A crudest prehistory of BLZ

Atsuo Kuniba

1/May/2008 (Thu) seminar@Komaba

1 非調和振動子

$$\left(-\frac{d^2}{dx^2}+x^{2M}-E\right)y(x)=0$$

Bohr-Sommerfeld 量子化． x_0 = positive turning point.

$$\left(\oint pdx=\right)\ 4\int_0^{x_0}\sqrt{E-x^{2M}}dx=2\pi(k+\frac{1}{2}),\quad k=0,1,2\ldots,$$

$$E=E_k=\nu\left(k+\frac{1}{2}\right)^{\frac{2M}{M+1}},\qquad \nu=\left(\frac{\sqrt{\pi}\Gamma\left(\frac{3}{2}+\frac{1}{2M}\right)}{\Gamma\left(1+\frac{1}{2M}\right)}\right)^{\frac{2M}{M+1}} \tag{1}$$

$M=1$ で exact．以下では $M>1$.

2 WKB

$\hbar$ (small parameter) を導入．最後に $\hbar=1$ とおく．

$$\hbar^2 y''(x)=V(x)y(y),\quad V(x)=x^{2M}-E,$$

$$y(x)=\exp\left(\frac{1}{\hbar}\sum_{n\geq 0}\hbar^n S_n(x)\right).$$

$$S'_0(x)=-V(x)^{\frac{1}{2}},$$

$$2S'_0S'_n+\sum_{j=1}^{n-1}S'_jS'_{n-j}+S''_{n-1}=0\ \ (n\geq 1).$$

$$S'_1=-\frac{V'}{4V},$$

$$S'_2=\frac{5V'^2}{32V^{5/2}}-\frac{V''}{8V^{3/2}},$$

$$S'_3=\frac{-15V'^3}{64V^4}+\frac{9V'V''}{32V^3}-\frac{V'''}{16V^2}$$

$$\frac{1}{2i}\oint\sum_{n\geq 0}S'_n(z)dz=\pi k\quad (k=0,1,2,\ldots)\qquad \text{J. Dunham (PR 1932)}$$

S'_0, S'_1 まで = BS 量子化． S'_1 から $-\pi/2$ が出る．

基底エネルギーの WKB 級数の解析性： Bender-Wu (PR 1969).
Higher S'_n の寄与の具体形. Bender, Olaussen, Wang (PRD 1977).　$n=7$ まで.

$$E^{\frac{1}{2}+\frac{1}{2M}}\sum_{n\geq 0}a_n E^{-n(1+1/M)}=(k+\frac{1}{2})\pi,$$

$$a_n=(-1)^n\frac{2\sqrt{\pi}\Gamma\left(1+\frac{1-2n}{2M}\right)P_n}{\Gamma\left(\frac{3-2n}{2}+\frac{1-2n}{2M}\right)(2n+2)!2^n}\quad (M=1\text{ だと }n\geq 1\text{ で }0)$$

$$\begin{aligned}
P_0 &= 1\\
P_1 &= 2(2M-1),\\
P_2 &= (2M-1)(2M-3)(2M+3),\\
P_3 &= \frac{4}{9}(2M-1)(2M-5)(192M^3+88M^2-234M-139).
\end{aligned}$$

$M=2$ case

1 次補正まで解くと

$$E_k=K^{4/3}(b_0+\frac{b_1}{K^2}),\qquad K=k+\frac{1}{2},\ \ b_0=2^{2/3}3^{4/3}\pi^2\Gamma(\frac{1}{4})^{-8/3},\ \ b_1=\frac{C_0}{9\pi}$$

$E_j^{(n)}$：　n 次 WKB 解による j 番目の励起状態のエネルギー

$$\begin{aligned}
E_0 &= 1.060362090, & E_4 &= 16.26182601\\
E_0^{(0)} &= 0.87 & E_4^{(0)} &= 16.2336147\\
E_0^{(2)} &= 0.98 & E_4^{(2)} &= 16.2619367\\
E_0^{(4)} &= 0.95 & E_4^{(4)} &= 16.2618286
\end{aligned}$$

一般には WKB series は asymptotic.

n	0	1	2	3	4	5	6	7
a_n	1.7	-0.15	0.038	0.092	-0.56	-5.1	72.5	1592

ある次数で最も良い精度を達成し，それ以上ではかえって悪くなる.

