Integrable Markov process, matrix products and the tetrahedron equation

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Non-equilibrium statistical mechanics

Stochastic dynamics,
Markov process, …

Integrable systems

Quantum groups,
Yang-Baxter equation, …

Integrable Markov process

Spectral problem of the Markov matrix: solvable by Bethe ansatz
Exact asymptotic analysis: connection to random matrices, etc.
Stationary states: matrix product structure (Today’s topic)

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Stationary states: matrix product structure (Today's topic)

Prototype examples

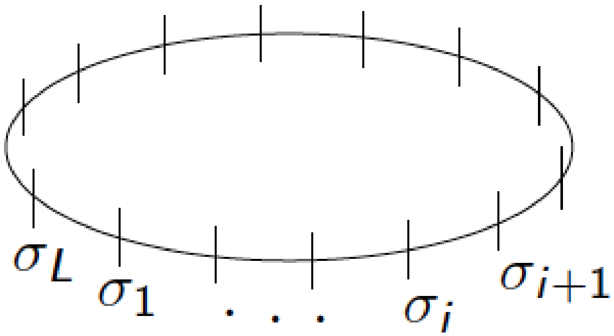
Totally asymmetric simple exclusion process (TASEP) ← today
Totally asymmetric zero range process (TAZRP) ← Tuesday -- Friday

Key feature

Hidden 3D structure in the matrix product related to the tetrahedron equation (= 3D generalization of the Yang-Baxter equation) which becomes manifest in multispecies versions of TASEP and TAZRP.

(K-Maruyama-Okado, 2015, 2016)

Totally Asymmetric Simple Exclusion Process (TASEP)



1D periodic chain with L sites

$\sigma_i \in \{0, 1, \ldots, n\}$ (n-TASEP)

Stochastic dynamics

$$(\sigma_i, \sigma_{i+1}) \to (\sigma'_i, \sigma'_{i+1})$$

$$(\alpha, \beta) \to (\beta, \alpha) \quad \text{if } \alpha > \beta$$

Master equation

$$\frac{d}{dt}|P\rangle = H|P\rangle, \quad |P\rangle = \sum_{\{\sigma_i\}} \mathbb{P}(\sigma_1, \ldots, \sigma_L)|\sigma_1, \ldots, \sigma_L\rangle \in (\mathbb{C}^{n+1})^{\otimes L}$$

$$H = \sum_{i \in \mathbb{Z}_L} h_{i.i+1}, \quad h_{i,i+1} = 1 \otimes \cdots \otimes 1 \otimes \overset{i,i+1}{h} \otimes 1 \otimes \cdots \otimes 1$$

$$h|\alpha, \beta\rangle = \begin{cases} |\beta, \alpha\rangle - |\alpha, \beta\rangle & (\alpha > \beta), \\ 0 & (\alpha \le \beta). \end{cases}$$

Sectors and Steady states

Sectors labeled by multiplicities $\mathbf{m} = (m_0, \ldots, m_n) \in \mathbb{Z}_+^{n+1}$:

$$S(\mathbf{m}) = \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_L) \in \{0, \ldots, n\}^L \mid \#_k(\boldsymbol{\sigma}) = m_k\}.$$

Each sector has the unique steady state (up to normalization)

$$|\bar{P}(\mathbf{m})\rangle = \sum_{\sigma \in S(\mathbf{m})} \underbrace{\mathbb{P}(\sigma_1, \ldots, \sigma_L)}|\sigma_1, \ldots, \sigma_L\rangle$$

Stationary probability

$$\begin{aligned}
|\bar{P}(1,1,1)\rangle &= 2|012\rangle + |021\rangle + |102\rangle + 2|120\rangle + 2|201\rangle + |210\rangle, \\
|\bar{P}(2,1,1)\rangle &= 3|0012\rangle + |0021\rangle + 2|0102\rangle + 3|0120\rangle + 2|0201\rangle + |0210\rangle \\
&\quad + |1002\rangle + 2|1020\rangle + 3|1200\rangle + 3|2001\rangle + 2|2010\rangle + |2100\rangle, \\
|\bar{P}(1,2,1)\rangle &= 2|0112\rangle + |0121\rangle + |0211\rangle + |1012\rangle + |1021\rangle + |1102\rangle \\
&\quad + 2|1120\rangle + 2|1201\rangle + |1210\rangle + 2|2011\rangle + |2101\rangle + |2110\rangle.
\end{aligned}$$

Steady states are non-trivial for $n \geq 2$.

Preceding results on stationary probability of n-TASEP

Combinatorial algorithm: Ferrari-Martin (2007)

Matrix product formula: Evans-Ferrari-Mallick (2009), . . .

What is a *matrix product formula* ?

Stationary probability

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}\left(X_{\sigma_1} \cdots X_{\sigma_L}\right)$$

Trace over the auxiliary space

Each X_{σ_i} is an operator acting on some *auxiliary space*

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Our case:

Auxiliary space = $F^{\otimes n(n-1)/2}$; F = Fock space of *q-boson* at q=0

X_{σ_i} = Piece of a *layer transfer matrix* of 3D lattice model satisfying the tetrahedron equation

Matrix product formula for stationary probability

$$\mathbb{P}(\sigma) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}\left(X_{\sigma_1} \cdots X_{\sigma_L}\right)$$

Zamolodchikov-Faddeev algebra

$$h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$$

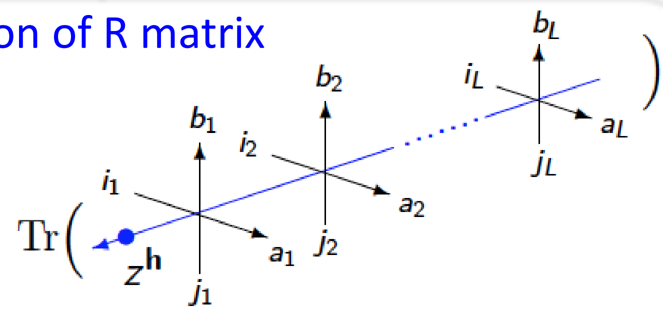
Ferrari-Martin algorithm

Bilinear identity among Layer transfer matrices of 3D lattice model

$$\sum x^{\cdots} y^{\cdots} T(x) T(y) = (x \leftrightarrow y)$$

Factorization of R matrix

$$\mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} =$$



Tetrahedron equation

Ferrari-Martin algorithm

Sector $S(\mathbf{m})$ with multiplicity $\mathbf{m} = (m_0, \ldots, m_n)$ with $\forall m_i > 0$.

