Integrable Markov process, matrix products and the tetrahedron equation

Atsuo Kuniba (Univ. of Tokyo)

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Non-equilibrium statistical mechanics

Integrable Markov process

Spectral problem of the Markov matrix: solvable by Bethe ansatz
Exact asymptotic analysis: connection to random matrices, etc.
Stationary states: matrix product structure (Today’s topic)

Integrable systems

Quantum groups, Yang-Baxter equation, ...
Non-equilibrium statistical mechanics
- Stochastic dynamics, Markov process, ...

Integrable Markov process
- Spectral problem of the Markov matrix: solvable by Bethe ansatz
- Exact asymptotic analysis: connection to random matrices, etc.
- Stationary states: matrix product structure (Today’s topic)

Prototype examples
- Totally asymmetric simple exclusion process (TASEP) - today
- Totally asymmetric zero range process (TAZRP) - Tuesday -- Friday

Key feature
- Hidden 3D structure in the matrix product related to the tetrahedron equation (= 3D generalization of the Yang-Baxter equation) which becomes manifest in multispecies versions of TASEP and TAZRP.
  (K-Maruyama-Okado, 2015, 2016)
1D periodic chain with $L$ sites

$\sigma_i \in \{0, 1, \ldots, n\} \quad (n\text{-TASEP})$

Stochastic dynamics

$(\sigma_i, \sigma_{i+1}) \rightarrow (\sigma'_i, \sigma'_{i+1})$

$(\alpha, \beta) \rightarrow (\beta, \alpha) \quad \text{if } \alpha > \beta$

Master equation

$$\frac{d}{dt} |P\rangle = H |P\rangle, \quad |P\rangle = \sum_{\{\sigma_i\}} \mathbb{P}(\sigma_1, \ldots, \sigma_L) |\sigma_1, \ldots, \sigma_L\rangle \in (\mathbb{C}^{n+1})^\otimes L$$

$$H = \sum_{i \in \mathbb{Z}_L} h_{i,i+1}, \quad h_{i,i+1} = 1 \otimes \cdots \otimes 1 \otimes h_{i,i+1} \otimes 1 \otimes \cdots \otimes 1$$

$$h |\alpha, \beta\rangle = \begin{cases} |\beta, \alpha\rangle - |\alpha, \beta\rangle & (\alpha > \beta), \\ 0 & (\alpha \leq \beta). \end{cases}$$
Sectors and Steady states

Sectors labeled by multiplicities \( m = (m_0, \ldots, m_n) \in \mathbb{Z}_{n+1}^+ \):

\[
S(m) = \{ \sigma = (\sigma_1, \ldots, \sigma_L) \in \{0, \ldots, n\}^L \mid \#_k(\sigma) = m_k \}.
\]

Each sector has the unique steady state (up to normalization)

\[
|\vec{P}(m)\rangle = \sum_{\sigma \in S(m)} \Pi P(\sigma_1, \ldots, \sigma_L) |\sigma_1, \ldots, \sigma_L\rangle
\]

Stationary probability

\[
|\vec{P}(1, 1, 1)\rangle = 2|012\rangle + |021\rangle + |102\rangle + 2|120\rangle + 2|201\rangle + |210\rangle,
|\vec{P}(2, 1, 1)\rangle = 3|0012\rangle + |0021\rangle + 2|0102\rangle + 3|0120\rangle + 2|0201\rangle + |0210\rangle
\]
\[
+ |1002\rangle + 2|1020\rangle + 3|1200\rangle + 3|2001\rangle + 2|2010\rangle + |2100\rangle,
|\vec{P}(1, 2, 1)\rangle = 2|0112\rangle + |0121\rangle + |0211\rangle + |1012\rangle + |1021\rangle + |1102\rangle
\]
\[
+ 2|1120\rangle + 2|1201\rangle + |1210\rangle + 2|2011\rangle + |2101\rangle + |2110\rangle.
\]

Steady states are non-trivial for \( n \geq 2 \).
Preceding results on stationary probability of n-TASEP


What is a matrix product formula?

