Tetrahedron equation and quantum R matrices for spin representations

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19 Sept. 2012 Kyushu University

Tetrahedron equations

- RRRR relation

$$R \in \mathrm{End}(F \otimes F \otimes F) \qquad \text{(3D } R\text{)}$$

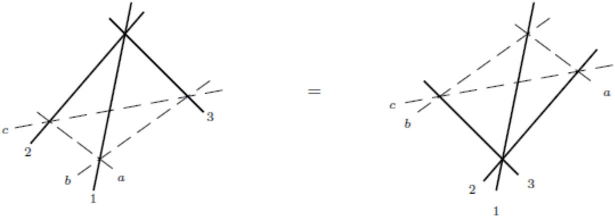
$$R_{123}R_{145}R_{246}R_{356} = R_{356}R_{246}R_{145}R_{123} \in \mathrm{End}(F^{\otimes 6})$$

- RLLL relation

$$L_{1ab} \in \mathrm{End}(\overset{1}{F} \otimes \overset{a}{V} \otimes \overset{b}{V}) \qquad \text{(3D } L\text{)}$$

$$R_{123}\, L_{1ab}\, L_{2ac} L_{3bc} = L_{3bc}\, L_{2ac}\, L_{1ab}\, R_{123}$$

$$\in \mathrm{End}(\overset{1}{F} \otimes \overset{2}{F} \otimes \overset{3}{F} \otimes \overset{a}{V} \otimes \overset{b}{V} \otimes \overset{c}{V})$$



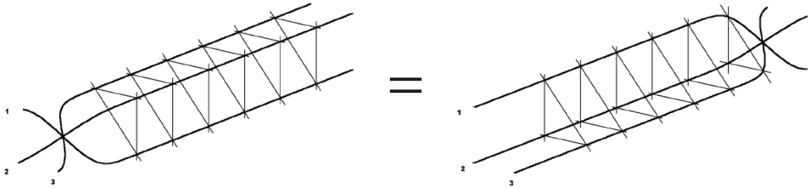
n-layer version

$$\mathbf{L}_{1,\mathbf{a},\mathbf{b}} = L_{1a_1b_1}L_{1a_2b_2}\cdots L_{1a_nb_n} \in \mathrm{End}(\overset{1}{F}\otimes\overset{\mathbf{a}}{\mathbf{V}}\otimes\overset{\mathbf{b}}{\mathbf{V}})$$

$$\overset{\mathbf{a}}{\mathbf{V}} = \overset{a_1}{V}\otimes\overset{a_2}{V}\otimes\cdots\otimes\overset{a_n}{V}$$

$$R_{123}\,\mathbf{L}_{1,\mathbf{a},\mathbf{b}}\,\mathbf{L}_{2,\mathbf{a},\mathbf{c}}\mathbf{L}_{3,\mathbf{b},\mathbf{c}} = \mathbf{L}_{3,\mathbf{b},\mathbf{c}}\,\mathbf{L}_{2,\mathbf{a},\mathbf{c}}\,\mathbf{L}_{1,\mathbf{a},\mathbf{b}}\,R_{123}$$

$$\in \mathrm{End}(\overset{1}{F}\otimes\overset{2}{F}\otimes\overset{3}{F}\otimes\overset{\mathbf{a}}{\mathbf{V}}\otimes\overset{\mathbf{b}}{\mathbf{V}}\otimes\overset{\mathbf{c}}{\mathbf{V}})$$



Solutions: R

Intertwiner of $A_q(SL_3)$-modules (A_q: q-deformed coordinate ring)
[Kapranov-Voevodsky 1994] (Bazhanov-Sergeev 2006)

$$F = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle \quad \text{(bosonic Fock space)}$$

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{abc} R_{ijk}^{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle.$$

$$R_{ijk}^{abc} = \delta_{a+b}^{i+j}\delta_{b+c}^{j+k} \sum_{\lambda,\mu\geq 0,\lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \begin{bmatrix} i, j, c+\mu \\ \mu, \lambda, i-\mu, j-\lambda, c \end{bmatrix}$$

$$(q)_i = \prod_{j=1}^{i}(1-q^j), \quad \begin{bmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_s \end{bmatrix} = \begin{cases} \frac{\prod_{m=1}^{r}(q^2)_{i_m}}{\prod_{m=1}^{s}(q^2)_{j_m}} & \forall i_m, j_m \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise} \end{cases}$$

$$q = \text{generic}, \qquad R = \oplus(\text{finite dimensional matrix})$$

Solutions: L

q-boson valued 6-vertex model [Bazhanov-Sergeev 2006]

$$V = \mathbb{C}^2, \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & \mathbf{a}^- & -i\mathbf{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}(F \otimes V \otimes V)$$

$$\mathbf{a}^{\pm}|m\rangle = \sqrt{1-q^{2m+1\pm 1}}|m \pm 1\rangle, \quad \mathbf{k}|m\rangle = q^{m+\frac{1}{2}}|m\rangle,$$
$$\langle m|\mathbf{a}^{\mp} = \langle m \pm 1|\sqrt{1-q^{2m+1\pm 1}}, \quad \langle m|\mathbf{k} = \langle m|q^{m+\frac{1}{2}}$$

2D reduction

Introduce F^* and linear pairing:

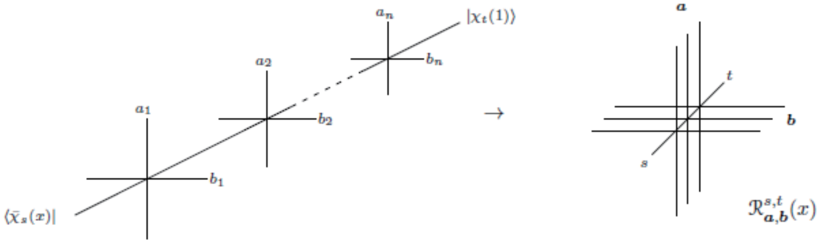
$$F^* = \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}\langle m|, \qquad F^* \times F \rightarrow \mathbb{C}: \quad \langle m|m'\rangle = \delta_{m,m'}$$

Introduce the following vectors in F and F^* (x: generic parameter):

$$|\chi_1(x)\rangle = \frac{1}{(x\,\mathbf{a}^+;q)_\infty}|0\rangle, \qquad |\chi_2(x)\rangle = \frac{1}{(x(\mathbf{a}^+)^2;q^4)_\infty}|0\rangle,$$
$$\langle\overline{\chi}_1(x)| = \langle 0|\frac{1}{(x\,\mathbf{a}^-;q)_\infty}, \qquad \langle\overline{\chi}_2(x)| = \langle 0|\frac{1}{(x(\mathbf{a}^-)^2;q^4)_\infty}$$

2D reduction of 3D L $(s, t = 1, 2)$

$$R^{s,t}(x) := \langle \overline{\chi}_s(x) | \mathbf{L}_{1,\mathbf{a},\mathbf{b}} | \chi_t(1) \rangle \in \mathrm{End}(\overset{\mathbf{a}}{\mathbf{V}} \otimes \overset{\mathbf{b}}{\mathbf{V}}) = \mathrm{End}((\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n})$$



Theorem (K-Sergeev, arXiv:1203.6436)

$$R^{2,1}(x) = R_{B_n^{(1)}}(x), \qquad R^{1,1}(x) = R_{D_{n+1}^{(2)}}(x), \qquad R^{2,2}(x) = R_{D_n^{(1)}}(x),$$

where the right hand sides are the (2D) quantum R matrices for spin representations [Okado 1990 (for $B_n^{(1)}, D_n^{(1)}$)].