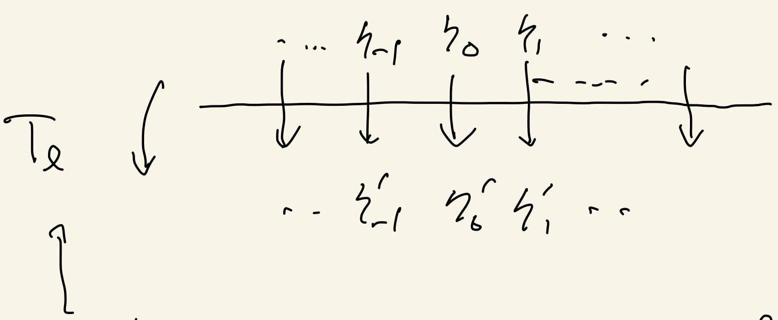
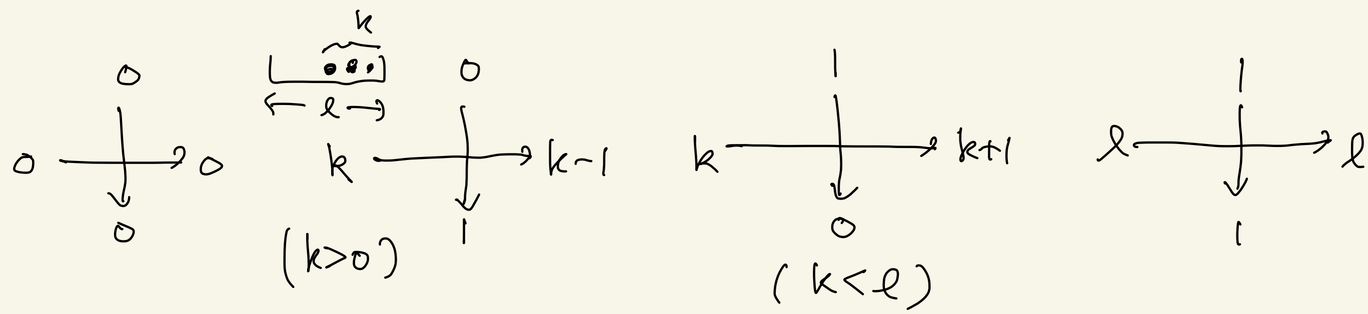
BBS (Box - ball system)



$\in \{0,1\}^L$

B.C. $\forall$ distant $\eta_i = 0$

time evolution by capacity ℓ carrier.



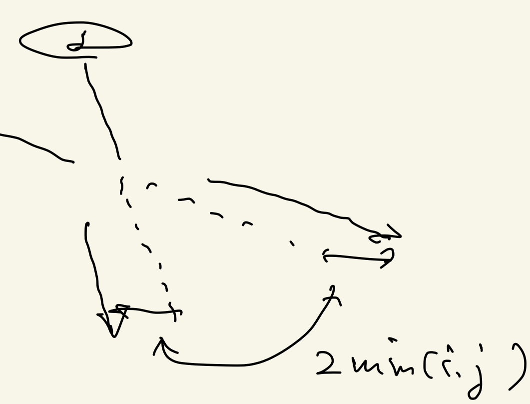
- $[T_\ell, T_m] = 0$

- $\cdots 0 0 \overbrace{1 1 \cdots 1}^{k} 0 \cdots 0$ ⋯ k-soliton

- bare speed of k-soliton under T_ℓ $= \min(\ell, k)$

- phase shift of i-soliton & j-soliton $= 2\min(i,j)$

- Conserved quantities



$E_1, E_2, \cdots, E_\infty$

$\#$solitons $= E_1$, $E_\infty = \#\text{balls} = \sum_i \eta_i$

$E_\ell \circ T_m = E_\ell \quad (\forall \ell, m)$

GGE (Generalized Gibbs Ensemble)

$$Z_L = \sum e^{-\beta_1 E_1 - \beta_2 E_2 - \cdots}$$

(B.C. Combinatorial Bethe ansatz, $L \to \infty$)

$$\cong \sum_{\{m_j\}} e^{-\sum_{i,j} \beta_i \underbrace{\min(i,j) m_j}_{E_i}} \prod_i \binom{p_i + m_i}{m_i}$$

$$E_\ell = \sum_j \min(\ell, j) m_j$$

m_i = # i-solitons

p_i = # i-holes $= L - 2\sum_j \min(i,j) m_j \cdots (\star)$

$$\mathcal{F} = -\lim_{L\to\infty} \frac{1}{L} \log Z_L$$

Randomized BBS

I.I.d distribution of balls with relative probability

empty	ball
1 :	q

density $\frac{q}{1+q}$

$\longleftrightarrow \ \forall \beta_i = 0$ except β_∞ $\quad q = e^{-\beta_\infty}$

