Tetrahedron and 3D reflection equations from quantized algebra of functions

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- **Yang-Baxter equation**

$$R : V \otimes V \to V \otimes V \qquad\qquad (2\mathrm{D}\ R)$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

- **Tetrahedron equation**

$$R : V \otimes V \otimes V \to V \otimes V \otimes V \qquad (3\mathrm{D}\ R)$$

$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123}$$

- **Relevant quantum algebra**

2D R : $U_q(\hat{g})$; quantized universal enveloping algebra of $g = \mathrm{Lie}\,\mathrm{G}$
3D R : $A_q(G)$; quantized algebra of functions on G

- **Simplest example:** $\mathbf{A_q(SL_2)} = \langle \mathbf{t_{11}, t_{12}, t_{21}, t_{22}} \rangle$

$$t_{11}t_{21} = qt_{21}t_{11},\ t_{12}t_{22} = qt_{22}t_{12},\ t_{11}t_{12} = qt_{12}t_{11},\ t_{21}t_{22} = qt_{22}t_{21},$$
$$[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}, \quad t_{11}t_{22} - qt_{12}t_{21} = 1.$$

Hopf algebra with coproduct $\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$.

- **Fock representation** $\pi_1 : \mathbf{A_q(SL_2)} \longrightarrow \mathbf{End(F_q)}$

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} \\ -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix} \text{ acting on } F_q = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle \oplus \cdots$$

$$\mathbf{k}|m\rangle = q^m|m\rangle,\ \mathbf{a}^+|m\rangle = |m+1\rangle,\ \mathbf{a}^-|m\rangle = (1 - q^{2m})|m-1\rangle.$$

Theorem (Classification of irreducible reps. Soibelman (1991))

1. *$A_q(G)$ has an irreducible representation π_i attached to each vertex i of the Dynkin diagram of G.*
2. *Irreducible representations are in 1:1 correspondence with elements of the Weyl group $W(G)$ (up to a "torus degree of freedom").*
3. *For any reduced expression $w = s_{i_1} \cdots s_{i_r} \in W(G)$, the irreducible representation for w is realized as $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$.*

Crucial Corollary

Unique (up to normalization) existence of the intertwiner for

$$\pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \simeq \pi_{j_1} \otimes \cdots \otimes \pi_{j_r}$$

for any other reduced expression $w = s_{j_1} \cdots s_{j_r}$.

Example

$A_q(\mathrm{SL}_3) = \langle t_{ij} \rangle_{i,j=1}^{3}$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \longmapsto \overset{\pi_1}{\begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}}$$

$$W(\mathrm{SL}_3) = \langle s_1, s_2 \rangle. \qquad s_i^2 = 1, \quad s_2 s_1 s_2 = s_1 s_2 s_1.$$

$$\Longrightarrow \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_1 \otimes \pi_2 \otimes \pi_1.$$

$\exists 1$ Intertwiner $\Phi : (F_q)^{\otimes 3} \to (F_q)^{\otimes 3}$ such that

$$(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi = \Phi \circ (\pi_1 \otimes \pi_2 \otimes \pi_1),$$

$$\Phi(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle.$$

Explicit form

$$R = \Phi P_{13}, \quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x,$$

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R^{abc}_{ijk} |a\rangle \otimes |b\rangle \otimes |c\rangle.$$

$$R^{abc}_{ijk} = \delta_{i+j,a+b}\delta_{j+k,b+c} \sum_{\lambda,\mu \geq 0, \lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \times \begin{bmatrix} i, j, c+\mu \\ \mu, \lambda, i-\mu, j-\lambda, c \end{bmatrix}.$$

$$(q)_i = \prod_{j=1}^{i}(1-q^j), \quad \begin{bmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{bmatrix} = \begin{cases} \frac{\prod_{m=1}^{r}(q^2)_{i_m}}{\prod_{m=1}^{s}(q^2)_{j_m}} & \forall i_m, j_m \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise.} \end{cases}$$

$$R = \oplus(\text{finite dimensional matrix}).$$

Theorem (Kapranov-Voevodsky 1994)

R satisfies the tetrahedron equation
$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123} \in \mathrm{End}((F_q)^{\otimes 6})$.

Sketch of proof. $W(\mathrm{SL}_4) = \langle s_1, s_2, s_3 \rangle$.

$s_2s_1s_2 = s_1s_2s_1, \;\; s_3s_2s_3 = s_2s_3s_2, \;\; s_1s_2s_3s_1s_2s_1 = s_3s_2s_3s_1s_2s_3$ (longest el.)

