

組合せ論的ベータ仮説
(Combinatorial Bethe ansatz)

東京大学 総合文化研究科 (Univ. Tokyo)
国場敦夫 (Atsuo Kuniba)

日本数学会 東北大学 21 Sep. 2007

“Unsere Method liefert also alle Lösungen des Problems.”

... H. Bethe (1931)

“ ... composition of our bijection with the Robinson-Schensted-Knuth correspondence may be viewed as a combinatorial version of the Bethe ansatz.”

... Kerov-Kirillov-Reshetikhin (1986)

1 dimensional Heisenberg chain

$$H = \sum_{k=1}^L (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)$$

$$\sigma_k^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_k, \quad \sigma_k^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_k, \quad \sigma_k^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_k, \quad (\sigma_{L+1}^a = \sigma_1^a)$$

H acts on

$$(\mathbb{C}^2)^{\otimes L} = \otimes (\text{spin up or down})$$

Symmetry: $[sl_2, H] = 0$.

In particular, H preserves $N := \#$ of down spins.

Diagonalization of H . (Bethe 1931)

$$H|u_1, \dots, u_N\rangle = E|u_1, \dots, u_N\rangle, \quad E = \sum_{j=1}^N \frac{-8}{u_j^2 + 1},$$

if u_1, \dots, u_N satisfy Bethe equation

$$\left(\frac{u_j + i}{u_j - i} \right)^L = \prod_{k=1, (k \neq j)}^N \frac{u_j - u_k + 2i}{u_j - u_k - 2i} \quad (j = 1, \dots, N).$$

$|u_1, \dots, u_N\rangle$: Bethe vector

- contains N down spins
- symmetric under permutations of u_j 's.

Example. Length $L = 6$ chain with $N = 3$ down spins.

Unknowns: u_1, u_2, u_3

$$\begin{aligned}\left(\frac{u_1 + i}{u_1 - i}\right)^6 &= \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)}, \\ \left(\frac{u_2 + i}{u_2 - i}\right)^6 &= \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)}, \\ \left(\frac{u_3 + i}{u_3 - i}\right)^6 &= \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.\end{aligned}$$

How many solutions up to permutations of u_1, u_2 and u_3 ?

We should have **5** vectors if Bethe ansatz is complete.

Reason:

- Bethe vector is **highest weight vector** with respect to sl_2 .
- Irreducible decomposition:

$$\square^{\otimes 6} = \square \square \square \square \square \square + 5 \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \end{array} + 9 \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \end{array} + 5 \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}$$

$$2^6 = 7 + 5 \times 5 + 9 \times 3 + 5 \times 1$$

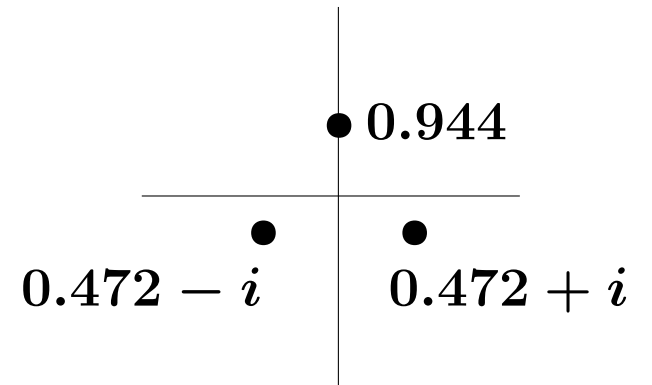
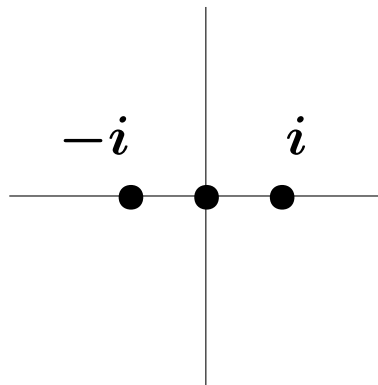
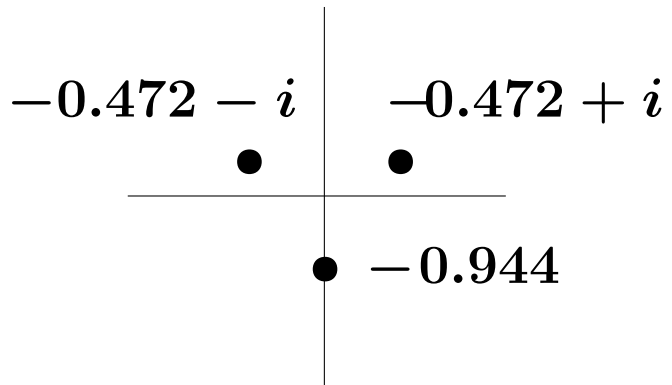
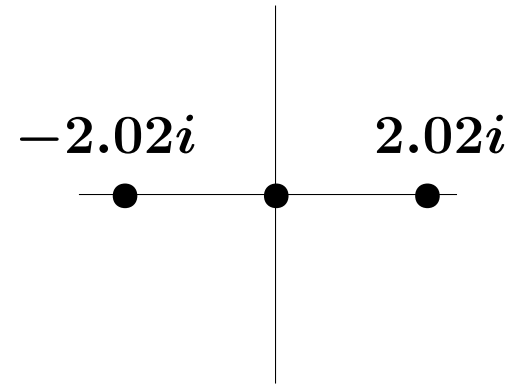
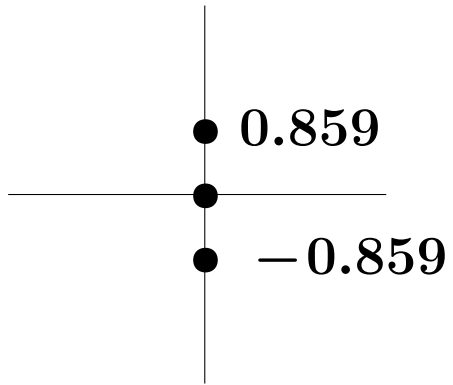
$$(N = 0)$$

$$(N = 1)$$

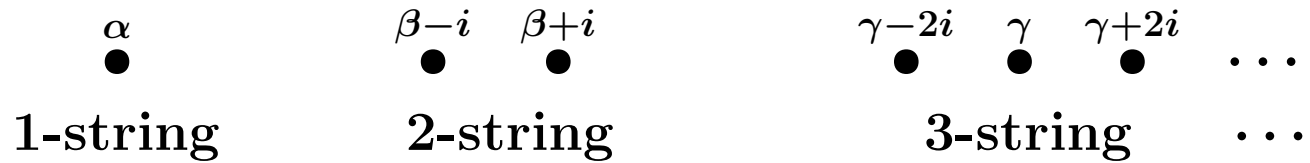
$$(N = 2)$$

$$(N = 3)$$

General case: **Kostka number** $K_{(L-N,N),(1^L)} = \binom{L}{N} - \binom{L}{N-1}$



String hypothesis: Bethe roots $\{u_1, \dots, u_N\} =$ collection of **strings**.