TBA (Thermodynamic Bethe Ansatz)

Assume $\quad m_i \sim L\rho_i \;,\; P_i \sim L\sigma_i \quad$ as $L \to \infty$

$$(\star) \to \quad \sigma_i = 1 - 2\sum_j \min(i,j)\,\rho_j \qquad \cdots (\#)$$

$$\mathcal{F} = \Big[\beta\infty \sum_i i\rho_i - \sum_i \big\{ \underbrace{(\sigma_i+\rho_i)\log(\sigma_i+\rho_i) - \sigma_i\log\sigma_i - \rho_i\log\rho_i}_{\text{Stirling}} \big\} \Big]_{\min}$$

min (equilibrium) condition

$$\frac{\partial \mathcal{F}}{\partial \rho_i} = 0 \iff y_i^{-2} = (1+y_{i-1}^{-1})(1+y_{i+1}^{-1}) \qquad i \geq 1$$

$$\lim_{i\to\infty} \frac{1+y_{i+1}^{-1}}{1+y_i^{-1}} = e^{\beta\infty} = q^{-1} \qquad y_0^{-1} = 0$$

for $\quad y_i = \dfrac{\rho_i}{\sigma_i}$

·· Y-system (algebraic form of TBA eq)

$$\mathcal{F} = -\sum_i \log(1+y_i)$$

Solution $\quad y_i = \dfrac{q^i(1-q)^2}{(1-q^i)(1-q^{i+2})}$

Matrix form

$$M = (2\min(i,j))_{i,j\geq 1} \quad \cdots \quad S\text{-matrix}$$

$$\mathbb{y} = \begin{pmatrix} y_1 & & 0 \\ & y_2 & \\ 0 & & \ddots \end{pmatrix} \qquad \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \end{pmatrix} \qquad \mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}$$

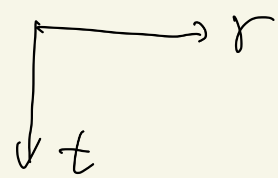
$$(\#) \Longleftrightarrow \quad (1 + M\mathbb{y})\sigma = \mathbb{1}$$

$$\longrightarrow \quad \sigma = (1 + M\mathbb{y})^{-1}\mathbb{1} = \sigma(\mathbb{y})$$

$$\rho = \rho(\mathbb{y}), \quad \cdots$$

GHD (Generalized Hydrodynamics)

Assume y_i, σ_i, ρ_i are (r,t)-dependent



$$\left.\begin{array}{l} \text{Speed eq} \\ \text{Continuity eq} \end{array}\right\} \longrightarrow \text{separated eq on } \underset{\substack{\uparrow \\ \text{Normal mode}}}{y_i}$$

[Speed]

$$v_i = \overbrace{\min(i,\ell)}^{\substack{\text{bare speed} \\ \text{under } T_\ell}} + \sum_j \overbrace{2\min(i,j)}^{\text{phase shift}} \overbrace{(v_i - v_j)\rho_j}^{\#\text{ Collisions}}$$

$$\downarrow (\#)$$

$$\sigma_i v_i + 2\sum_j \min(i,j)\underbrace{\rho_j v_j}_{\substack{\parallel \\ y_j \sigma_j v_j}} = \min(i,\ell) \qquad \left(y_j = \frac{\rho_j}{\sigma_j} \right)$$

$$(1+M\boldsymbol{y})(\sigma * v) = k_\ell$$

$$\sigma * v = \begin{pmatrix} \sigma_1 v_1 \\ \sigma_2 v_2 \\ \vdots \end{pmatrix} \qquad k_\ell = \begin{pmatrix} \min(1,\ell) \\ \min(2,\ell) \\ \vdots \end{pmatrix}$$

$$\longrightarrow \quad v = \frac{(1+M\boldsymbol{y})^{-1} k_\ell}{(1+M\boldsymbol{y})^{-1} \mathbb{1}} = v(\boldsymbol{y})$$

$$v_i = \frac{1+q^{\ell+1}}{1-q^{\ell+1}} \left(\frac{1+q}{1-q} k - \frac{2q(1+q)(1-q^k)}{(1-q)^2 (1+q^{k+1})} \right) \qquad (k = \min(\ell, i))$$

