1 \leqslant a \leqslant r\}$. Setting $k_{i,j} = k_i - k_j$, the central charge is given by

$$c = \mathscr{L}\big(X_r^{(1)}, k_{1,0}\big) + \dots + \mathscr{L}\big(X_r^{(1)}, k_{m+1,m}\big) - \mathscr{L}\big(X_r^{(1)}, l\big) + mr, \tag{4.3}$$

which corresponds to the coset pair

$$\begin{array}{lcccc} & X_r^{(1)} \oplus & \cdots & \oplus\, X_r^{(1)} & \supset X_r^{(1)} \\ \text{level} & k_{1,0} & \cdots & k_{m+1,m} & l. \end{array} \tag{4.4}$$

The formula (3.10a) corresponds to the choice $s_a = s\delta_{pa}$, $s/t_p \in \mathbb{Z}$. Remarkably, for $X_r^{(1)} = A_r^{(1)}$, the above coset pair precisely realizes the conjecture proposed in ref. [38] (see also ref. [34]) based on computer experiments on the one point functions.

Finally, it should be interesting to study the scattering theories relevant to the off-critical RSOS models. See e.g. refs. [39,40] for the recent development in the thermodynamic Bethe ansatz approach in this direction. We hope that the generality considered in this paper will carry over to such theories.

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Note added

After completing the manuscript, the author was informed from A.N. Kirillov [41] that he had noticed that eq. (7) of ref. [12] needs to be replaced by (A.1) of the present paper for non-simply-laced algebras. In addition, the author learned that Kirillov now has the proof of (A.1) for $X_r^{(1)} = C_r^{(1)}$ and $D_r^{(1)}$.

The Bethe equation (2.6) is associated with the quantum groups in the sense of ref. [1]. Some interrelations between a modified Bethe equation and the quantum group invariant spin chain hamiltonian have been noted in ref. [42].

Appendix A. Conjecture on the Rogers dilogarithmic function identity for the pair $(X_r^{(1)}, l)$

Let $X_r^{(1)}$ be an arbitrary non-twisted affine Lie algebra and fix an integer $l \geqslant 1$. Denote the Cartan matrix of the classical part X_r by $C^{(X_r)}$, the dual Coxeter number by g and define the integer $t_a = 1$, 2 or 3 $(1 \leqslant a \leqslant r)$ as in (2.2). For each pair $(X_r^{(1)}, l)$ we propose

Conjecture 1.

$$\frac{6}{\pi^2} \sum_{a=1}^{r} \sum_{m=1}^{t_a l-1} L\left(f_m^{(a)}\right) = \frac{l \dim X_r}{l+g} - r, \tag{A.1a}$$

$$L(x) = -\frac{1}{2} \int_0^x \left(\frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right) \mathrm{d}y. \tag{A.1b}$$

The entry $0 < f_m^{(a)} < 1$ in (A.1a) is the solution of the simultaneous equations

$$\sum_{m=1}^{t_a l-1} C_{jm}^{(\mathrm{A}_{t_a l-1})} \log f_m^{(a)} = \sum_{b=1}^{r} \Bigg(C_{ba}^{(\mathrm{X}_r)} \log\left(1 - f_{t_b j/t_a}^{(b)}\right) + C_{ab}^{(\mathrm{X}_r)} \sum_{n=1}^{t_b/t_a - 1} n \log\left(\left(1 - f_{t_b(j-1)/t_a + n}^{(b)}\right)\left(1 - f_{t_b(j+1)/t_a - n}^{(b)}\right)\right)\Bigg), \tag{A.1c}$$

for $1 \leqslant a \leqslant r$, $1 \leqslant j \leqslant t_a l - 1$, where, by convention, $f_m^{(a)} = 0$ if $m \notin \mathbb{Z}$ and $\Sigma_{n=1}^{t_b/t_a - 1} = 0$ on the right-hand side when $t_a \geqslant t_b$.