Stationary probability

\[ \mathbb{P}(\sigma_1, \ldots, \sigma_L) = \text{Tr} \left( X_{\sigma_1} \cdots X_{\sigma_L} \right) \]

Trace over the auxiliary space

Each \( X_{\sigma_i} \) is an operator acting on some auxiliary space
Preceding results on stationary probability of n-TASEP


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Our case:

Auxiliary space = \(F^{\otimes n(n-1)/2}\); \(F\) = Fock space of \(q\)-boson at \(q=0\)

\(X_{\sigma_i}\) = Piece of a layer transfer matrix of 3D lattice model satisfying the tetrahedron equation
Matrix product formula for stationary probability

\[ \mathbb{P}(\sigma) = \text{Tr}_{F \otimes n(n-1)/2} \left( X_{\sigma_1} \cdots X_{\sigma_L} \right) \]

Zamolodchikov-Faddeev algebra

\[ h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X \]

Bilinear identity among Layer transfer matrices of 3D lattice model

\[ \sum x^{i_1 \cdots i_L} y^{j_1 \cdots j_L} T(x) T(y) = (x \leftrightarrow y) \]

Factorization of R matrix

\[ \mathcal{R}(z)_{i,j}^{a,b} = \text{Tr} \left( Z_z^{i_1 \cdots i_L} b_1 \cdots b_L \right) \]

Tetrahedron equation
Ferrari-Martin algorithm

Sector $S(m)$ with multiplicity $m = (m_0, \ldots, m_n)$ with $\forall m_i > 0$.

$$s_i := m_{n-i+1} + \cdots + m_{n-1} + m_n, \quad 0 < s_1 < \cdots < s_n < L,$$

$$B^s := \{b = (b_1, \ldots, b_L) \in \{0, 1\}^L \mid b_1 + \cdots + b_L = s\},$$

e.g. $B^1 = \{100, 010, 001\}$, $B^2 = \{110, 101, 011\}$ for $L = 3$,

$B(m) := B^{s_1} \otimes \cdots \otimes B^{s_n}$.

Ferrari-Martin gave a combinatorial construction of the surjective map

$$\pi : B(m) \to S(m)$$

such that $P(\sigma) = \#(\pi^{-1}(\sigma))$.

Example. $n = 3$, $m = (2, 1, 2, 2)$, $b_1 \otimes b_2 \otimes b_3 \in B(m) = B^2 \otimes B^4 \otimes B^5$

$L = 7$

$$B^5 \ni b_3 = \begin{array}{cccccc}
\bullet & \bullet & \bullet & & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\end{array}$$

$$B^4 \ni b_2 = \begin{array}{ccccccc}
\bullet & \bullet & \bullet & & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\end{array}$$

$$B^2 \ni b_1 = \begin{array}{ccccccc}
\bullet & \bullet & \bullet & & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\end{array}$$

$b_1 = (0010100)$, etc.
Ferrari-Martin algorithm

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\[
\begin{align*}
B^5 \ni b_3 &= \begin{align*}
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\end{array}
\end{align*} \\
B^4 \ni b_2 &= \begin{align*}
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\end{array}
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\begin{array}{|c|c|c|c|c|c|c|}
\hline
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\end{array}
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\[ \begin{array}{c|c|c|c} 3 & 3 \\
\end{array} \]

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\end{array} \]

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\hline
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\hline
\end{array}$

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\hline
\end{array}$

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\[
\begin{array}{cccc}
& & 3 & \\
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& & 3 & \\
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& & & \\
B^2 & \ni b_1 & & \\
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\( L = 7 \)

\[
\begin{array}{cccc}
3 & 3 & 2 & 2 \\
\end{array}
\]

\( B^5 \ni \mathbf{b}_3 : \)

\( B^4 \ni \mathbf{b}_2 : \)

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Example. $n = 3$, $m = (2, 1, 2, 2)$, $b_1 \otimes b_2 \otimes b_3 \in B(m) = B^2 \otimes B^4 \otimes B^5$

$L = 7$

\begin{array}{cccc}
3 & 1 & 3 & 2 \ 2
\end{array}

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$L = 7$

$$0 \ 3 \ 1 \ 3 \ 0 \ 2 \ 2 = \pi(b_1 \otimes b_2 \otimes b_3) \in S(m)$$

- $B^5 \ni b_3$:
- $B^4 \ni b_2$:
- $B^2 \ni b_1$:

$b_1 = (0010100)$, etc.
The Ferrari-Martin algorithm is a composition of the map

\[ R : B^s \otimes B^r \to B^r \otimes B^s \]

\[ i \otimes j \mapsto b \otimes a \]

which is identified(!) with the Combinatorial R of \( U_q(\hat{s}/L) \)