[Continuity] $\quad \partial_t \sigma_i + \partial_r (\sigma_i v_i) = 0$

$$\partial_t (1+M\boldsymbol{y})^{-1} \mathbb{1} + \partial_r (1+M\boldsymbol{y})^{-1} k_\ell = 0$$

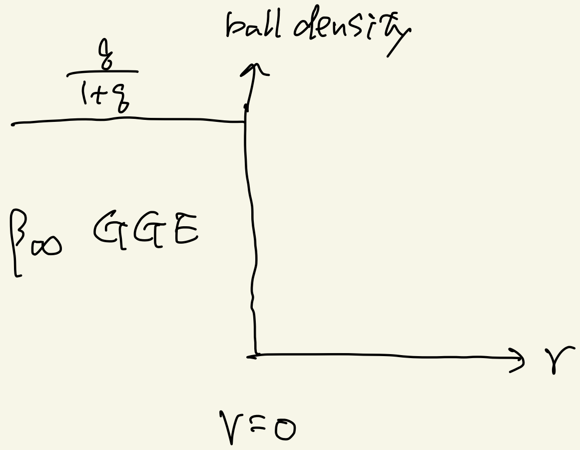
$$G = 1+M\boldsymbol{y} \qquad \partial_\alpha G^{-1} = -G^{-1} M \partial_\alpha \boldsymbol{y}\, G^{-1}$$

$$-G^{-1} M \partial_t \boldsymbol{y}\, G^{-1} \mathbb{1} - G^{-1} M \partial_r \boldsymbol{y}\, G^{-1} k_\ell = 0$$

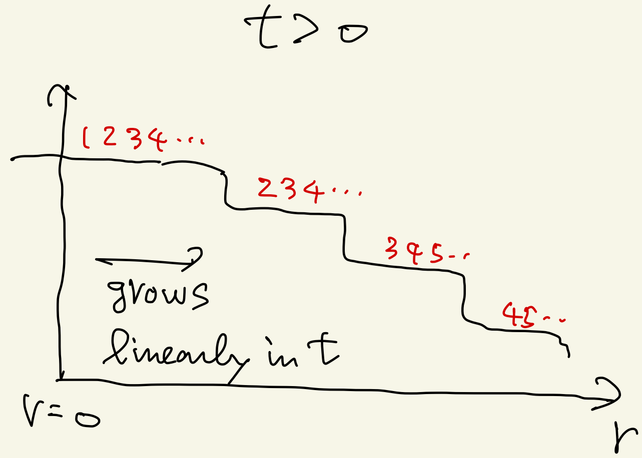
$$(\partial_t \boldsymbol{y})\sigma + (\partial_r \boldsymbol{y})(\sigma * v) = 0$$

$$\longrightarrow \quad \partial_t y_i + v_i \partial_r y_i = 0 \quad \ldots \text{ Separated.}$$

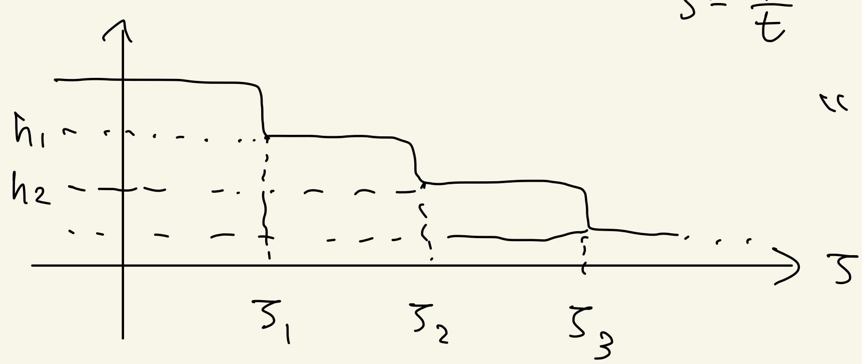
Application to non-equilibrium dynamics started from domain-wall initial condition