As mentioned in sect. 1, (A.1) coincides with the original conjecture by Kirillov [12] if $\mathrm{X}_r^{(1)} = \mathrm{A}_r^{(1)}$, $\mathrm{D}_r^{(1)}$ and $\mathrm{E}_{6,7,8}^{(1)}$, where one has $\forall t_a = 1$, hence (A.1c) reduces to an especially simple form,

$$\sum_{m=1}^{l-1} C_{jm}^{(\mathrm{A}_{l-1})} \log f_m^{(a)} = \sum_{b=1}^{r} C_{ab}^{(\mathrm{X}_r)} \log\left(1 - f_j^{(b)}\right). \tag{A.1d}$$

So far, the proof has been obtained for $\mathrm{X}_r^{(1)} = \mathrm{A}_r^{(1)}$ in ref. [12]. (See note added.) In addition, the case $l = 1$ can be verified easily for an arbitrary $\mathrm{X}_r^{(1)}$ by reducing it to $\mathrm{A}_{r'}^{(1)}$ for some r' by means of (3.8) and (3.12). We remark that (A.1c) has arisen from the Bethe ansatz calculation in sects. 2 and 3 and in fact is equivalent to

$$\sum_{m=1}^{t_a l-1} \hat{K}_a^{jm}(0) \log f_m^{(a)} = \sum_{b=1}^{r} \sum_{k=1}^{t_b l-1} \hat{J}_{ab}^{jk}(0) \log\left(1 - f_k^{(b)}\right), \tag{A.1c$'$}$$

where the functions $\hat{K}_a^{mj}(x)$ and $\hat{J}_{ab}^{jk}(x)$ are defined by (2.11) and (2.18), respectively. Our main purpose here is to characterize the quantity $f_m^{(a)}$ in terms of the q-dimension of the X_r modules at $q = \exp(2\pi i/(l+g))$. The description is partly based on the finite-dimensional representation theory of yangians [20]. As for the dilogarithmic functions, we refer to ref. [15].

A.1. QUANTITY $f_m^{(a)}$ FOR THE PAIR $(\mathrm{X}_r^{(1)}, l)$

Hereafter we shall write the Cartan matrix of X_r simply as C and use the notations (2.1) and (2.2) for the simple root α_a, the fundamental weight ω_a and the integer $t_a (1 \leqslant a \leqslant r)$.

Consider the following recursion relation for the infinitely many variables $\{Q_m^{(a)} \mid 1 \leqslant a \leqslant r,\ m \geqslant -1\}$:

$$Q_m^{(a)2} = Q_{m+1}^{(a)} Q_{m-1}^{(a)} + \prod_{\substack{b=1 \\ b \neq a,\ C_{ab} \neq 0}} \prod_{j=0}^{-C_{ab}-1} Q_{[(C_{ba}m-j)/C_{ab}]}^{(b)} \quad \text{for } 1 \leqslant a \leqslant r,\ m \geqslant 0, \qquad \text{(A.2a)}$$

$$Q_0^{(a)} = 1, \quad Q_{-1}^{(a)} = 0 \quad \text{for } 1 \leqslant a \leqslant r, \qquad \text{(A.2b)}$$

where the symbol $[x]$ stands for the greatest integer not exceeding x. A remarkable solution to this equation is known in terms of the character of the finite-dimensional X_r modules [20],

$$Q_m^{(a)} = \sum_{\boldsymbol{n}} Z(a, m, \boldsymbol{n}) \chi(\omega_{a,m}(\boldsymbol{n})), \quad \boldsymbol{n} = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geqslant 0}^{\otimes r}, \qquad \text{(A.3a)}$$

$$\omega_{a,m}(\boldsymbol{n}) = m\omega_a - \sum_{b=1}^{r} n_b \alpha_b, \qquad \text{(A.3b)}$$

$$Z(a, m, \boldsymbol{n}) = \sum_{\nu} \prod_{b=1}^{r} \prod_{k=1}^{\infty} \binom{\mathscr{P}_k^{(b)}(\nu) + \nu_k^{(b)}}{\nu_k^{(b)}}, \qquad \text{(A.3c)}$$

$$\mathscr{P}_k^{(b)}(\nu) = \min(m, k)\delta_{ab} - 2\sum_{j \geqslant 1} \min(k, j)\nu_j^{(b)} + \sum_{\substack{c=1 \\ c \neq b}}^{r} \sum_{j \geqslant 1} \min(-kC_{cb}, -jC_{bc})\nu_j^{(c)}, \qquad \text{(A.3d)}$$