Example. \( B^4 \otimes B^6 \quad B^6 \otimes B^4 \)

\[
1100100100 \otimes 0010111110 \mapsto 1110110100 \otimes 0000101110
\]

\( i \quad j \quad b \quad a \)

\[
\begin{array}{c}
\begin{array}{cccccccc}
. & . & . & . & . & . & . & . \\
\end{array} \\
\begin{array}{cccccccc}
. & . & . & . & . & . & . & . \\
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\end{array}
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\begin{array}{cccccccc}
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\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
. & . & . & . & . & . & . & . \\
\end{array} \\
\begin{array}{cccccccc}
. & . & . & . & . & . & . & . \\
\end{array}
\end{array}
\]

\( = a \quad = b \)
The Ferrari-Martin algorithm is a composition of the map

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**Example.**

\[ B^4 \otimes B^6 \quad B^6 \otimes B^4 \]

\[ 1100100100 \otimes 0010111110 \mapsto 1110110100 \otimes 0000101110 \]

The combinatorial R admits a matrix product formula having a 3D structure

**BBQ stick** with X shape sausages
\[ F = \bigoplus_m \mathbb{C} |m\rangle \]

- \((L_{i,j}^{a,b}) = 3D L\text{-operator at } q = 0. \)

\[
L_{i,j}^{a,b} = i \begin{array}{c}
|b\rangle \\
\downarrow \\
|a\rangle \\
\downarrow \\
|j\rangle
\end{array} \in \text{End}(F) \quad (q = 0)\text{-boson valued 5vertex}
\]

- \((q = 0)\text{-boson acts on } F = \bigoplus_{m \geq 0} \mathbb{C} |m\rangle \text{ as}\)

\[
\begin{array}{ccccccc}
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & a^+ & a^- & k
\end{array}
\]

This is a consequence of the tetrahedron equation satisfied by 3D L-operator.
Ferrari-Martin map $\pi$ as Corner transfer matrix

$$\pi(b_1 \otimes b_2 \otimes b_3) = \begin{bmatrix} 0 & 3 & 1 & 3 & 0 & 2 & 2 \end{bmatrix}$$

\begin{align*}
b_3 &= \\
b_2 &= \\
b_1 &=
\end{align*}
Ferrari-Martin map $\pi$ as Corner transfer matrix

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\pi(b_1 \otimes b_2 \otimes b_3) = \begin{array}{cccccc}
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\end{array}
\]

\[
\begin{array}{cccccccc}
b_3 = & & & & & & & \\
b_2 = & & & & & & & \\
b_1 = & & & & & & & \\
\end{array}
\]

Elementary relabeling of $n$-TASEP $\sigma$ by $\varphi_k(\sigma) \in B^{s_k}$ ($k = 1, \ldots, n$) defined as

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$0$</th>
<th>$3$</th>
<th>$1$</th>
<th>$3$</th>
<th>$0$</th>
<th>$2$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_1(\sigma)$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\varphi_2(\sigma)$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\varphi_3(\sigma)$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Ferrari-Martin map \( \pi \) as Corner transfer matrix

\[
\pi(b_1 \otimes b_2 \otimes b_3) = 0 \quad 3 \quad 1 \quad 3 \quad 0 \quad 2 \quad 2
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
 & & & & & & \\
\hline
b_3 & & & & & & \\
\hline
b_2 & & & & & & \\
\hline
b_1 & & & & & & \\
\hline
\end{array}
\]

Elementary relabeling of \( n \)-TASEP \( \sigma \) by \( \varphi_k(\sigma) \in B^s_k \) \( (k = 1, \ldots, n) \) defined as

<table>
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<tr>
<td>( \varphi_1(\sigma) )</td>
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Ferrari-Martin map $\pi$ as Corner transfer matrix

$$\pi(b_1 \otimes b_2 \otimes b_3) = 0 \ 3 \ 1 \ 3 \ 0 \ 2 \ 2$$

Elementary relabeling of $n$-TASEP $\sigma$ by $\varphi_k(\sigma) \in B^{sk}$ ($k = 1, \ldots, n$) defined as

$$\begin{array}{c|cccccc}
\sigma & 0 & 3 & 1 & 3 & 0 & 2 \\
\varphi_1(\sigma) & 0 & 1 & 0 & 1 & 0 & 0 \\
\varphi_2(\sigma) & 0 & 1 & 0 & 1 & 0 & 1 \\
\varphi_3(\sigma) & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{array}$$