$$s_i := m_{n-i+1} + \cdots + m_{n-1} + m_n, \quad 0 < s_1 < \cdots < s_n < L,$$

$$B^s := \{\mathbf{b} = (b_1, \ldots, b_L) \in \{0,1\}^L \mid b_1 + \cdots + b_L = s\},$$

$$\text{e.g. } B^1 = \{100, 010, 001\}, \quad B^2 = \{110, 101, 011\} \text{ for } L = 3,$$

$$\mathcal{B}(\mathbf{m}) := B^{s_1} \otimes \cdots \otimes B^{s_n}.$$

Ferrari-Martin gave a combinatorial construction of the surjective map

$$\pi : \mathcal{B}(\mathbf{m}) \to S(\mathbf{m}) \quad \text{such that} \quad \mathbb{P}(\sigma) = \#(\pi^{-1}(\sigma)).$$

Example. $n = 3$, $\mathbf{m} = (2,1,2,2)$, $\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \in \mathcal{B}(\mathbf{m}) = B^2 \otimes B^4 \otimes B^5$
$L = 7$

$B^5 \ni \mathbf{b}_3 =$		●	●	●		●	●
$B^4 \ni \mathbf{b}_2 =$	●	●			●		●
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$\mathbf{b}_1 = (0010100)$, etc.

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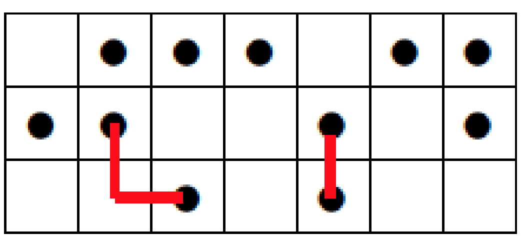

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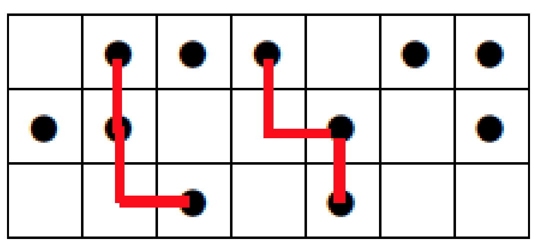

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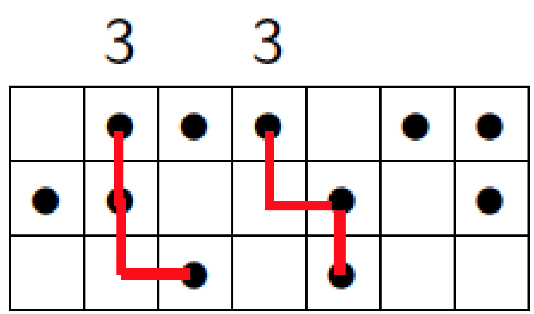

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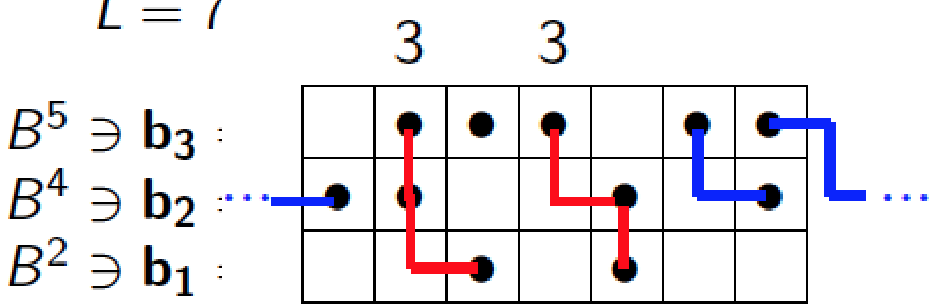

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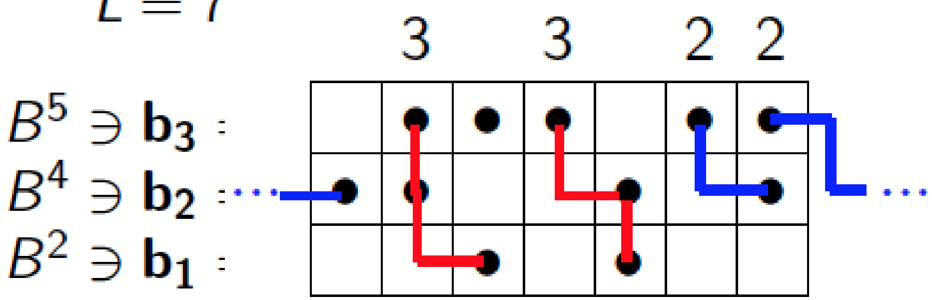

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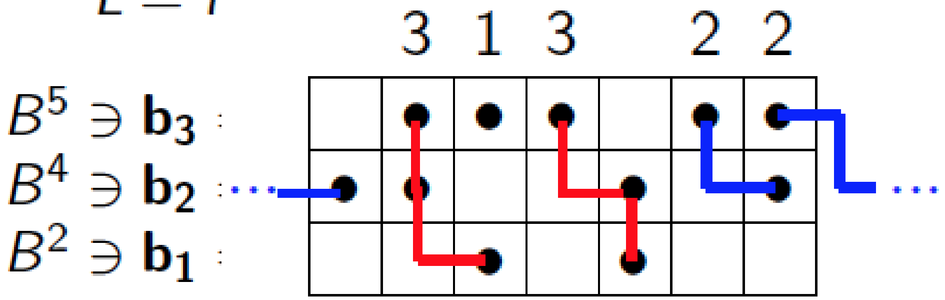