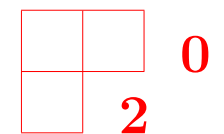
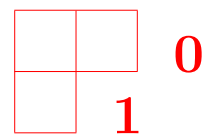
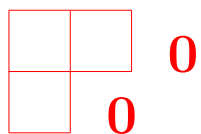
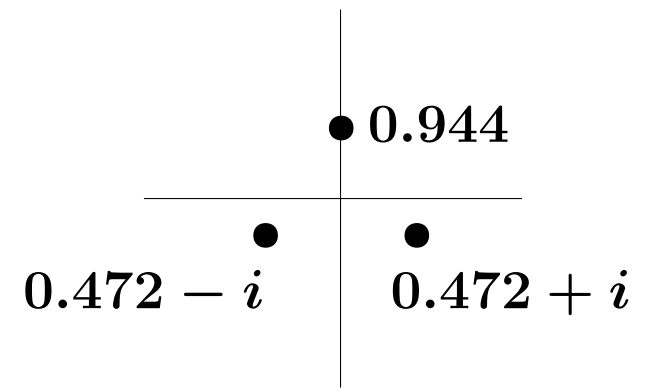
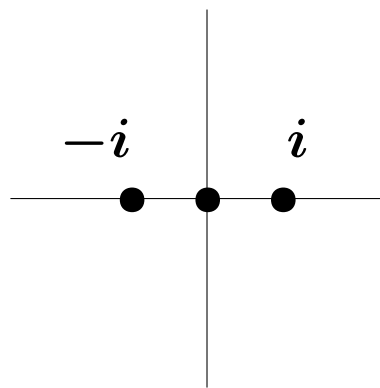
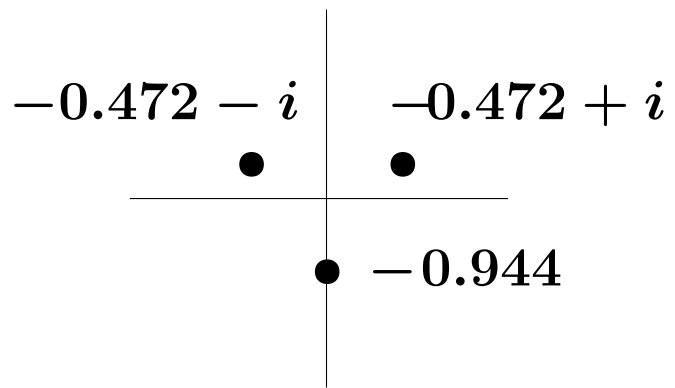
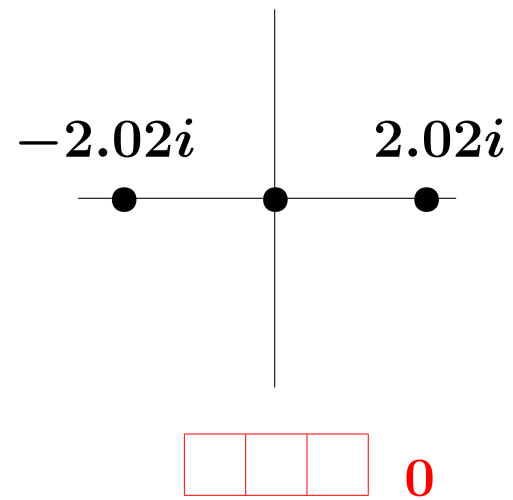
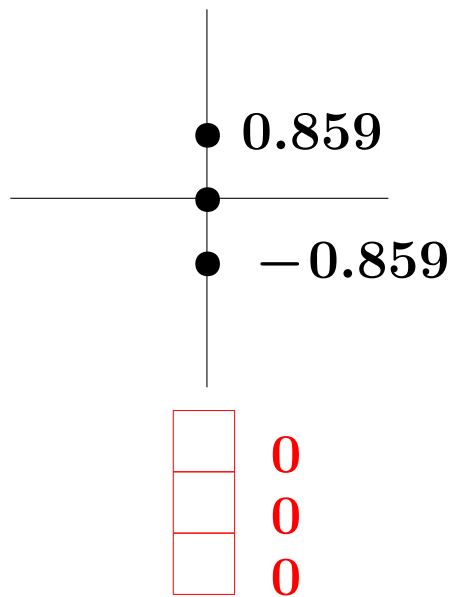


with small distortions ($\alpha, \beta, \gamma, \dots \in \mathbb{R}$: string center).