Steady state probability $\mathbb{P}(\sigma) = \#(\pi^{-1}(\sigma))$ is expressed as CTM

$$\mathbb{P}(\sigma) = \sum_{b_1 \otimes b_2 \otimes b_3 \in B(m)}$$

$(n = 3$ case)
To each vertex, substitute the BBQ stick formula.

\[ \mathbb{P}(\sigma) = \text{Tr}_{F^{\otimes 3}}(X_{\sigma_1} \cdots X_{\sigma_L}) \quad (n = 3 \text{ case}) \]

where \( X_i \)'s are CTMs of the 5V-model with the boundary conditions:

\[
\begin{align*}
X_0 &= \sum_0 \quad X_1 &= \sum_1 \quad X_2 &= \sum_1 \quad X_3 &= \sum_1 \\
&= \sum_0 & &+ \sum_0 & &+ \sum_0 & &+ \sum_1 & &+ \sum_1 \\
&= 1 \otimes 1 \otimes 1 & &+ a^+ \otimes 1 \otimes 1 & &+ k \otimes a^+ \otimes 1 & &+ a^- \otimes a^+ \otimes a^+ & &+ 1 \otimes a^+ \otimes a^+
\end{align*}
\]

Example.

\[
X_0 = \quad + \quad + \quad + \quad + \\
= 1 \otimes 1 \otimes 1 & &+ a^+ \otimes 1 \otimes 1 & &+ k \otimes a^+ \otimes 1 & &+ a^- \otimes a^+ \otimes a^+ & &+ 1 \otimes a^+ \otimes a^+
\]

\[
X_1 = \quad + \quad + \\
= k \otimes k \otimes 1 & &+ a^- \otimes k \otimes a^+ & &+ 1 \otimes k \otimes a^+
\]
Theorem (Matrix product formula for stationary prob. of $n$-TASEP)

$$\mathbb{P}(\sigma) = \text{Tr}_{F \otimes n(n-1)/2} (X_{\sigma_1} \cdots X_{\sigma_L})$$

$X_0, \ldots, X_n$ are CTMs of $(q = 0)$-boson valued 5 vertex model:

$$X_i = \sum \begin{array}{c}
\vdots \\
1 \\
1 \\
1 \\
\vdots \\
\cdots \\
1 \\
0 \\
0 \\
\end{array} 
\in \text{End}(F \otimes n(n-1)/2)$$
Theorem (Matrix product formula for stationary prob. of $n$-TASEP)

\[ \mathbb{P}(\sigma) = \text{Tr}_{F \otimes n(n-1)/2} (X_{\sigma_1} \cdots X_{\sigma_L}) \]

$X_0, \ldots, X_n$ are CTMs of $(q = 0)$-boson valued 5 vertex model:

\[ X_i = \sum \begin{array}{cccccc}
\uparrow & \uparrow & \cdots & \uparrow & \uparrow & \uparrow \\
\vdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & 1 & 1 & 1 & 1 & 1 \\
\vdots & 1 & 1 & 1 & 1 & 1 \\
\vdots & 1 & 1 & 1 & 1 & 1 \\
\vdots & 1 & 1 & 1 & 1 & 1 \\
\end{array} \]

\[ \in \text{End}(F \otimes n(n-1)/2) \]

- $X_i =$ Layer transfer matrix with $\ \downarrow \ $ shape
- $\mathbb{P}(\sigma) =$ Partition function of a 3D system with prism shape
Theorem (Matrix product formula for stationary prob. of $n$-TASEP)

\[ \mathbb{P}(\sigma) = \text{Tr}_{F^\otimes n(n-1)/2} (X_{\sigma_1} \cdots X_{\sigma_L}) \]

$X_0, \ldots, X_n$ are CTMs of $(q = 0)$-boson valued 5 vertex model:

\[ X_i = \sum \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & \cdot \\
\end{array} \in \text{End}(F^\otimes n(n-1)/2) \]

- $X_i = \text{Layer transfer matrix with } \bigtriangleup \text{ shape}$
- $\mathbb{P}(\sigma) = \text{Partition function of a 3D system with prism shape}$