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$$0\ 3\ 1\ 3\ 0\ 2\ 2 = \pi(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) \in S(\mathbf{m})$$

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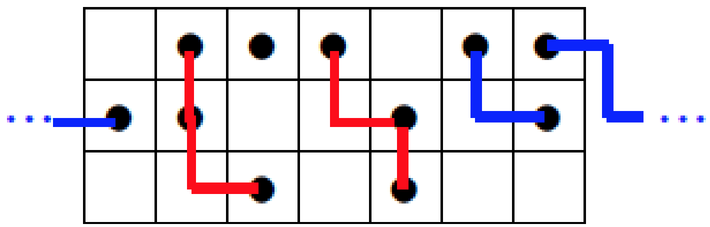

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The Ferrari-Martin algorithm is a composition of the map

$$R : B^s \otimes B^r \to B^r \otimes B^s$$

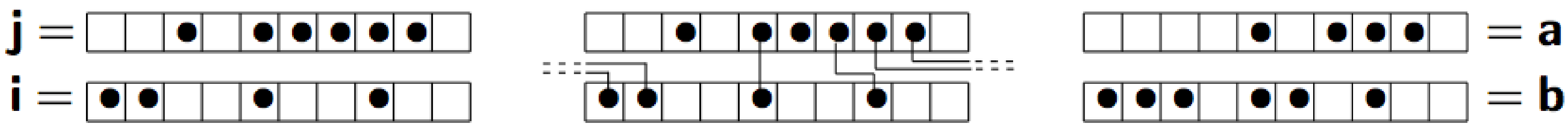
$$\mathbf{i} \otimes \mathbf{j} \quad \mapsto \quad \mathbf{b} \otimes \mathbf{a}$$

which is identified(!) with the Combinatorial R of $U_q(\widehat{sl}_L)$

Example. $B^4 \otimes B^6 \qquad\qquad B^6 \otimes B^4$

$$1100100100 \otimes 0010111110 \mapsto 1110110100 \otimes 0000101110$$

$$\mathbf{i} \qquad\qquad \mathbf{j} \qquad\qquad \mathbf{b} \qquad\qquad \mathbf{a}$$



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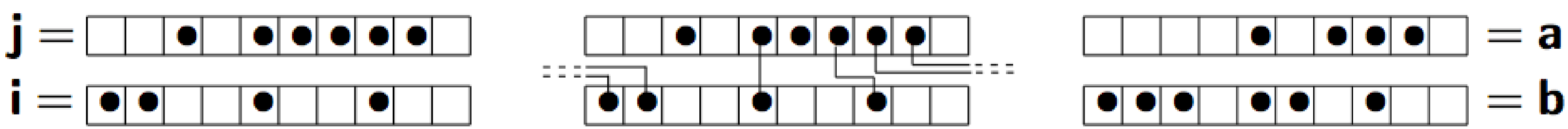
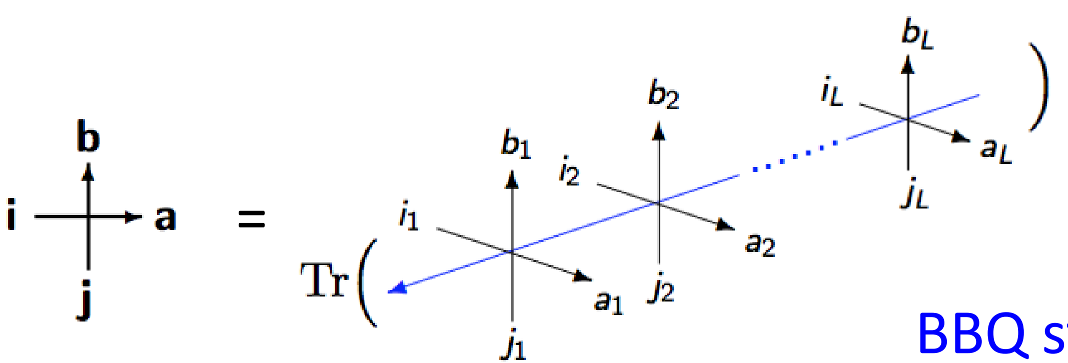
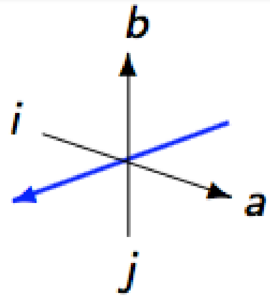


The combinatorial R admits a matrix product formula having a 3D structure



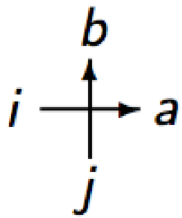
BBQ stick with X shape sausages



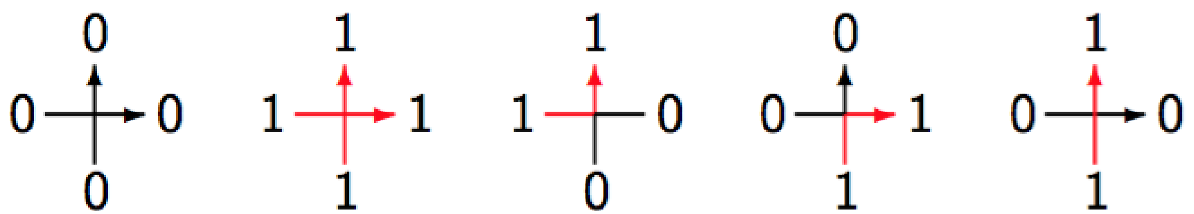
= 3D L-operator acting on the Fock space (blue arrow)

$$F = \oplus_m \mathbb{C}|m\rangle$$

- $(L^{a,b}_{i,j})$ = 3D L-operator at $q = 0$.

$L^{a,b}_{i,j} =$ 

$\in \mathrm{End}(F)$ (q = 0)-boson valued 5vertex



$1 \qquad 1 \qquad \mathbf{a}^+ \qquad \mathbf{a}^- \qquad \mathbf{k}$

- $(q = 0)$-boson acts on $F = \bigoplus_{m \ge 0} \mathbb{C}|m\rangle$ as

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-\delta_{m,0})|m-1\rangle, \quad \mathbf{k}|m\rangle = \delta_{m,0}|m\rangle$$

This is a consequence of the tetrahedron equation satisfied by 3D L-operator.