Bethe equation + string hypothesis

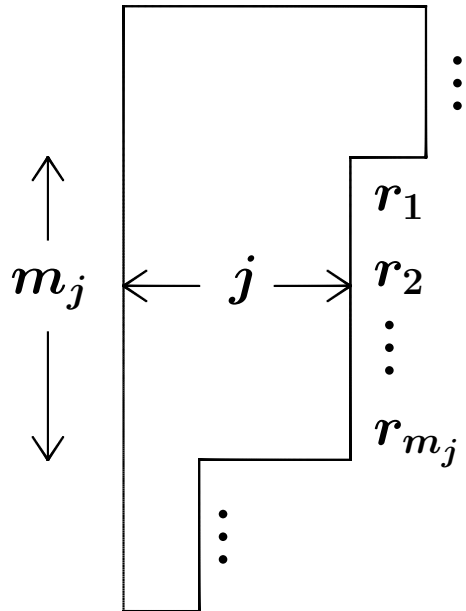
↓ physicist's argument

$\{\text{Bethe roots}\} \xleftrightarrow{1:1?} \{\text{Rigged configuration}\}$



Rigged configuration for sl_2 (spin $\frac{1}{2}$) $^{\otimes L}$

Young diagram = configuration, $\{r_i\}$ = rigging



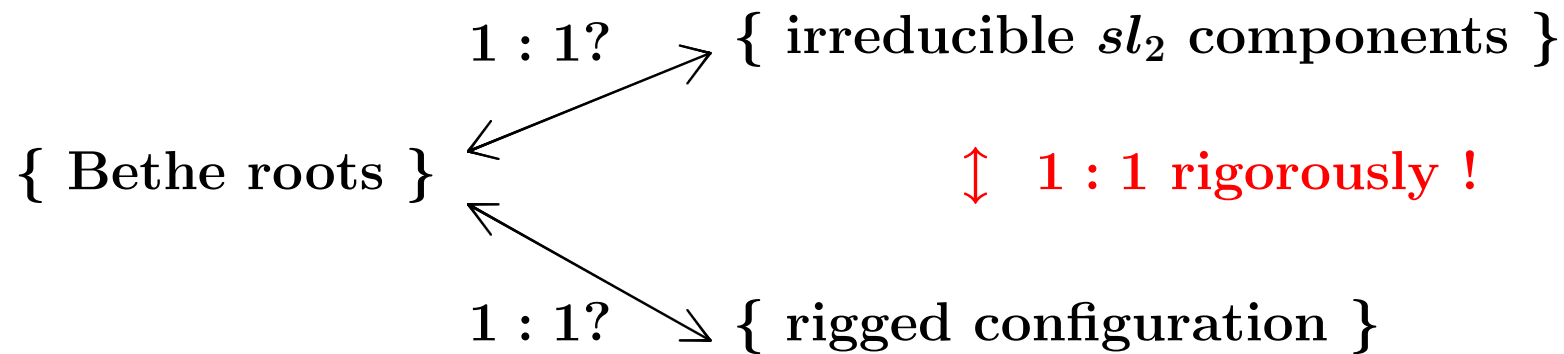
$$0 \leq r_1 \leq \dots \leq r_{m_j} \leq p_j$$

... **(fermionic) selection rule**

$$p_j = L - 2 \sum_{k \geq 1} \min(j, k) m_k$$

... **vacancy number**

$$\# \text{ of rigged configurations} = \sum_{\{m_i\}} \prod_{i \geq 1} \binom{p_i + m_i}{m_i}$$



Theorem. (“alle Lösungen”, Bethe’s fermionic formula 1931)

$$\text{Kostka number } K_{(L-N, N), (1^L)} = \sum_{\{m_i\}} \prod_{i \geq 1} \binom{p_i + m_i}{m_i}$$

KKR theory. (Kerov-Kirillov-Reshetikhin 1986)

Canonical bijection and q -analogue of Bethe’s formula from integrable spin chain with sl_n symmetry.

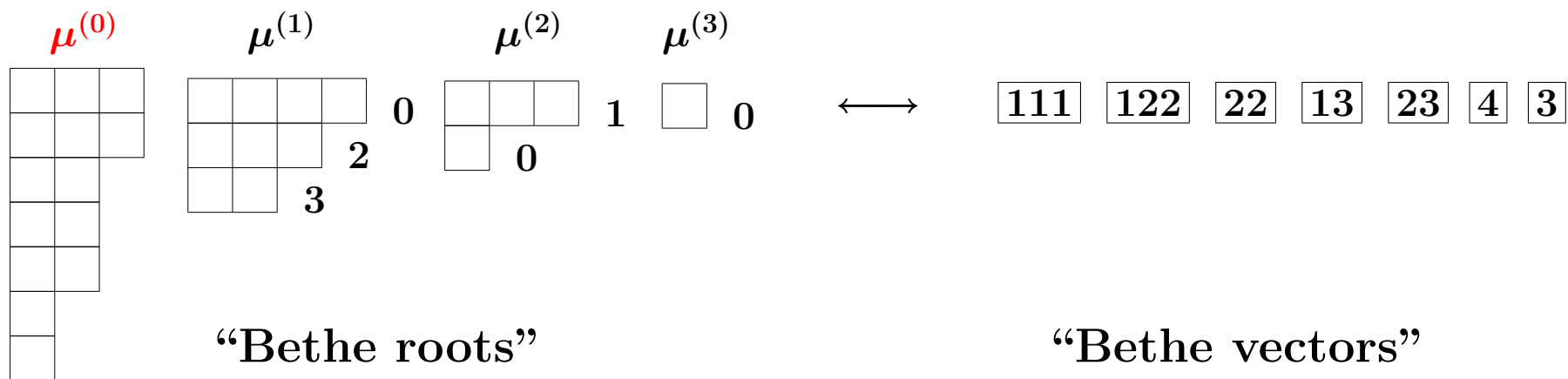
{rigged configurations} $\xleftrightarrow{\text{KKR}}$ {standard tableaux} $\xleftrightarrow{\text{RS}}$ {highest paths}

<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td></tr> </table>		0		0		0		<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">1</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">3</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">5</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">2</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">4</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">6</td></tr> </table>	1	3	5	2	4	6		121212
	0															
	0															
	0															
1	3	5														
2	4	6														
<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td></tr> </table>				0		<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">1</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">2</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">3</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">4</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">5</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">6</td></tr> </table>	1	2	3	4	5	6		111222		
			0													
1	2	3														
4	5	6														
<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td><td></td></tr> </table>			0		0			<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">1</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">3</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">4</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">2</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">5</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">6</td></tr> </table>	1	3	4	2	5	6		121122
		0														
	0															
1	3	4														
2	5	6														
<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">1</td><td></td></tr> </table>			0		1			<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">1</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">2</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">5</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">3</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">4</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">6</td></tr> </table>	1	2	5	3	4	6		112212
		0														
	1															
1	2	5														
3	4	6														
<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">0</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="padding: 0 5px;">2</td><td></td></tr> </table>			0		2			<table style="border-collapse: collapse; margin: auto;"> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">1</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">2</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">4</td></tr> <tr><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">3</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">5</td><td style="border: 1px solid black; width: 20px; height: 20px; text-align: center;">6</td></tr> </table>	1	2	4	3	5	6		112122
		0														
	2															
1	2	4														
3	5	6														

“composition of **our bijection** with the **R**obinson-**S**chensted-Knuth correspondence may be viewed as a combinatorial version of the Bethe ansatz.” (KKR 1986)