<table>
<thead>
<tr>
<th>Physical space</th>
<th>initial setup</th>
<th>cross channel (we are here!)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_L$ ring</td>
<td>{0, \ldots, n}</td>
<td>size $n$ \bigtriangleup $U_q(\hat{sl}_L)$ at $q = 0$</td>
</tr>
</tbody>
</table>
Matrix product formula for stationary probability

\[ \mathbb{P}(\sigma) = \text{Tr}_{F \otimes n(n-1)/2} (X_{\sigma_1} \cdots X_{\sigma_L}) \]

Ferrari-Martin algorithm

Factorization of R matrix

Zamolodchikov-Faddeev algebra

Bilinear identity among Layer transfer matrices of 3D lattice model

Tetrahedron equation
n-TASEP Markov matrix $H$ consists of local (pairwise) term $h$:

$$H = h \otimes 1 \otimes 1 \otimes \cdots + 1 \otimes h \otimes 1 \otimes + \cdots$$

Suppose the operators $X = (X_\alpha)_{0 \leq \alpha \leq n}$, $\hat{X} = (\hat{X}_\alpha)_{0 \leq \alpha \leq n}$ satisfy the “hat relation”

$$h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$$

Then the stationary states under the periodic B.C. is constructed as $\text{Tr}(X X \cdots X)$ due to the cancellation mechanism:

$$H \text{Tr}(XX \cdots X) = \text{Tr} \left( (X \hat{X} - \hat{X}X)XX \cdots X \right)$$
$$+ \text{Tr} \left( X(X \hat{X} - \hat{X}X)X \cdots X \right)$$
$$+ \text{Tr} \left( XX(X \hat{X} - \hat{X}X) \cdots X \right) + \cdots = 0$$
$$h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X$$

is the infinitesimal version of Zamolodchikov-Faddeev (ZF) algebra

$$S(x/y) (X(x) \otimes X(y)) = X(y) \otimes X(x)$$

with spectral parameters $x, y$ via the correspondence

$$h = S'(1), \quad X = X(1), \quad \hat{X} = X'(1).$$
\[ h(X \otimes X) = X \otimes \hat{X} - \hat{X} \otimes X \]

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with \textit{spectral parameters} \( x, y \) via the correspondence

\[ h = S'(1), \quad X = X(1), \quad \hat{X} = X'(1). \]

In our \( n \)-TASEP, \( X(z) = (X_0(z), \ldots, X_n(z)) \) and the ZF relation are given by

\[ X_i(z) = \sum z^{\alpha_1 + \cdots + \alpha_n} \]

\[ xX_i(y)X_j(x) = yX_i(x)X_j(y) \quad (i > j), \quad [X_i(x), X_j(y)] = [X_i(y), X_j(x)] \quad (\forall i, j). \]
Strategy of the proof

ZF relation . . . Highly non-local relation

\[ x X_i(y) X_j(x) = y X_i(x) X_j(y) \quad (i > j), \quad [X_i(x), X_j(y)] = [X_i(y), X_j(x)] \quad (\forall i, j). \]

**Tetrahedron equation**
(Single local relation)

**Bilinear relations of layer transfer matrices**

**q \to 0**
Introduce the 3D $L$-operator $\mathcal{L}(z)$ involving $q, z$ by

\begin{align*}
&0 \quad 1 \quad 1 \quad 0 \quad 1 \\
&0 \quad 1 \quad 1 \quad 0 \quad 1 \\
&1 \quad 1 \quad z^{-1}a^+ \quad za^- \quad k \quad qk
\end{align*}

$a^+, a^-, k$ are $q$-boson operators on the Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$

\begin{align*}
a^+|m\rangle &= |m + 1\rangle, \quad a^-|m\rangle = (1 - q^{2m})|m - 1\rangle, \quad k|m\rangle = (-q)^m|m\rangle
\end{align*}

Define $T(z)_j^a \in \text{End}(F \otimes n^2)$ by

\begin{align*}
T(z)_j^a := \sum_{j_1, j_2, j_3}
\end{align*}

$n = 3$ Example

\begin{align*}
a = (a_1, \ldots, a_n), \quad j = (j_1, \ldots, j_n)
\end{align*}

All vertices are $\mathcal{L}(z)$

Sum over all edge spins except $a, j$

Layer transfer matrix with **SE-fixed/NW-free** boundary condition
Prop. Bilinear relations of the layer-to-layer transfer matrices

\[ \sum x^{|\alpha|+|\beta|} y^{\overline{\alpha}+\overline{\beta}} T(x)_{.,\alpha_1,\ldots,\alpha_r,\ldots} T(y)_{.,\overline{\beta_1},\ldots,\overline{\beta_s},\ldots} = (x \leftrightarrow y). \]

- Sum over \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \{0, 1\}^r \), \( \beta = (\beta_1, \ldots, \beta_s) \in \{0, 1\}^s \).
- \( 0 \leq r \leq n \), \( 0 \leq s \leq n \) are arbitrary, \( \overline{\alpha}_i = 1 - \alpha_i \), \( |\alpha| = \alpha_1 + \cdots + \alpha_r \), etc.
- Arrays “….“ are arbitrary, but to be taken common for \( T(x) \) and \( T(y) \).
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Example.