wherein (A.3c), the symbol ( ) means the binomial coefficient and the sum extends over all possible decompositions $\{\nu_k^{(b)} \mid n_b = \sum_{k=1}^{\infty} k\nu_k^{(b)},\ \nu_k^{(b)} \in \mathbb{Z}_{\geqslant 0},\ 1 \leqslant b \leqslant r,\ k \geqslant 1\}$ such that $\mathscr{P}_k^{(b)}(\nu) \geqslant 0$ for all $1 \leqslant b \leqslant r$ and $k \geqslant 1$. The quantity $\chi(\omega)$ in (A.3a) denotes the character of the irreducible X_r module $V(\omega)$ with highest weight ω and therefore the above $Q_m^{(a)}$ is a (Laurent) polynomial solution with r variables. As argued in ref. [20], one may consider that there underlies the yangian $\mathscr{Y}(X_r) \supset X_r$ and its finite-dimensional module $W(a, m)$ that decomposes as

$$W(a, m) = \oplus_{\boldsymbol{n}} Z(a, m, \boldsymbol{n}) V(\omega_{a,m}(\boldsymbol{n})) = V(m\omega_a) \oplus \ldots, \qquad \text{(A.4)}$$

as a X_r module, hence the $Q_m^{(a)}$ corresponds to its character. Here, the latter equality means that the $W(a, m)$ contains the irreducible X_r module $V(m\omega_a)$ as a "top term" with multiplicity one. The explicit form of $Q_m^{(a)}$ is available in (A.9)–

(A.14) for all the classical series $X_r = A_r$, B_r, C_r and D_r as well as some results and conjectures on exceptional Lie algebras.

Next introduce the integer $l \geqslant 1$ and specialize the character $\chi(\omega)$ into the q-dimension as

$$\chi(\omega) = \prod_{\alpha \in \Delta_+} \frac{q^{(\alpha | \omega+\rho)/2} - q^{-(\alpha|\omega+\rho)/2}}{q^{(\alpha|\rho)/2} - q^{-(\alpha|\rho)/2}}, \quad q = \exp\left(\frac{2\pi i}{l+g}\right), \tag{A.5}$$

where Δ_+ denotes the set of positive roots of X_r, $\rho = \frac{1}{2}\Sigma_{\alpha \in \Delta_+}\alpha$ and g is the dual Coxeter number. Then we have

Conjecture 2. Under the above specialization (A.5), the $Q_m^{(a)}$ in (A.3) has the properties

$$Q_m^{(a)} = Q_{t_a l - m}^{(a)} \quad \text{for } 0 \leqslant m \leqslant t_a l \tag{A.6a}$$

$$Q_m^{(a)} < Q_{m+1}^{(a)} \quad \text{for } 0 \leqslant m \leqslant \left[\frac{t_a l}{2}\right] - 1, \tag{A.6b}$$

for all $1 \leqslant a \leqslant r$.

Supported by numerical tests, we hereafter assume this. From (A.2b) $Q_0^{(a)} = 1$, therefore conjecture 2 implies $Q_m^{(a)} \geqslant 1$ for all $1 \leqslant a \leqslant r, 0 \leqslant m \leqslant t_a l$. Now the solution $f_m^{(a)}$ to the equation (A.1c) is given as follows:

$$f_m^{(a)} = 1 - \frac{Q_{m+1}^{(a)} Q_{m-1}^{(a)}}{Q_m^{(a)2}}, \quad 1 \leqslant a \leqslant r, \quad 1 \leqslant m \leqslant t_a l - 1, \tag{A.7}$$

where the $Q_m^{(a)}$ means the one under the specialization (A.5). Admitting (A.6), each term in the recursion relation (A.2a) is strictly positive for $1 \leqslant m \leqslant t_a l - 1$, hence we have $0 < f_m^{(a)} < 1$ indeed for these m's. Finally (A.1c) can be proved just by combining (A.2) and (A.7).