KKR bijection for sl_n

{rigged configurations} $\xleftrightarrow{1:1}$ {highest paths}

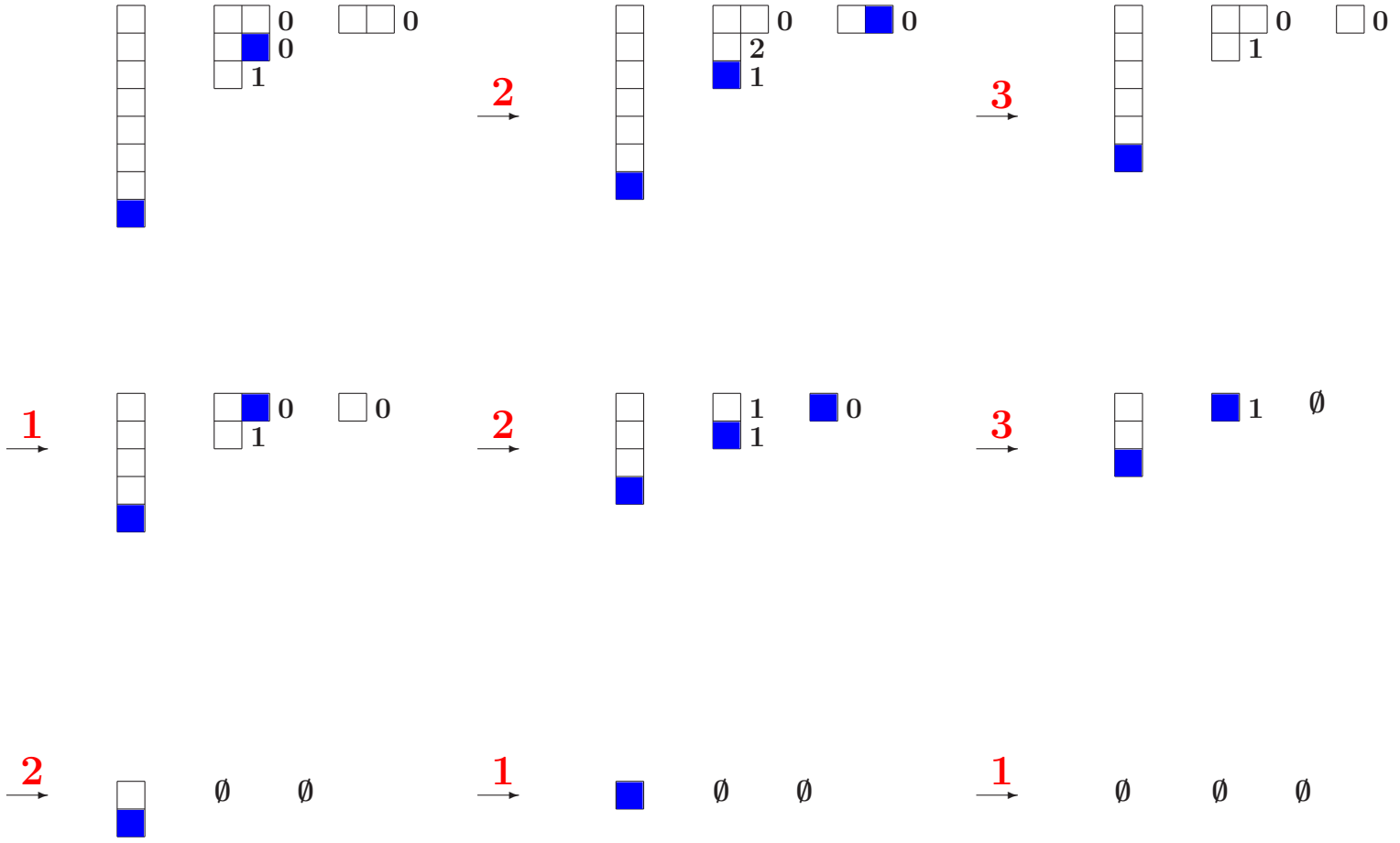


- highest path = $b_1 b_2 \dots b_L$, $b_i = \text{shape } (\mu_i^{(0)}) \text{ semistandard tab. (+ highest condition)}$

- rigged configuration: $(\mu, r) = (\mu^{(0)}, (\mu^{(1)}, r^{(1)}), \dots, (\mu^{(n-1)}, r^{(n-1)}))$

$$\left. \begin{array}{l} \mu^{(0)}, \mu^{(1)}, \dots, \mu^{(n-1)} : \text{configuration} \\ r^{(1)}, \dots, r^{(n-1)} : \text{rigging} \end{array} \right\} (+\text{selection rule})$$

Example of KKR algorithm



Top left rigged configuration $\xrightarrow{\text{KKR}}$ **11232132**

- Lascoux-Schützenberger charge on semistandard tableaux

↓ KKR

charge function $c(\mu, r)$

$$c(\mu, r) = \frac{1}{2} \sum_{a,b=1}^{n-1} C_{ab} \min(\mu^{(a)}, \mu^{(b)}) - \min(\mu^{(0)}, \mu^{(1)}) + \sum_{a=1}^{n-1} |r^{(a)}|$$

$$\left(\begin{array}{l} \min(\lambda, \mu) = \sum_{ij} \min(\lambda_i, \mu_j), \quad |r| = \sum_i r_i \\ (C_{ab}) = \text{Cartan matrix of } sl_n \end{array} \right)$$

Corollary. (Fermionic formula for Kostka-Foulkes polynomial)

$$q^* K_{\lambda, \mu^{(0)}}(q) = \sum_{(\mu, r)} q^{c(\mu, r)} = \sum_{\mu} q^{c(\mu, 0)} \prod_{a,i} \left[\begin{array}{c} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{array} \right]_q$$

Summary so far

KKR bijection

$$\{\text{rigged configurations}\} \xleftrightarrow{1:1} \{\text{highest paths}\}$$

is a combinatorial analogue of the Bethe ansatz

$$\{\text{Bethe roots}\} \longrightarrow \{\text{Bethe vectors}\}$$

that establishes the Fermionic formula.

∃ Conjectural fermionic formula for \forall affine Lie algebra
(Hatayama et al. 1999, 2002).

Box-ball system on ∞ lattice (Takahashi-Satsuma 1990)

$\dots 1111111144322111111111111111111111111111111111 \dots$
 $\dots 1111111111111144322111111111111111111111111111 \dots$
 $\dots 1111111111111111111144322111111111111111111111 \dots$

1 =empty box, 2, 3, 4 = color of balls

time evolution: $T_\infty = (\text{move } 2)(\text{move } 3)(\text{move } 4)$

(move i) · Pick the leftmost ball with color i and move it to the nearest right empty box.

· Do the same for the other color i balls.

- soliton=consecutive balls $i_1 \dots i_a$ with color $i_1 \geq \dots \geq i_a \geq 2$.
- velocity=amplitude.

- Collisions of 2 solitons

... 111**44322**1111**433**111111111111111111111111 ...
... 11111111144**322**114**33**111111111111111111111111 ...
... 1111111111111144**322**4**33**1111111111111111111111 ...
... 11111111111111111111**322**44**433**11111111111111 ...
... 111111111111111111111111**322**1144**433**1111111111 ...
... 1111111111111111111111111111**322**1111**44433**11 ...

- Amplitudes are individually conserved.

- Two body scattering:

Exchange of internal labels (colors) like quarks in hadrons

Phase shift

Collision of 3 solitons

... 11**432**11**42**1111**3**111111111111111111 ...
... 11111**432**1**42**111**3**111111111111111111 ...
... 11111111**43**1**422**1**3**111111111111111111 ...
... 1111111111**43**11**4232**111111111111111111 ...
... 111111111111**43**11**21432**1111111111111111 ...
... 1111111111111111**43**1**211432**11111111111111 ...
... 111111111111111111**4132**1111**432**11 ...