Ferrari-Martin map π as Corner transfer matrix

$$\pi(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = 0\ \ 3\ \ 1\ \ 3\ \ 0\ \ 2\ \ 2$$

$\mathbf{b}_3 =$

$\mathbf{b}_2 =$

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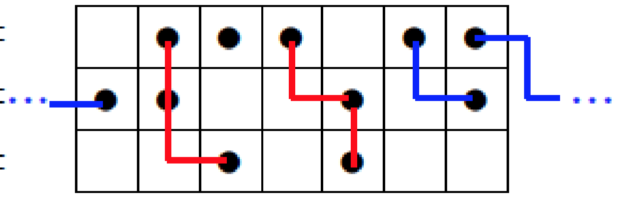

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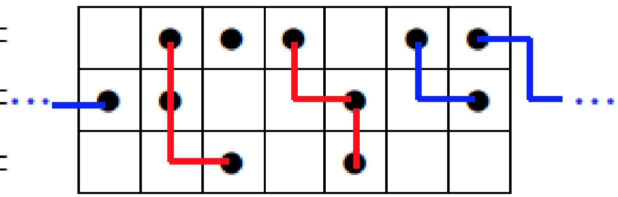

Elementary relabeling of n-TASEP $\boldsymbol{\sigma}$ by $\varphi_k(\boldsymbol{\sigma}) \in B^{s_k}$ $(k = 1, \ldots, n)$ defined as

$\boldsymbol{\sigma}$	0	3	1	3	0	2	2
$\varphi_1(\boldsymbol{\sigma})$	0	1	0	1	0	0	0
$\varphi_2(\boldsymbol{\sigma})$	0	1	0	1	0	1	1
$\varphi_3(\boldsymbol{\sigma})$	0	1	1	1	0	1	1

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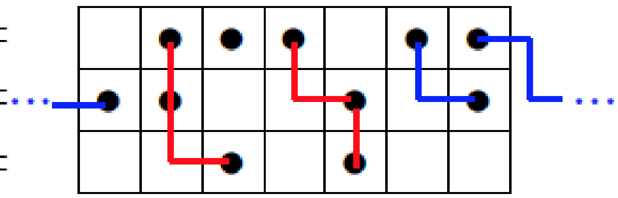

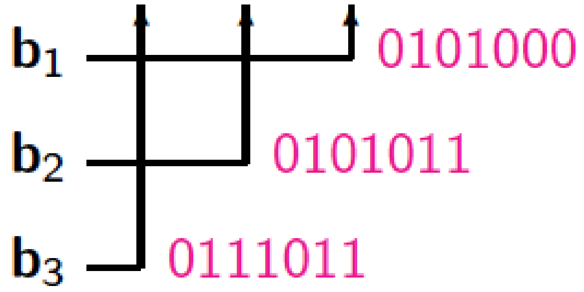

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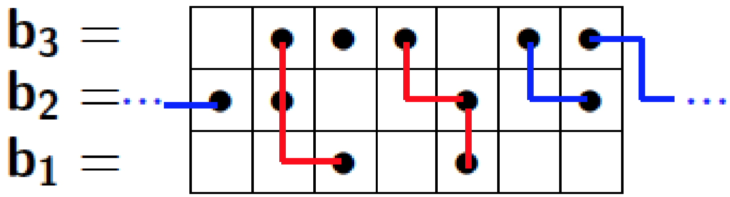
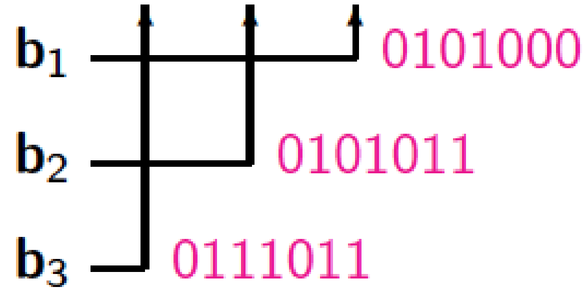
Ferrari-Martin map π as Corner transfer matrix

$\pi(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = 0\ 3\ 1\ 3\ 0\ 2\ 2$



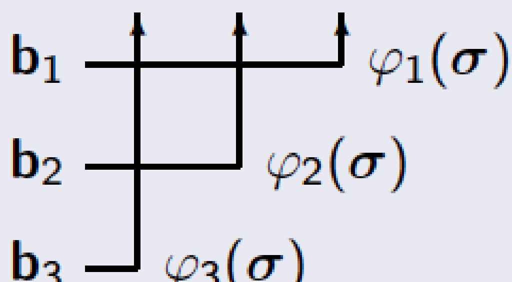


Elementary relabeling of n-TASEP $\boldsymbol{\sigma}$ by $\varphi_k(\boldsymbol{\sigma}) \in B^{s_k}$ $(k = 1, \ldots, n)$ defined as

$\boldsymbol{\sigma}$	0	3	1	3	0	2	2
$\varphi_1(\boldsymbol{\sigma})$	0	1	0	1	0	0	0
$\varphi_2(\boldsymbol{\sigma})$	0	1	0	1	0	1	1
$\varphi_3(\boldsymbol{\sigma})$	0	1	1	1	0	1	1

Steady state probability $\mathbb{P}(\boldsymbol{\sigma}) = \#(\pi^{-1}(\boldsymbol{\sigma}))$ is expressed as CTM

$$\mathbb{P}(\boldsymbol{\sigma}) = \sum_{\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \in \mathcal{B}(\mathbf{m})}$$