1. \( r = s = 0 : T(x)_a T(y)_a = T(y)_a T(x)_a \) \( \cdots \) Commutativity
2. \( r = s = 1 : y^2 T(x)_{0,\ldots} T(y)_{1,\ldots} + xy T(x)_{1,\ldots} T(y)_{0,\ldots} + xy T(x)_{0,\ldots} T(y)_{1,\ldots} + x^2 T(x)_{1,\ldots} T(y)_{0,\ldots} = (x \leftrightarrow y). \)
Prop. Bilinear relations of the layer-to-layer transfer matrices

\[ \sum_{x^{\mid \alpha \mid + \mid \beta \mid}} y^{\mid \bar{\alpha} \mid + \mid \bar{\beta} \mid} T(x)_{..., \alpha_1, ..., \alpha_r, ...} T(y)_{..., \bar{\alpha}_1, ..., \bar{\alpha}_r, ...} = (x \leftrightarrow y). \]

- Sum over \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \{0, 1\}^r \), \( \beta = (\beta_1, \ldots, \beta_s) \in \{0, 1\}^s \).
- \( 0 \leq r \leq n, \ 0 \leq s \leq n \) are arbitrary, \( \bar{\alpha}_i = 1 - \alpha_i \), \( |\alpha| = \alpha_1 + \cdots + \alpha_r \), etc.
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Example.

1. \( r = s = 0 : \ T(x)^a_j \ T(y)^a_j = T(y)^a_j \ T(x)^a_j \quad \cdots \ \text{Commutativity} \)
2. \( r = s = 1 : \ y^2 T(x)^0_{0, \cdots} \ T(y)^1_{1, \cdots} + xy T(x)^0_{1, \cdots} \ T(y)^1_{0, \cdots} \)
   \[ \quad + xy T(x)^0_{0, \cdots} \ T(y)^1_{0, \cdots} + x^2 T(x)^1_{1, \cdots} \ T(y)^0_{0, \cdots} = (x \leftrightarrow y). \]

Fact: ZF relations are included(!) in (1) & (2) at \( q = 0 \).
Final task: Proof of the bilinear relations for $T(z)^a_j$

Introduce a variant of the 3D $L$-operator: $\mathcal{M}(z) := \mathcal{L}(z)|_{q \to -q}$.

**Theorem ($\mathcal{L}(z)$ and $\mathcal{M}(z)$ satisfy the tetrahedron equation)**

\[
\mathcal{M}\left(\frac{ux}{y}\right)_{126} \mathcal{M}(u)_{346} \mathcal{L}(x)_{135} \mathcal{L}(y)_{245} = \mathcal{L}(y)_{245} \mathcal{L}(x)_{135} \mathcal{M}(u)_{346} \mathcal{M}\left(\frac{ux}{y}\right)_{126}
\]
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**Background and relevant topics:**

- Factorized scattering theory in 2+1 dimension (Zamolodchikov 1980)
- Intertwiner of Soibelman’s reps. of quantized coordinate ring (Kapranov-Voevodsky 1994)
- Quantum geometry interpretation (Bazhanov-Mangazeev-Sergeev 2008)
The bilinear relations are proved by evaluating this identity between various left/right eigenvectors of $M(\cdot)$ running along the green arrow.
Concluding remarks

A parallel story holds for n-species *Totally Asymmetric Zero Range Process* (n-TAZRP)

TAZRP and TASEP are sister models corresponding to the quantum R matrices that are factorizable to solutions of tetrahedron equations:

Tetrahedron eq. : 3D R operator 3D L operator


Markov process: n-TAZRP n-TASEP
(K-Maruyama-Okado 2016) (today’s talk)

q-versions of these models are formulated by *stochastic R matrices* (Tuesday—Friday)