... 11**432**1111**42**11**3**111111111111111111 ...
... 11111**432**111**42**1**3**111111111111111111 ...
... 11111111**432**11**423**111111111111111111 ...
... 111111111111**432**1**243**111111111111111111 ...
... 1111111111111111**4132432**1111111111111111 ...
... 111111111111111111**411321432**11111111111111 ...
... 1111111111111111111**411132**11**432**11 ...

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

Double origin of integrability

(1) **UltraDiscretization (UD)** of soliton equations

- Key formula

$$\lim_{\varepsilon \rightarrow +0} \varepsilon \log \left(\exp\left(\frac{a}{\varepsilon}\right) + \exp\left(\frac{b}{\varepsilon}\right) \right) = \max(a, b)$$

$$(+, \times) \longrightarrow (\max, +)$$

keeps distributive law:

$$AB + AC = A(B + C) \rightarrow \max(a + b, a + c) = a + \max(b, c)$$

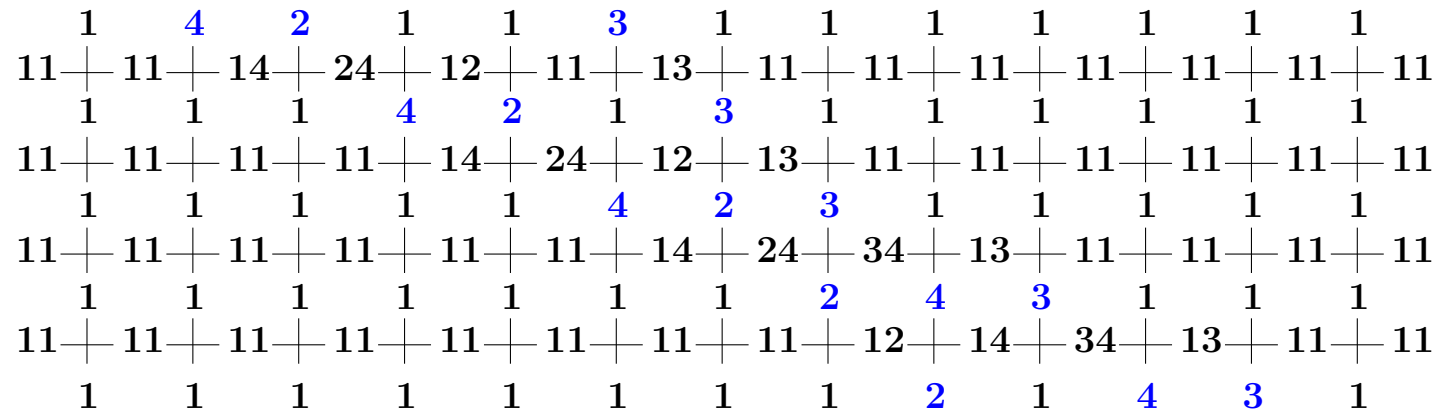
- UD of a discrete KdV equation turns out to be an evolution eq. of the 2-state box-ball system. (Tokihiro et al. 1996)

(2) Solvable lattice model at “temperature 0”

Time evolution pattern

$\dots 1421131111111 \dots$
 $\dots 1114213111111 \dots$
 $\dots 1111142311111 \dots$
 $\dots 1111111243111 \dots$
 $\dots 1111111121431 \dots$

is a configuration of a 2D lattice model in statistical mechanics

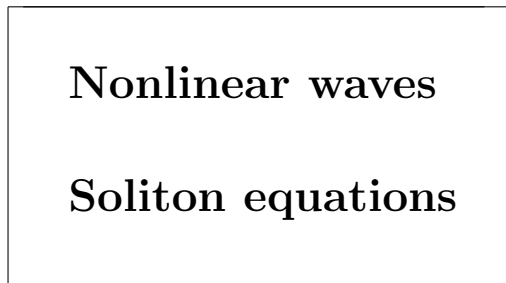


- n -state box-ball system
 - = 2D solvable vertex model associated with quantum group $U_q(\widehat{sl}_n)$ at $q = 0$. ($q \sim$ temperature)
- row transfer matrix at $q = 0$
 - = deterministic map
 - = time evolution of box-ball system
(forms a commuting family $T_1, T_2, \dots, T_\infty$.)
- proper formulation by Kashiwara's crystal base theory
(Hatayama et al. 2000, Fukuda et al. 2000)

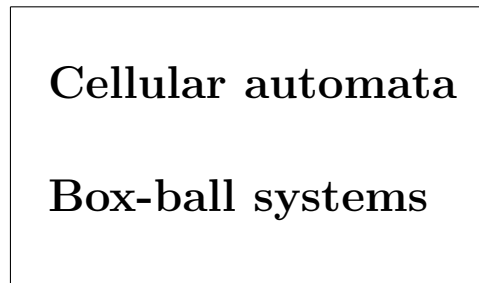
Classical
integrable system

Ultradiscrete
integrable system

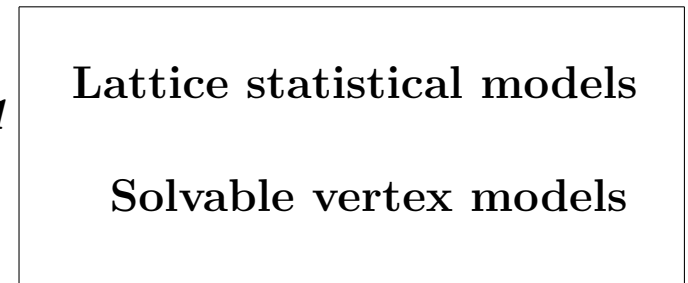
Quantum
integrable system



UD
→



$0 \leftarrow q$
←



Inverse scattering method

Combinatorial Bethe ansatz

Bethe ansatz

Natural to invest KKR theory to box-ball systems.

Dynamics of box-ball system in terms of rigged configuration

$t = 0:$ 11112222111113321143111111111111111111111111111111
 $t = 1:$ 1111111122221111332143111111111111111111111111111111
 $t = 2:$ 111111111111222211133243111111111111111111111111111111
 $t = 3:$ 11111111111111112222113243311111111111111111111111111
 $t = 4:$ 111111111111111111112221322433111111111111111111111111
 $t = 5:$ 1111111111111111111111122113224332111111111111111111111
 $t = 6:$ 1111111111111111111111111221113221433211111111111111111
 $t = 7:$ 11111111111111111111111111111221111322114332111111111111

$$\begin{array}{cccc}
 \mu^{(0)} & \mu^{(1)} & \mu^{(2)} & \mu^{(3)} \\
 (1^{48}) & \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \begin{array}{l} 4t \\ 6 + 3t \\ 11 + 2t \end{array} & \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \begin{array}{l} 1 \\ 0 \end{array} & \begin{array}{|c|} \hline \\ \hline \end{array} 0
 \end{array}$$

configuration \dots conserved quantity (action variable)

rigging \dots linear flow (angle variable)

KKR bijection \dots direct/inverse scattering map (separation of variables)

Theorem. (KO-Sakamoto-TY 2006)

- KKR theory (a version of combinatorial Bethe ansatz) is the inverse scattering scheme of the box-ball system on ∞ lattice.
- Solution of the initial value problem.