3　Spectrum の exact な関係式

- Parisi (LNM 1982)

$$Z(s):=\sum_{k\geq 0}E_k^{-s}=\frac{\sin\pi s}{\pi}\int_0^\infty E^{-s}\,\mathrm{Tr}\left(\frac{1}{H+E}\right)ds,$$

$$\text{trace identity}\ :\ Z(m)=0\qquad \text{if}\quad m\in\mathbb{Z}_{<0}\setminus\{\frac{1+M}{2M}(1-2j)|j=1,2,\ldots\}$$

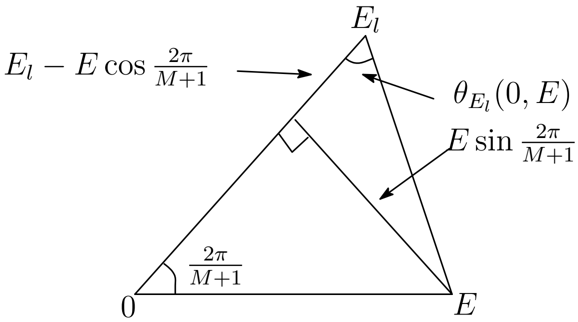
$M=1$ なら Riemann zeta の自明な零点.

- Voros (JPA 1994)

$$\Sigma_\pm(E_k) = 2\pi\left(k+\frac{1}{2}\pm C_M\right) \quad \begin{cases} k \text{ even} \\ k \text{ odd} \end{cases} \quad C_M = \frac{M-1}{2M+2} \quad \text{(Exact !)}$$

$$\Sigma_\pm(E) = 4 \sum_{l\geq 0 \ \text{(even or odd)}} \theta_{E_l}(0,E), \qquad \Sigma'_\pm(E) > 0.$$

$$\theta_{E_l}(0,E) = \mathrm{Arctan}\frac{E\sin\frac{2\pi}{M+1}}{E_l - E\cos\frac{2\pi}{M+1}} \in (0,\pi).$$



E_k の初期値＝ WKB 値とおいて $E_0, E_1, \ldots, E_{k_0}$ について数値的に再帰的に解ける.
基底エネルギーは 20 step 程度で 1.06039209 になる.

$$\exp(2i\mathrm{Arctan}(x/y)) = \frac{y+ix}{y-ix},$$

$$\exp\left(2i\theta_{E_l}(0,E)\right) = \frac{E_l - E\cos\omega + iE\sin\omega}{E_l - E\cos\omega - iE\sin\omega} = \frac{E_l - q^{-2}E}{E_l - q^2 E} \quad \left(q = \exp\frac{\pi i}{M+1}\right)$$

$$\therefore \prod_{l \text{ even or odd}} \frac{E_l - q^{-2}E_k}{E_l - q^2 E_k} = e^{i\pi(k+\frac{1}{2}\pm C_M)} \qquad \cdots \text{ Bethe 方程式！} \tag{2}$$

E_l を WKB 近似すると Bohr-Sommerfeld 量子化条件に帰着.

- Dorey-Tateo (JPA 1999)

非調和振動子のスペクトル行列式 　　 Baxter Q-function

$$D_\pm(E) = \prod_{k\geq 0,\text{even,odd}}\left(1-\frac{E}{E_k}\right) \quad \sim \quad Q_\pm(E)$$

$$Q_\pm(E) = \langle \mathrm{vac}|\mathbf{Q}_\pm(E)|\mathrm{vac}\rangle$$

$\mathbf{Q}_\pm(E)$: Q-operator (BLZ I-III, CMP 1996-9)

acting on Virasoro Fock space with $c = 1 - \dfrac{6M^2}{M+1}, \quad \Delta = \dfrac{1-4M^2}{16(M+1)}$