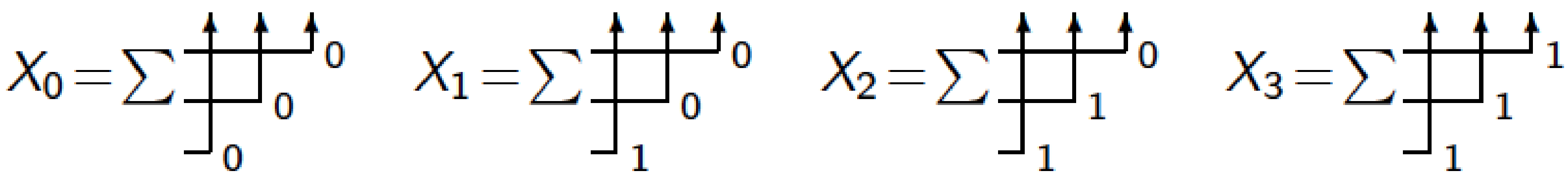


$(n = 3$ case$)$

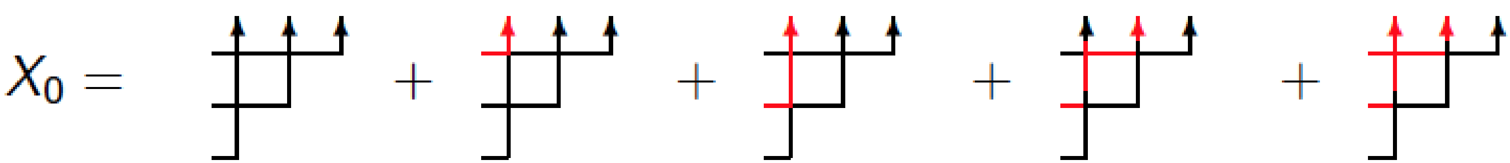
To each vertex, substitute the BBQ stick formula.

Result : $\mathbb{P}(\sigma) = \mathrm{Tr}_{F^{\otimes 3}}\left(X_{\sigma_1} \cdots X_{\sigma_L}\right)$ $(n = 3 \text{ case})$

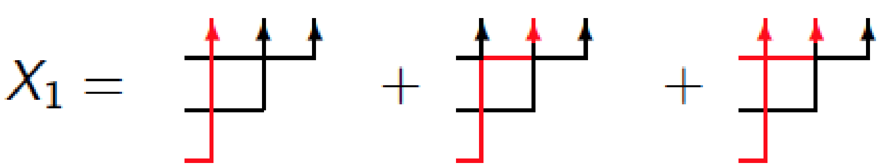
where X_i's are CTMs of the 5V-model with the boundary conditions:



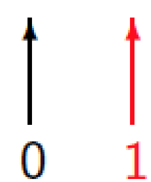
Example.



$= \quad 1 \otimes 1 \otimes 1 \; + \; \mathbf{a}^+ \otimes 1 \otimes 1 \; + \; \mathbf{k} \otimes \mathbf{a}^+ \otimes 1 \; + \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ \; + \; 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+$



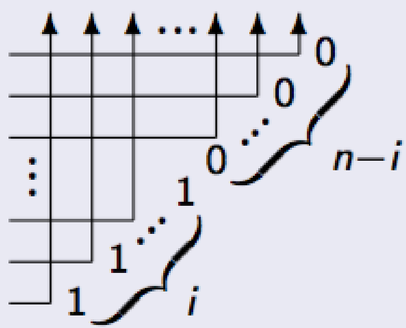
$= \quad \mathbf{k} \otimes \mathbf{k} \otimes 1 \; + \; \mathbf{a}^- \otimes \mathbf{k} \otimes \mathbf{a}^+ + \; 1 \otimes \mathbf{k} \otimes \mathbf{a}^+$



Theorem (Matrix product formula for stationary prob. of n-TASEP)

$$\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}(X_{\sigma_1} \cdots X_{\sigma_L})$$

$X_0, \ldots, X_n$ are CTMs of $(q = 0)$-boson valued 5 vertex model:

$X_i = \sum$  $\in \mathrm{End}(F^{\otimes n(n-1)/2})$

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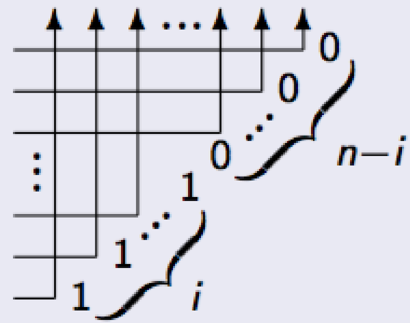
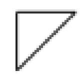
$X_i = \sum$ 

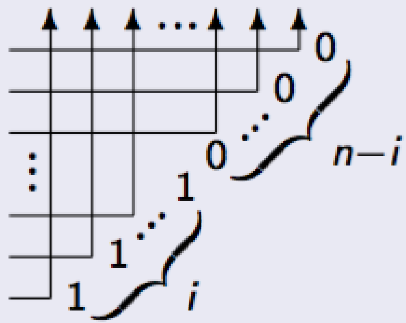
$\in \mathrm{End}(F^{\otimes n(n-1)/2})$

- X_i = Layer transfer matrix with  shape
- $\mathbb{P}(\boldsymbol{\sigma})$ = Partition function of a 3D system with prism shape

Theorem (Matrix product formula for stationary prob. of n-TASEP)

$$\mathbb{P}(\boldsymbol{\sigma}) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}(X_{\sigma_1} \cdots X_{\sigma_L})$$

$X_0, \ldots, X_n$ are CTMs of ($q = 0$)-boson valued 5 vertex model:

$X_i = \sum$  $\in \mathrm{End}(F^{\otimes n(n-1)/2})$

- X_i = Layer transfer matrix with ◸ shape
- $\mathbb{P}(\boldsymbol{\sigma})$ = Partition function of a 3D system with prism shape

	initial setup	cross channel (we are here!)
Physical space	$\mathbb{Z}_L$ ring	size n ◸
Internal degree	$\{0, \ldots, n\}$	$U_q(\widehat{sl}_L)$ at $q = 0$

Matrix product formula for stationary probability

$$\mathbb{P}(\sigma) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}\left(X_{\sigma_1}\cdots X_{\sigma_L}\right)$$