KKR theory	box-ball system	crystal theory
rigged configuration	scattering data	\otimes (affine crystals)
KKR bijection	direct/inverse scattering	composition of R

Ultradiscrete tau function

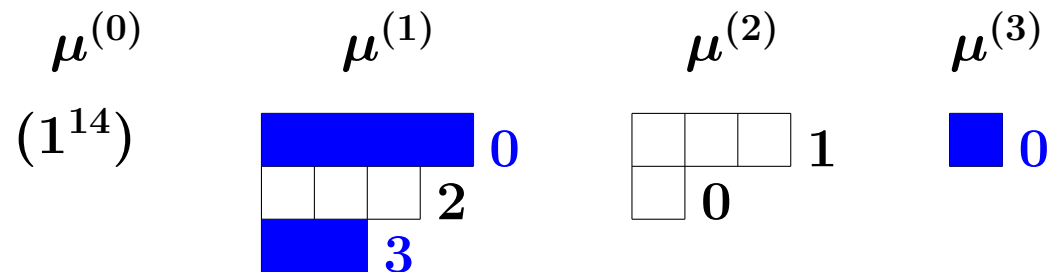
rigged configuration: $(\mu, r) = (\mu^{(0)}, (\mu^{(1)}, r^{(1)}), \dots, (\mu^{(n-1)}, r^{(n-1)}))$

charge function: $c(\mu, r) = \frac{1}{2} \sum_{a,b=1}^{n-1} C_{ab} \min(\mu^{(a)}, \mu^{(b)}) - \min(\mu^{(0)}, \mu^{(1)}) + \sum_{a=1}^{n-1} |r^{(a)}|$

UD tau function :

$$\tau_{k,i} = - \min_{(\nu, s)} \{ c(\nu, s) + |\nu^{(i)}| \} \quad (k \geq 1, 1 \leq i \leq n)$$

min : over all subsets $(\nu, s) \subseteq (\mu, r)$ with $\nu^{(0)} = (\mu_1^{(0)}, \dots, \mu_k^{(0)})$.



Theorem. (KSY 2007) $\tau_{k,i}$ admits characterizations:

- (1) UD of tau functions for KP hierarchy
- (2) “corner transfer matrix” of box-ball system ■

$$(1) \quad \tau_{k,i} = \lim_{\epsilon \rightarrow 0} \epsilon \log \langle i | \exp \left(\sum_j c_j \psi(p_j) \psi^*(q_j) \right) | i \rangle$$

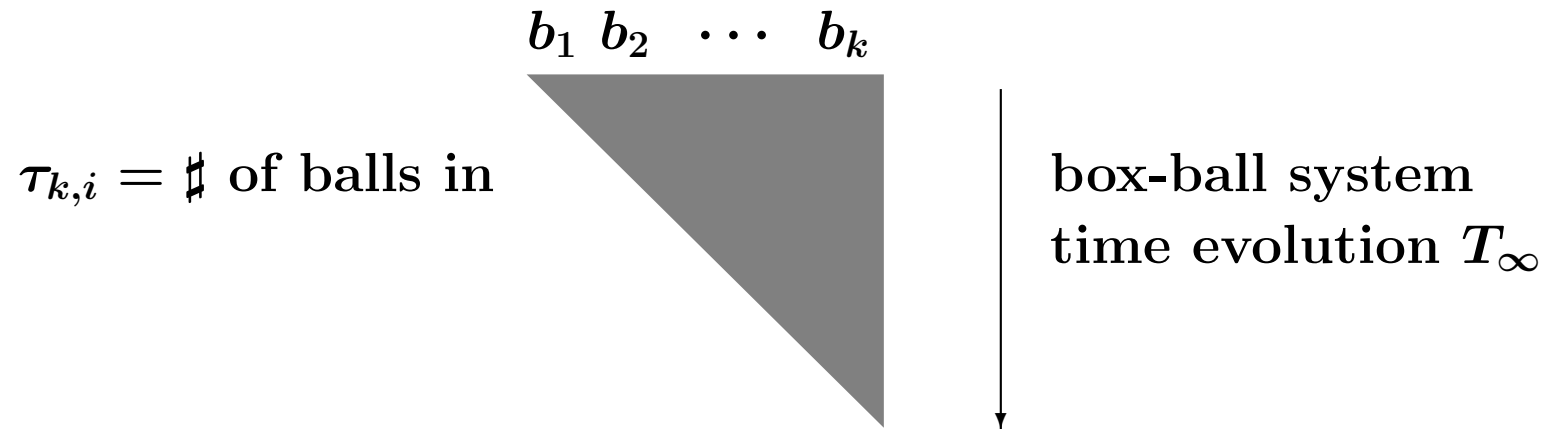
$(c_j, p_j, q_j) \longleftrightarrow$ strings in rigged configuration

Corollary of (1). (UD Hirota bilinear eq.)

$$\bar{\tau}_{k,i-1} + \tau_{k-1,i} = \max(\bar{\tau}_{k,i} + \tau_{k-1,i-1}, \bar{\tau}_{k-1,i-1} + \tau_{k,i} - \mu_k^{(0)})$$

$$\bar{\tau}_{k,i} = \tau_{k,i} \text{ associated with } T_\infty((\mu, r))$$

(2) Suppose $b_1 b_2 \cdots b_L \xleftrightarrow{\text{KKR}} (\mu, r) \longrightarrow \{\tau_{k,i}\}$. Then,



... “analogue” of Baxter’s corner transfer matrix

UD Hirota bilinear = eq. of motion of box-ball system.

Corollary of (1)&(2).