Remark. One can show $Q_{t_a l+1}^{(a)} = 0$ by using (A.2a) with $m = t_a l$ and $Q_{t_a l}^{(a)} = Q_0^{(a)} = 1$ from (A.6a). Thus the $f_m^{(a)}$ in (A.7) naturally extends to the range $0 \leqslant m \leqslant t_a l$ as $f_0^{(a)} = f_{t_a l}^{(a)} = 1$. Noting that $L(1) = \pi^2/6$, the conjecture (A.1a) can also be written as

$$\frac{6}{\pi^2} \sum_{a=1}^{r} \sum_{m=1}^{t_a l} L(f_m^{(a)}) = \frac{l \dim X_r}{l+g}. \tag{A.8}$$

A.2. EXPLICIT FORM OF $Q_m^{(a)}$

It is possible to write down the explicit form of the $Q_m^{(a)}$ $(m \geqslant 0)$ for the classical series $X_r = A_r$, B_r, C_r and D_r (cf. ref. [20]),

$X_r = A_r$;

$$Q_m^{(a)} = \chi(m\omega_a), \quad 1 \leqslant a \leqslant r, \tag{A.9a}$$

$X_r = C_r$;

$$Q_m^{(a)} = \begin{cases} \sum \chi(k_1\omega_1 + k_2\omega_2 + \ldots + k_a\omega_a), & 1 \leqslant a \leqslant r-1, \\ \chi(m\omega_r), & a = r, \end{cases} \tag{A.9b}$$

$X_r = B_r$ and D_r;

$$Q_m^{(a)} = \sum \chi(k_{a_0}\omega_{a_0} + k_{a_0+2}\omega_{a_0+2} + \ldots + k_a\omega_a), \quad 1 \leqslant a \leqslant r', \tag{A.9c}$$

$$r' = \begin{cases} r & \text{for } B_r \\ r-2 & \text{for } D_r \end{cases} \quad a_0 \equiv a \bmod 2, \quad a_0 = 0 \text{ or } 1, \tag{A.9d}$$

$$Q_m^{(a)} = \chi(m\omega_a), \quad a = r-1,\, r \quad \text{only for } D_r, \tag{A.9e}$$

where by convention $\omega_0 = 0$.

The sum in (A.9b) is taken over non-negative integers $k_1, k_2, \ldots, k_a$ that satisfy $k_1 + k_2 + \ldots + k_a \leqslant m$, $k_j \equiv m\delta_{ja}$ mod 2 for all $1 \leqslant j \leqslant a$. The sum in (A.9c) extends over non-negative integers $k_{a_0}, k_{a_0+2}, \ldots, k_a$ obeying the constraint $t_a(k_{a_0} + k_{a_0+2} + \ldots + k_{a-2}) + k_a = m$.

For the exceptional algebras $X_r = E_{6,7,8}$, F_4 and G_2, an explicit formula like (A.9) is not known so far in general. Here we present some conjectures and the results for a few simple cases. They serve as the "initial condition" that allows one to uniquely determine all the $Q_m^{(a)}$ by the recursion relation (A.2a). Practically, using the recursion relation is much simpler than the direct calculation of (A.3) in order to obtain the $Q_m^{(a)}$ and the $f_m^{(a)}$ numerically, hence to check the dilogarithm identity (A.1a).

$X_r = E_6$ (Conjecture except for small values of m);

$$Q_m^{(1)} = \chi(m\omega_1), \tag{A.10a}$$

$$Q_m^{(2)} = \sum_{j=0}^{m} \chi(j\omega_2 + (m-j)\omega_5), \tag{A.10b}$$

$$Q_m^{(3)} = \sum (j_4+1)(\min(m-j_1-2j_2-j_3-j_4, j_3)+1)$$
$$\times \chi(j_1(\omega_1+\omega_5)+j_2(\omega_2+\omega_4)+j_3\omega_3+j_4\omega_6), \qquad \text{(A.10c)}$$

$$Q_m^{(6)} = \sum_{j=0}^{m} \chi(j\omega_6), \qquad \text{(A.10d)}$$

where the sum in (A.10c) is taken over non-negative integers $j_1, \dots, j_4$ under the condition $j_1+2j_2+j_3+j_4 \leqslant m$. The other $Q_m^{(a)}$ are obtained from $Q_m^{(6-a)}$ by the replacement $\omega_a \leftrightarrow \omega_{6-a}$ $(1 \leqslant a \leqslant 5)$, which is consistent with the order-2 symmetry of the Dynkin diagram. The conjecture (A.10c) have been checked for $1 \leqslant m \leqslant 5$.