4　Bazhanov-Lukyanov-Zamolodchikov による一般化

$$\mathcal{V} = \oplus_{n_k<0}\, \mathbb{C}L_{n_1}\cdots L_{n_j}|\mathrm{vac}\rangle, \quad L_0|\mathrm{vac}\rangle = \Delta|\mathrm{vac}\rangle, \quad L_{n>0}|\mathrm{vac}\rangle = 0$$

$$c = 1 - \frac{6\alpha^2}{\alpha+1}, \quad \Delta = \frac{(2l+1)^2 - 4\alpha^2}{16(\alpha+1)^2} \quad (\alpha > 0,\ l \geq -1/2)$$

$$\mathcal{V} = \oplus_{L\geq 0}\mathcal{V}_L, \quad L_0\mathcal{V}_L = (\Delta + L)\mathcal{V}_L \qquad \text{(level subspace)}$$

$$\mathbf{Q}_\pm(s)\ :\ \mathcal{V}_L \to \mathcal{V}_L, \qquad [\mathbf{Q}_\varepsilon(s), \mathbf{Q}_{\varepsilon'}(s')] = 0.$$

$Q_\pm(s) =$ eigenvalue of $\mathbf{Q}_\pm(s)$, $\quad A_\pm(s) = (-s)^{\pm(2l+1)/4}Q_\pm(s)$

$A_\pm(s)$ は s の整関数, $A_\pm(0) = 1$.　量子 Wronskian 関係式を満たす

$$q^{l+1/2}A_+(sq)A_-(s/q) - q^{-l-1/2}A_+(s/q)A_-(sq) = q^{l+1/2} - q^{-l-1/2}.$$

$A_\pm(s) = \prod_j (1 - s/s_j^\pm)$, 　Wronskian relation で $s \to q^{\pm 1}s_k^+$

$$q^{l+1/2}A_+(q^2 s_k^+)A_-(s_k^+) = q^{l+1/2} - q^{-l-1/2} = -q^{-l-1/2}A_+(q^{-2}s_k^+)A_-(s_k^+),$$

$$\frac{A_+(q^2 s_k^+)}{A_+(q^{-2}s_k^+)} = -q^{-(2l+1)} \qquad \text{同様に} \quad \frac{A_-(q^2 s_k^-)}{A_-(q^{-2}s_k^-)} = -q^{2l+1}$$

$l = 0$ とし，$s_k^+ = \mathrm{const}\, E_{k\ \mathrm{odd}}$，$s_k^- = \mathrm{const}\, E_{k\ \mathrm{even}}$ とすると (2) に一致.

Claim (BLZ, Adv. Theor. Math. Phys. 2003)

Virasoro level L 状態での Q-operator の固有値 $A_\pm(E/\nu)$

$=$ Monstrous potential Schrödinger 方程式のスペクトル行列式 $D_\pm(E)$

$$\left(-\frac{d^2}{dx^2} + U(x) - E\right)y(x) = 0,$$

$$U(x) = x^{2\alpha} + \frac{l(l+1)}{x^2} - 2\frac{d^2}{dx^2}\sum_{j=1}^{L}\log(x^{2\alpha+2} - z_j)$$

$\nu = (1)|_{M=\alpha}$. $z_1, \ldots, z_L$ は以下の代数方程式の根.

$$\sum_{j=1, j\neq k}^{L} \frac{z_k\big(z_k^2 + (3+\alpha)(1+2\alpha)z_k z_j + \alpha(1+2\alpha)z_j^2\big)}{(z_k - z_j)^3} - \frac{\alpha z_k}{4(1+\alpha)} + \triangle = 0 \quad (k = 1, 2, \cdots, L).$$

$L = 0$ の場合は証明されているといってよい．$\pm$ に応じて $\pm l > (1+\alpha)L - 3/2$ の範囲に限定.

$L = 0, l = 0, \alpha = M \quad \to$ 非調和振動子.