$\exists 1$ intertwiners $\Phi^{(i)}$ for $A_q(\mathrm{SL}_4)$ modules:

$$(\pi_2 \otimes \pi_1 \otimes \pi_2) \circ \Phi^{(1)} = \Phi^{(1)} \circ (\pi_1 \otimes \pi_2 \otimes \pi_1),$$

$$(\pi_3 \otimes \pi_2 \otimes \pi_3) \circ \Phi^{(2)} = \Phi^{(2)} \circ (\pi_2 \otimes \pi_3 \otimes \pi_2),$$

$$(\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1) \circ \Phi^{(3)} = \Phi^{(3)} \circ (\pi_3 \otimes \pi_2 \otimes \pi_3 \otimes \pi_1 \otimes \pi_2 \otimes \pi_3)$$

- $\Phi^{(1)} = \Phi^{(2)} = RP_{13}$ as matrices.
- $\Phi^{(3)} = \Phi^{(2)}\Phi^{(1)}\Phi^{(2)}\Phi^{(1)} = \Phi^{(2)}\Phi^{(1)}\Phi^{(2)}\Phi^{(1)}$.
  ($\because \exists 2$ ways to rewrite the longest element.) $\square$

Summary so far (type A case)

$$\begin{aligned} \text{Weyl group elements} &\longleftrightarrow \text{“Particle (string) states”} \\ \text{Cubic Coxeter relation} &\longleftrightarrow \text{3D } R \text{ matrix} \\ \text{Transformation of longest element} &\longleftrightarrow \text{Tetrahedron equation} \end{aligned}$$

Remark. 3D R here = [Bazhanov, Sergeev, Mangazeev (2006, 2008)].

Our result (K & Okado, arXiv:1208.1586)

1. Type B, C, F_4 cases (mainly along C).
2. Explicit intertwiner K for the quartic Coxeter relation.
3. 3D reflection equation involving R and K.
4. Birational and Combinatorial counterparts of R and K.

$A_q(\mathrm{Sp}_6) = \langle t_{ij} \rangle_{i,j=1}^6$: [Reshetikhin, Takhtadzhyan, Faddeev (1990)]

Representations $\pi_1(t_{ij}), \pi_2(t_{ij}), \pi_3(t_{ij})$.

$$\pi_1 : \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} \\ 0 & 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 \\ 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} & 0 \\ 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \langle \mathbf{A}^\pm, \mathbf{K} \rangle = \langle \mathbf{a}^\pm, \mathbf{k} \rangle|_{q \to q^2}.$$

$W(\mathrm{Sp}_6) = \langle s_1, s_2, s_3 \rangle$

$$s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2.$$

Write $\pi_{i_1,\dots,i_r}$ to mean $\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$ to save space.

Equivalence	Intertwiner	
$\pi_{13} \simeq \pi_{31}$,	$P_{12}(x \otimes y) = y \otimes x$,	
$\pi_{121} \simeq \pi_{212}$,	$\Phi = R P_{13}$	(same as type A),
$\pi_{2323} \simeq \pi_{3232}$,	$\Psi = K P_{14} P_{23}$	(New object).

$$K \in \mathrm{End}(F_{q^2} \otimes F_q \otimes F_{q^2} \otimes F_q), \quad R \in \mathrm{End}((F_q)^{\otimes 3}).$$

Matrix elements

$$K(|a\rangle \otimes |i\rangle \otimes |b\rangle \otimes |j\rangle) = \sum_{c,m,d,n} K^{cmdn}_{a\,i\,b\,j} |c\rangle \otimes |m\rangle \otimes |d\rangle \otimes |n\rangle.$$

$$K^{cmdn}_{a\,i\,b\,j} = 0 \;\text{ unless }\; c+m+d = a+i+b, \;\; d+n-c = b+j-a.$$