$X_r = E_7$ (Conjecture except for small values of m);

$$Q_m^{(1)} = \sum_{j=0}^{m} \chi(j\omega_1), \qquad \text{(A.11a)}$$

$$Q_m^{(5)} = \sum_{\substack{j_1, j_2 \geqslant 0 \\ j_1+j_2 \leqslant m}} \chi(j_1\omega_1+j_2\omega_5), \qquad \text{(A.11b)}$$

$$Q_m^{(6)} = \chi(m\omega_1), \qquad \text{(A.11c)}$$

$$Q_m^{(7)} = \sum_{j=0}^{m} \chi((m-j)\omega_6+j\omega_7). \qquad \text{(A.11d)}$$

In addition, the following forms are derivable from (A.3):

$$Q_1^{(2)} = 1+2\chi(\omega_1)+\chi(\omega_2)+\chi(\omega_5), \qquad \text{(A.11e)}$$

$$Q_1^{(3)} = 2+4\chi(\omega_1)+3\chi(\omega_2)+\chi(2\omega_6)+\chi(\omega_3)$$
$$+\chi(2\omega_1)+4\chi(\omega_5)+\chi(\omega_1+\omega_5)+2\chi(\omega_6+\omega_7), \qquad \text{(A.11f)}$$

$$Q_1^{(4)} = \chi(\omega_4)+2\chi(\omega_6)+2\chi(\omega_7)+\chi(\omega_1+\omega_6). \qquad \text{(A.11g)}$$

For E_8, we regard (A.2a) as an evolution equation with respect to the a-index. Then the following information is sufficient as the initial condition:

$X_r = E_8$ (Conjecture except for small values of m);

$$Q_m^{(1)} = \sum_{j=0}^{m} \chi(j\omega_1), \qquad \text{(A.12a)}$$

$$Q_m^{(7)} = \sum_{\substack{j_1, j_2 > 0 \\ j_1+j_2 \leqslant m}} \chi(j_1\omega_1+j_2\omega_7). \qquad \text{(A.12b)}$$

The consistency of these relations has been numerically checked for the levels $2 \leqslant l \leqslant 19$. For F_4, we have only derived the initial condition

$$X_r = F_4;$$

$$Q_1^{(1)} = 1 + \chi(\omega_1), \tag{A.13a}$$

$$Q_1^{(2)} = 1 + 2\chi(\omega_1) + \chi(\omega_2) + \chi(2\omega_4), \tag{A.13b}$$

$$Q_1^{(3)} = \chi(\omega_3) + \chi(\omega_4), \tag{A.13c}$$

$$Q_1^{(4)} = \chi(\omega_4), \tag{A.13d}$$

Finally, for G_2 we have

$$X_r = G_2 \quad \text{(Conjecture except for small values of } m\text{)};$$

$$Q_m^{(1)} = \sum_{j=0}^{m} \chi(j\omega_1), \tag{A.14a}$$

in addition to a few results on $Q_m^{(2)}$

$$Q_1^{(2)} = \chi(\omega_2), \tag{A.14b}$$

$$Q_2^{(2)} = \chi(\omega_2) + \chi(2\omega_2), \tag{A.14c}$$

$$Q_3^{(2)} = 1 + 2\chi(\omega_1) + \chi(2\omega_2) + \chi(3\omega_2), \tag{A.14d}$$

$$Q_4^{(2)} = \chi(\omega_2) + \chi(2\omega_2) + \chi(3\omega_2) + \chi(4\omega_2) + 2\chi(\omega_1 + \omega_2). \tag{A.14e}$$

Throughout the formulas (A.9)–(A.14), $Q_m^{(a)}$ contains the term $\chi(m\omega_a)$ with the coefficient 1 in agreement with (A.4).

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