Zamolodchikov-Faddeev algebra

$$h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$$

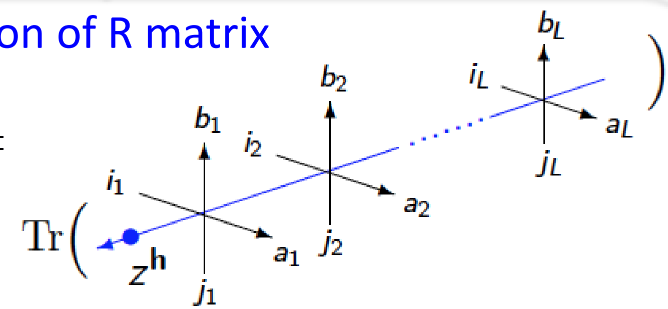
Bilinear identity among Layer transfer matrices of 3D lattice model

$$\sum x^{\cdots} y^{\cdots} T(x)T(y) = (x \leftrightarrow y)$$

Ferrari-Martin algorithm

Factorization of R matrix

$$\mathcal{R}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} =$$



Tetrahedron equation

n-TASEP Markov matrix H consists of local (pairwise) term h:

$$H = h \otimes 1 \otimes 1 \otimes \cdots + 1 \otimes h \otimes 1 \otimes + \cdots$$

Suppose the operators $X = (X_\alpha)_{0 \le \alpha \le n}, \quad \hat{X} = (\hat{X}_\alpha)_{0 \le \alpha \le n}$

satisfy the "hat relation" $h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$

Then the stationary states under the periodic B.C. is constructed as Tr(X X .... X) due to the cancellation mechanism:

$$\begin{aligned} H\,\mathrm{Tr}(XX\cdots X) &= \mathrm{Tr}\left((X\hat{X} - \hat{X}X)XX\cdots X\right) \\ &+ \mathrm{Tr}\left(X(X\hat{X} - \hat{X}X)X\cdots X\right) \\ &+ \mathrm{Tr}\left(XX(X\hat{X} - \hat{X}X)\cdots X\right) + \cdots = 0 \end{aligned}$$

$$h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$$

is the infinitesimal version of Zamolodchikov-Faddeev (ZF) algebra

$$S(x/y)\,(X(x) \otimes X(y)) = X(y) \otimes X(x)$$

with *spectral parameters* x, y via the correspondence

$$h = S'(1), \qquad X = X(1), \qquad \hat{X} = X'(1).$$

$$h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$$

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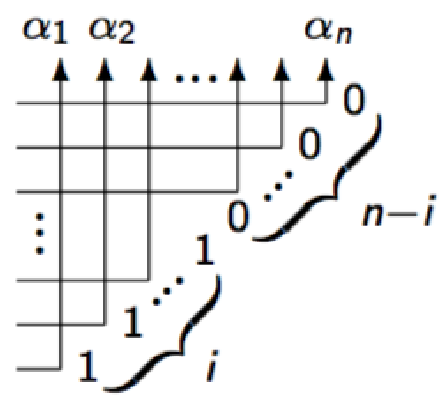
$$h = S'(1), \qquad X = X(1), \qquad \hat{X} = X'(1).$$

In our n-TASEP, $X(z) = (X_0(z), \ldots, X_n(z))$ and the ZF relation are given by

$$X_i(z) = \sum z^{\alpha_1 + \cdots + \alpha_n}$$



$$xX_i(y)X_j(x) = yX_i(x)X_j(y) \ (i > j), \quad [X_i(x), X_j(y)] = [X_i(y), X_j(x)] \ (\forall i, j).$$

Strategy of the proof

ZF relation . . . Highly non-local relation

$$xX_i(y)X_j(x) = yX_i(x)X_j(y) \ (i > j), \quad [X_i(x), X_j(y)] = [X_i(y), X_j(x)] \ (\forall i, j).$$

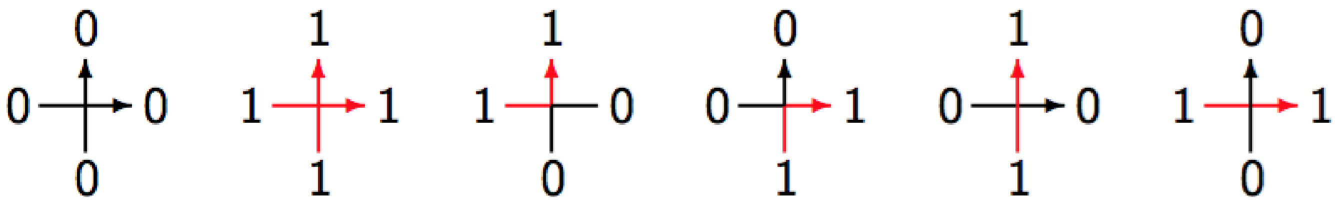
$q \longrightarrow 0$

Bilinear relations of
layer transfer matrices

Tetrahedron equation
(Single local relation)

q-Melting : 5 vertex → 6 vertex & ◸ → □

Introduce the 3D L-operator $\mathcal{L}(z)$ involving q, z by



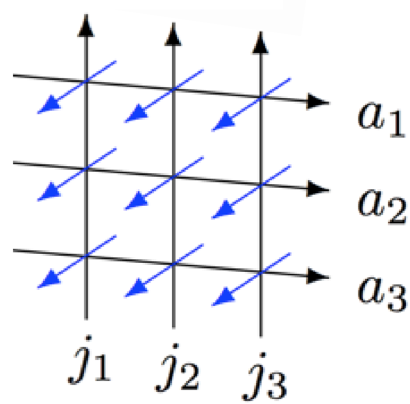
$1 \qquad 1 \qquad z^{-1}\mathbf{a}^+ \qquad z\mathbf{a}^- \qquad \mathbf{k} \qquad q\mathbf{k}$

$\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ are q-boson operators on the Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$