- **piecewise linear formula for KKR map:** $(\mu, r) \xrightarrow{\text{KKR}} b_1 \dots b_L$

semistandard tab. $b_k = (\overbrace{1 \dots 1}^{x_{k,1}} \dots \overbrace{n \dots n}^{x_{k,n}})$ is specified by

$$x_{k,i} = \tau_{k,i} - \tau_{k-1,i} - \tau_{k,i-1} + \tau_{k-1,i-1}.$$

- **general** N soliton solution of box-ball system

	Bethe ansatz	Corner transfer matrix
main combinatorial object	rigged configuration	charge (energy) in affine crystal
role in box-ball system	action-angle variable	tau function
dynamics	linear	bilinear

Periodic box-ball system $(sl_2 \text{ spin } \frac{1}{2}, \text{ size } L)$

evolution under T_2

```

1 1 2 1 1 1 2 2 2 1 1 1 2 2
2 2 1 2 1 1 1 1 2 2 2 1 1 1
1 1 2 1 2 2 1 1 1 1 2 2 2 1
2 1 1 2 1 1 2 2 1 1 1 1 2 2
2 2 2 1 2 1 1 1 2 2 1 1 1 1
1 1 2 2 1 2 2 1 1 1 2 2 1 1
1 1 1 1 2 1 2 2 2 1 1 1 2 2
2 2 1 1 1 2 1 1 2 2 2 1 1 1
    
```

evolution under T_3

```

1 1 2 1 1 1 2 2 2 1 1 1 2 2
2 2 1 2 1 1 1 1 1 2 2 2 1 1
1 1 2 1 2 2 2 1 1 1 1 1 2 2
2 2 1 2 1 1 1 2 2 2 1 1 1 1
1 1 2 1 2 2 1 1 1 1 2 2 2 1
2 2 1 2 1 1 2 2 1 1 1 1 1 2
1 1 2 1 2 2 1 1 2 2 2 1 1 1
1 1 1 2 1 1 2 2 1 1 1 2 2 2
    
```

commuting family of time evolutions T_1, T_2, \dots

$T_\infty =$ “ball moving” procedure (Yoshihara et al. 2003).

proper formulation \dots crystals and combinatorial R .

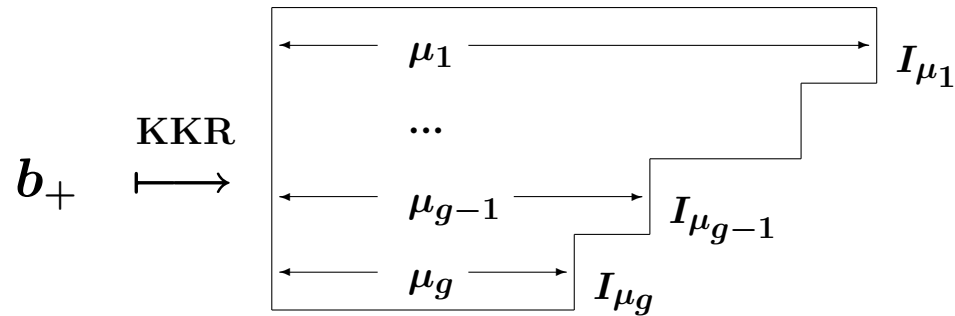
- Action-angle variables

Any path b is expressed as

$$b = T_1^d(b_+) \quad (T_1 : \text{cyclic shift, } b_+ : \text{highest path, } d \in \mathbb{Z}),$$

where (d, b_+) is not unique.

But anyway let (μ, \mathbf{I}) be the rigged configuration for b_+ :



$$\mu = (\mu_1, \dots, \mu_g), \quad \mathbf{I} = (I_{\mu_1}, \dots, I_{\mu_g}).$$

$$\left(\begin{array}{l} \text{For simplicity we assume } \mu_1 > \mu_2 > \dots > \mu_g \\ p_i = L - 2 \sum_{j \in \mu} \min(i, j) : \text{vacancy number} \end{array} \right)$$

Lemma.

- μ and $(\mathbf{I} + d\mathbf{h}_1)/A\mathbb{Z}^g$ are unique, where

$$\mathbf{h}_l = (\min(l, i))_{i \in \mu} \in \mathbb{Z}^g, \quad \mathbf{A} = (\delta_{ij} p_i + 2 \min(i, j))_{i, j \in \mu}.$$

- μ is invariant under $\{T_l\}$ (action variable).

Define

$\mathcal{P}(\mu) := \{\text{paths whose action variable} = \mu\}$: iso-level set

$\mathcal{J}(\mu) := \mathbb{Z}^g / A\mathbb{Z}^g$: set of angle variables

$\Phi(b) := (\mathbf{I} + d\mathbf{h}_1)/A\mathbb{Z}^g : \mathcal{P}(\mu) \longrightarrow \mathcal{J}(\mu)$

Theorem. (KT-Takenouchi 2006)

$\Phi : \mathcal{P}(\mu) \rightarrow \mathcal{J}(\mu)$ is a bijection.

$$\begin{array}{ccc} \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \\ T_l \downarrow & & \downarrow T_l \\ \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \end{array}$$

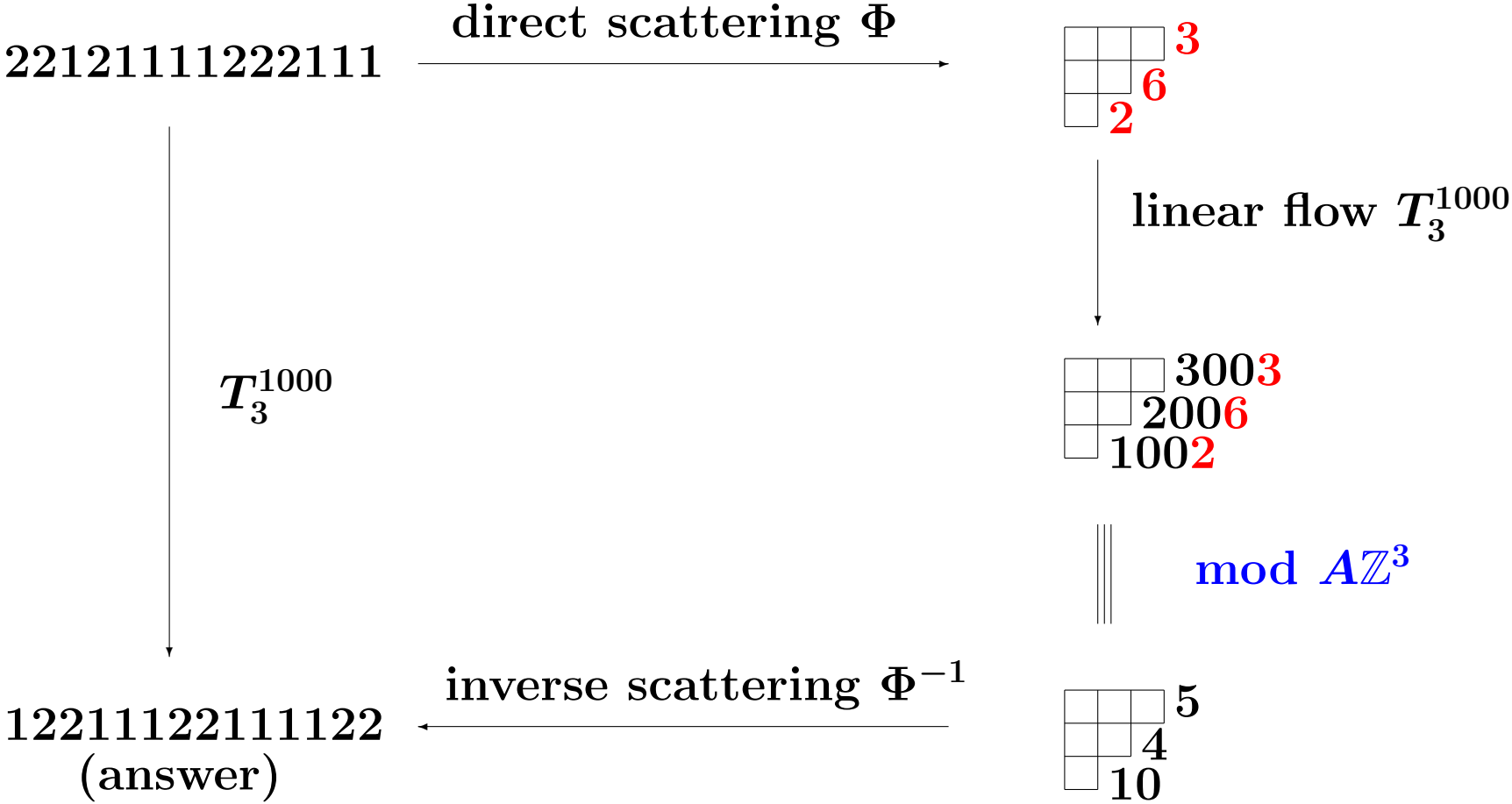
is commutative, where $T_l(\mathbf{J}) = \mathbf{J} + \mathbf{h}_l$ on $\mathcal{J}(\mu)$ ■