$(T_\infty)^t$ $\longrightarrow$



$\swarrow$ $\xi = \frac{r}{t}$



"static" plateaux

Ballistic approx. $\longrightarrow$ $h_1, h_2, \dots, \xi_1, \xi_2 \dots$

Diffusive correction $\longrightarrow$ broadening of Plateau edges

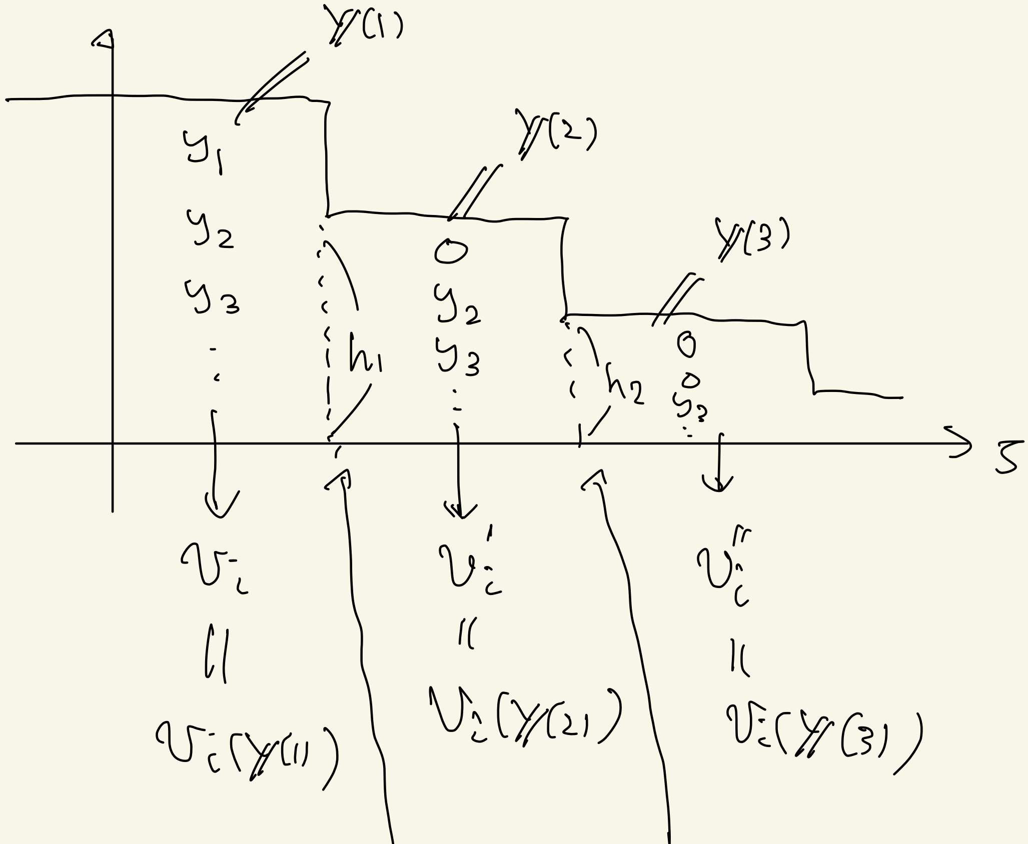
<u>Ballistic</u>

$$y_i = y_i(r,t) \to y_i\left(\xi = \frac{r}{t}\right)$$

separated eq $\longrightarrow$ $(\xi - v_i)\,\partial_\xi y_i = 0$

$$\longrightarrow \quad y_i(\xi) = \begin{cases} y_i(-\infty) & \xi < v_i \\ y_i(+\infty) & \xi > v_i \end{cases}$$

"weak sol"



$\zeta_1 = v_1 \qquad \zeta_2 = v_2'$

$$h_1 = \sum_j j \rho_j(\mathbb{Y}(1))$$

$$h_2 = \sum_j j \rho_j(\mathbb{Y}(2)) \qquad \text{etc.}$$