$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = (-q)^m|m\rangle$

Define $T(z)^{\mathbf{a}}_{\mathbf{j}} \in \mathrm{End}(F^{\otimes n^2})$ by

$T(z)^{\mathbf{a}}_{\mathbf{j}} := \sum$



n = 3 Example

$\mathbf{a} = (a_1, \ldots, a_n), \; \mathbf{j} = (j_1, \ldots, j_n)$

All vertices are $\mathcal{L}(z)$

Sum over all edge spins except $\mathbf{a}, \mathbf{j}$

Layer transfer matrix with SE-fixed/NW-free boundary condition

Prop. Bilinear relations of the layer-to-layer transfer matrices

$$\sum x^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} y^{|\overline{\boldsymbol{\alpha}}|+|\overline{\boldsymbol{\beta}}|} T(x)^{..,\alpha_1,\ldots,\alpha_r,..}_{..,\beta_1,\ldots,\beta_s,..} T(y)^{..,\bar{\alpha}_1,\ldots,\bar{\alpha}_r,..}_{..,\bar{\beta}_1,\ldots,\bar{\beta}_s,..} = (x \leftrightarrow y).$$

- Sum over $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \{0,1\}^r$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s) \in \{0,1\}^s$.
- $0 \le r \le n$, $0 \le s \le n$ are arbitrary, $\bar{\alpha}_i = 1 - \alpha_i$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_r$, etc.
- Arrays "...." are arbitrary, but to be taken common for $T(x)$ and $T(y)$.

Prop. Bilinear relations of the layer-to-layer transfer matrices

$$\sum x^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} y^{|\overline{\boldsymbol{\alpha}}|+|\overline{\boldsymbol{\beta}}|} T(x)^{..,\alpha_1,\ldots,\alpha_r,..}_{..,\beta_1,\ldots,\beta_s,..} T(y)^{..,\bar{\alpha}_1,\ldots,\bar{\alpha}_r,..}_{..,\bar{\beta}_1,\ldots,\bar{\beta}_s,..} = (x \leftrightarrow y).$$

- Sum over $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \{0,1\}^r$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s) \in \{0,1\}^s$.
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Example.

(1) $r = s = 0 : T(x)^{\mathbf{a}}_{\mathbf{j}} T(y)^{\mathbf{a}}_{\mathbf{j}} = T(y)^{\mathbf{a}}_{\mathbf{j}} T(x)^{\mathbf{a}}_{\mathbf{j}}$ $\cdots$ Commutativity

(2) $r = s = 1 :$
$$y^2 T(x)^{0,\ldots}_{0,\ldots} T(y)^{1,\ldots}_{1,\ldots} + xy T(x)^{0,\ldots}_{1,\ldots} T(y)^{1,\ldots}_{0,\ldots} + xy T(x)^{1,\ldots}_{0,\ldots} T(y)^{0,\ldots}_{1,\ldots} + x^2 T(x)^{1,\ldots}_{1,\ldots} T(y)^{0,\ldots}_{0,\ldots} = (x \leftrightarrow y).$$

Prop. Bilinear relations of the layer-to-layer transfer matrices

$$\sum x^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} y^{|\overline{\boldsymbol{\alpha}}|+|\overline{\boldsymbol{\beta}}|} T(x)^{..,\alpha_1,\dots,\alpha_r,..}_{..,\beta_1,\dots,\beta_s,..} T(y)^{..,\bar{\alpha}_1,\dots,\bar{\alpha}_r,..}_{..,\bar{\beta}_1,\dots,\bar{\beta}_s,..} = (x \leftrightarrow y).$$

- Sum over $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \{0,1\}^r$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s) \in \{0,1\}^s$.
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Example.

(1) $r = s = 0: \; T(x)^{\mathbf{a}}_{\mathbf{j}} T(y)^{\mathbf{a}}_{\mathbf{j}} = T(y)^{\mathbf{a}}_{\mathbf{j}} T(x)^{\mathbf{a}}_{\mathbf{j}}$ $\cdots$ Commutativity

(2) $r = s = 1: \; y^2 T(x)^{0,\dots}_{0,\dots} T(y)^{1,\dots}_{1,\dots} + xy T(x)^{0,\dots}_{1,\dots} T(y)^{1,\dots}_{0,\dots}$
$+ xy T(x)^{1,\dots}_{0,\dots} T(y)^{0,\dots}_{1,\dots} + x^2 T(x)^{1,\dots}_{1,\dots} T(y)^{0,\dots}_{0,\dots} = (x \leftrightarrow y).$

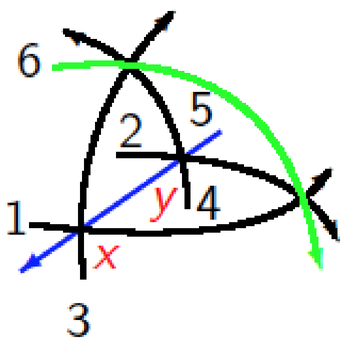
Fact: ZF relations are included(!) in (1) & (2) at $q = 0$.

Final task: Proof of the bilinear relations for $T(z)^{\mathrm{a}}_{\mathrm{j}}$

Introduce a variant of the 3D L-operator: $\mathcal{M}(z) := \mathcal{L}(z)|_{q \to -q}$.

Theorem ($\mathcal{L}(z)$ and $\mathcal{M}(z)$ satisfy the tetrahedron equation)

$$\mathcal{M}(\tfrac{ux}{y})_{126}\mathcal{M}(u)_{346}\mathcal{L}(x)_{135}\mathcal{L}(y)_{245} = \mathcal{L}(y)_{245}\mathcal{L}(x)_{135}\mathcal{M}(u)_{346}\mathcal{M}(\tfrac{ux}{y})_{126}$$



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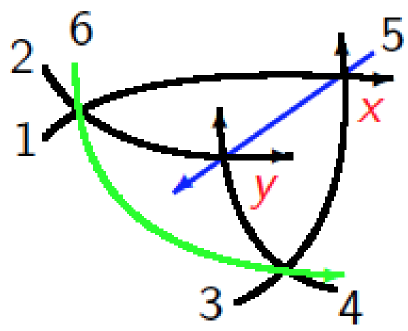