Nonlinear dynamics becomes straight motions in

$$\mathcal{J}(\mu) = \mathbb{Z}^g / \mathbf{A}\mathbb{Z}^g,$$

which is an ultradiscrete analogue of Jacobi variety.

Solution of initial value problem (inverse method)



Riemann theta (with pure imaginary period matrix) :

$$\vartheta(\mathbf{z}) := \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp\left(-\frac{{}^t \mathbf{n} \mathbf{A} \mathbf{n} / 2 + {}^t \mathbf{n} \mathbf{z}}{\epsilon}\right)$$

UD Riemann theta ($\mathbf{z} \in \mathbb{R}^g$):

$$\Theta(\mathbf{z}) := \lim_{\epsilon \rightarrow +0} \epsilon \log \vartheta(\mathbf{z}) = - \min_{\mathbf{n} \in \mathbb{Z}^g} \{{}^t \mathbf{n} \mathbf{A} \mathbf{n} / 2 + {}^t \mathbf{n} \mathbf{z}\}$$

Theorem. (KS 2006)

Action-angle variable $(\mu, \mathbf{I}) \mapsto \text{path } b_1 b_2 \dots b_L \in \{1, 2\}^L$ is given by

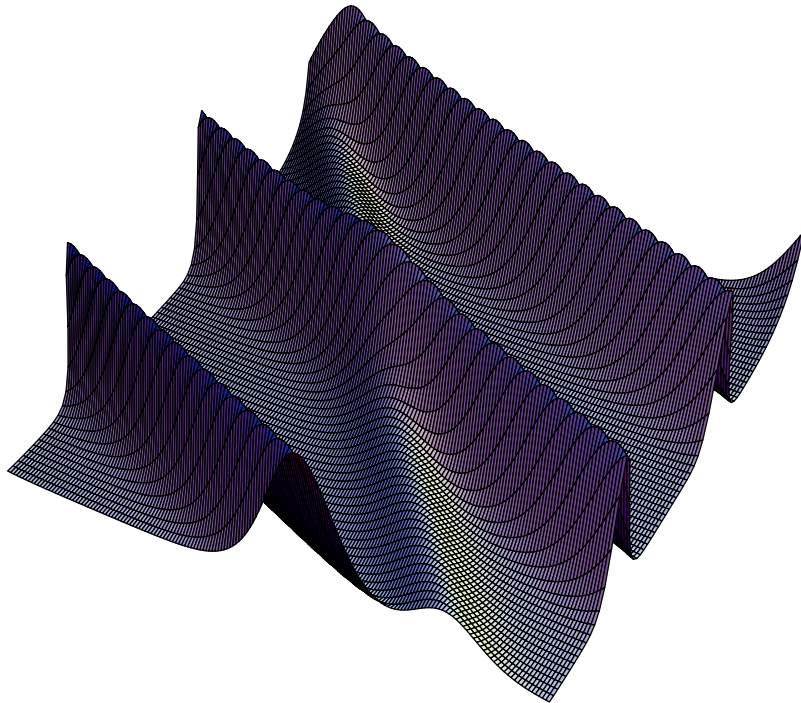
$$b_k = 1 + \Theta(\mathbf{J} - k\mathbf{h}_1) - \Theta(\mathbf{J} - (k-1)\mathbf{h}_1) \\ - \Theta(\mathbf{J} - k\mathbf{h}_1 + \mathbf{h}_\infty) + \Theta(\mathbf{J} - (k-1)\mathbf{h}_1 + \mathbf{h}_\infty),$$

with $\mathbf{J} = \mathbf{I} - {}^t(p_{\mu_1}, \dots, p_{\mu_g})/2$.

Inverse UD: double difference of $\Theta \longrightarrow$ double ratio of ϑ

$$b(k, t) = \frac{\vartheta(J + th_\infty - kh_1)\vartheta(J + (t + 1)h_\infty - (k - 1)h_1)}{\vartheta(J + th_\infty - (k - 1)h_1)\vartheta(J + (t + 1)h_\infty - kh_1)}.$$

Same structure as the quasi-periodic solution of the KdV/Toda eq. by Date-Tanaka and Kac-Moerbeke (1976).



Two soliton state with amplitudes 6 and 2.
System size $L = 170$, duration $0 \leq t \leq 70$.

Remarks.

- Origin of UD period matrix A :
 $U_q(\widehat{sl}_2)$ Bethe equation at $q = 0$ (K-Nakanishi 2000)
- General case:
UD Riemann theta with rational characteristics.
- $\mathcal{J}(\mu)$ from Jacobian of tropical hyperelliptic curve.
(Inoue-Takenawa 2007)

“I haven’t a slightest idea of what people did with it.”

... H. Bethe