Theorem (Explicit form)

$$K^{c,m,0,n}_{a,i,0,j} = \sum_{\lambda \geq 0} (-1)^{m+\lambda} \frac{(q^4)_{c+\lambda}}{(q^4)_c} q^{\phi_2} \begin{bmatrix} i,j \\ \lambda, j-\lambda, m-\lambda, i-m+\lambda \end{bmatrix},$$

$$\phi_2 = (a+c+1)(m+j-2\lambda)+m-j.$$

$$K^{cmdn}_{a\,i\,b\,j} = \frac{(q^4)_a}{(q^4)_c} \sum_{\alpha,\beta,\gamma \geq 0} \frac{(-1)^{\alpha+\gamma}}{(q^4)_{d-\beta}} q^{\phi_1} K^{a,i+b-\alpha-\beta-\gamma,0,j+b-\alpha-\beta-\gamma}_{c,m+d-\alpha-\beta-\gamma,0,n+d-\alpha-\beta-\gamma}$$

$$\times \begin{bmatrix} b, d-\beta, i+b-\alpha-\beta, j+b-\alpha-\beta \\ \alpha, \beta, \gamma, m-\alpha, n-\alpha, b-\alpha-\beta, d-\beta-\gamma \end{bmatrix},$$

$$\phi_1 = \alpha(\alpha+2d-2\beta-1)+(2\beta-d)(m+n+d)+\gamma(\gamma-1)-b(i+j+b).$$

Theorem

R and K satisy the **3D reflection equation**

$$R_{456}R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234} = K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}R_{456}.$$

- The proof is parallel with type A.
- Uses the reduced expressions of the longest element $s_1s_2s_3s_2s_1s_2s_3s_2s_3 = s_3s_2s_3s_2s_1s_2s_3s_2s_1 \in W(\mathrm{Sp}_6)$.
- The two sides come from the intertwiners for $\pi_{123212323} \simeq \pi_{323212321}$.

Physical interpretation of the 3D reflection equation

$$R_{456}R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234} = K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}R_{456}.$$

is a "factorization" of 3 string scattering with boundary reflections.

R : Scattering amplitude of 3 strings.
K: Reflection amplitude with boundary freedom signified by spaces $1, 3, 7$.

Trivializing them leads to

$$R_{456}R_{489}K_{59}R_{269}R_{258}K_{68}K_{24} = K_{24}K_{68}R_{258}R_{269}K_{59}R_{489}R_{456}.$$

This coincides with (constant version of) **Tetrahedron reflection equation** by Isaev-Kulish (1997), for which no solution is known yet.

Our 3D reflection equation
= a matrix version of tetrahedron reflection eq. with an explicit solution.

Classical analogue: Birational maps **R** and **K**.

$$\mathbf{R}:(c,b,a)\mapsto\left(\frac{bc}{a+c},\, a+c,\, \frac{ab}{a+c}\right),$$
$$\mathbf{K}:(d,c,b,a)\mapsto\left(\frac{bcd}{A},\frac{A^2}{B},\frac{B}{A},\frac{ab^2c}{B}\right),$$
$$A=ab+ad+cd,\quad B=ab^2+2abd+ad^2+cd^2.$$

The map **R** is due to Berenstein-Fomin-Zelevinsky(96), Kashaev-Korepanov-Sergeev(98), Y.Yamada(01), etc. **K** is new.

Theorem

R *and* **K** *satisfy the 3D reflection equation on 9 variables.*

$$\mathbf{R}_{456}\mathbf{R}_{489}\mathbf{K}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{K}_{1678}\mathbf{K}_{1234}=\mathbf{K}_{1234}\mathbf{K}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{K}_{3579}\mathbf{R}_{489}\mathbf{R}_{456}.$$

Proof. Independence of the Schubert cell for $w\in W(\mathrm{Sp}_6)$ on reduced expressions of w. □

Tropicalization: $ab \mapsto a+b, \;\; a+b \to \min(a,b)$.

Theorem (Combinatorial analogue: Bijections $\mathcal{R}$ and $\mathcal{K}$)

$$\lim_{q\to 0} R = \textit{tropicalization of } \mathbf{R}, \qquad \lim_{q\to 0} K = \textit{tropicalization of } \mathbf{K}.$$

They yield bijections $\mathcal{R}, \mathcal{K}$ on finite sets, still satisfying 3D reflection eq.

$$\mathcal{R}(c,b,a) = (b+c-\min(a,c), \min(a,c), a+b-\min(a,c)) : (\mathbb{Z}_{\ge 0})^3 \to (\mathbb{Z}_{\ge 0})^3,$$
$$\mathcal{K}(d,c,b,a) = (\text{similar formula}) : \qquad (\mathbb{Z}_{\ge 0})^4 \to (\mathbb{Z}_{\ge 0})^4.$$

Conclusion: Triad of 3D R and K

Quantum		Combinatorial		Birational
R, K	$\overset{q\to 0}{\longrightarrow}$	$\mathcal{R}, \mathcal{K}$	$\overset{\textit{tropicalization}}{\longleftarrow}$	$\mathbf{R}, \mathbf{K}$