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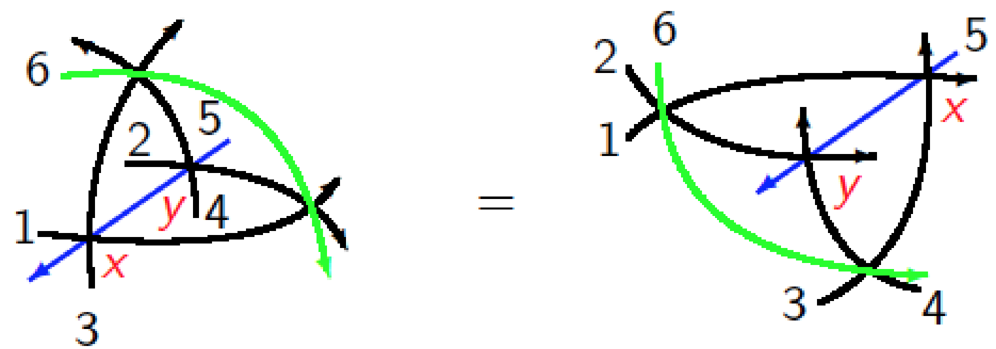


Background and relevant topics:

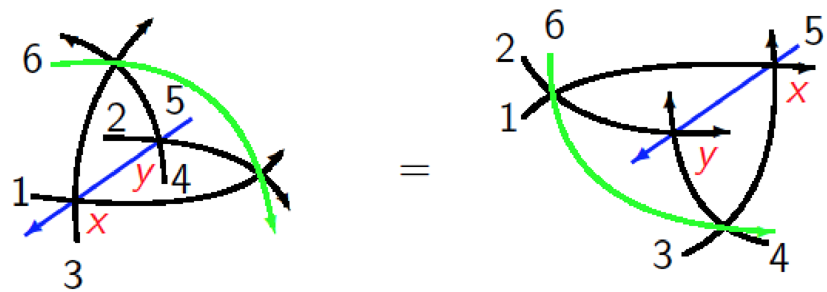
Factorized scattering theory in 2+1 dimension (Zamolodchikov 1980)

Intertwiner of Soibelman's reps. of quantized coordinate ring (Kapranov-Voevodsky 1994)

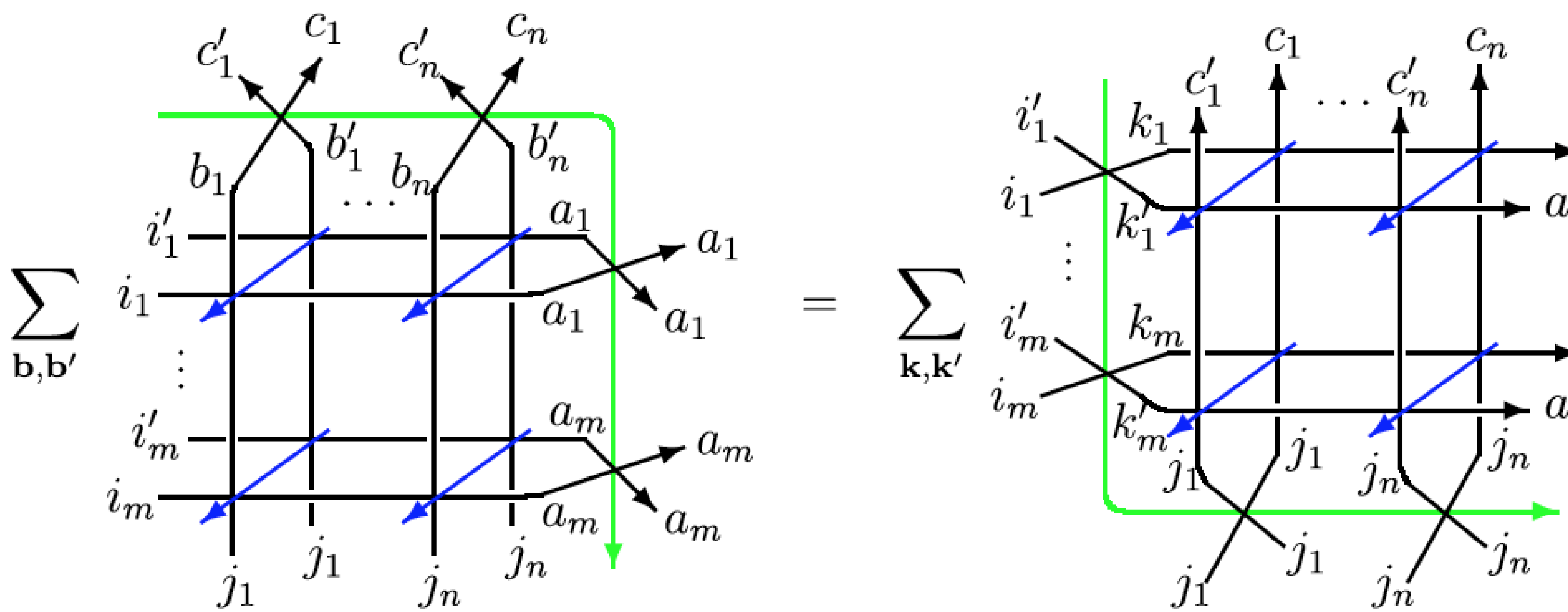
Transition coefficients of the PBW bases (Sergeev 2008, K-Okado-Yamada 2013)

Quantum geometry interpretation (Bazhanov-Mangazeev-Sergeev 2008)

Repeated application of the tetrahedron eq.



leads to



The bilinear relations are proved by evaluating this identity between various left/right eigenvectors of M(・) running along the green arrow.

Concluding remarks

A parallel story holds for
n-species *Totally Asymmetric Zero Range Process* (n-TAZRP)

TAZRP and TASEP are sister models corresponding to the quantum R matrices that are factorizable to solutions of tetrahedron equations:

Tetrahedron eq. :	3D R operator	3D L operator
Yang-Baxter eq. :	R matrix for symm. tensor rep.	R matrix for anti-symm. tensor rep.
Markov process:	n-TAZRP (K-Maruyama-Okado 2016)	n-TASEP (today's talk)

q-versions of these models are formulated by *stochastic R matrices* (Tuesday—Friday)