Nonlinear Waves in Optics and Fluid Dynamics

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Outline

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   -Classical vs. DM “solitons”
   -Lattice and discrete localized modes–soltions
   -Ground states, collapse profiles in quadradic NL media
3. Computational method
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   Kadomtsev-Petviashvili (KP) equations
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Introduction

- In water waves and NL optics solitary waves ("solitons") play an important role.
- Shallow water waves-long history: Boussinesq 1870's; Korteweg-deVries(KdV) 1895
- 1+1 dimensions, KdV equation (normalized)

$$u_t + 6uu_x + u_{xxx} = 0$$

soliton–elastic interaction:

$$u = 2\kappa^2 sech^2 \kappa(x - 4\kappa^2 t)$$

- 2+1 dimensions, KP equation (1970's; normalized)

$$\partial_x(u_t + 6uu_x + u_{xxx}) - 3sgn(\sigma)u_{yy} = 0$$

$\sigma > 0$ lumps; σ normalized surface tension

Introduction–con't

- NL Optics -Nonlinear Schrödinger (NLS) type Eq. central role

$$iu_z + \frac{1}{2}D(z)u_{tt} + g(z)|u|^2u = 0 \ (PNLS)$$

$D(z)$ dipsersion; $g(z)$: damping and amplification
"Classical" NLS eq.: $D = 1, g = 1$

- Classical soliton: $u = \eta \operatorname{sech}(\eta x)e^{-i\eta^2 t}$
- Solitons proposed in: fiber optics–Hasegawa and Tappert (1973) experiments –Molleneauer et al 1980's
- Dispersion-Management (DM) reduces penalties in communications;
DM commercial: 2001-present
DM -localized pulses: solitons; quasi-linear ...
New applications: M-L lasers

Dispersion-Managed System

Perturbed NLS (P-NLS, nondimensional)

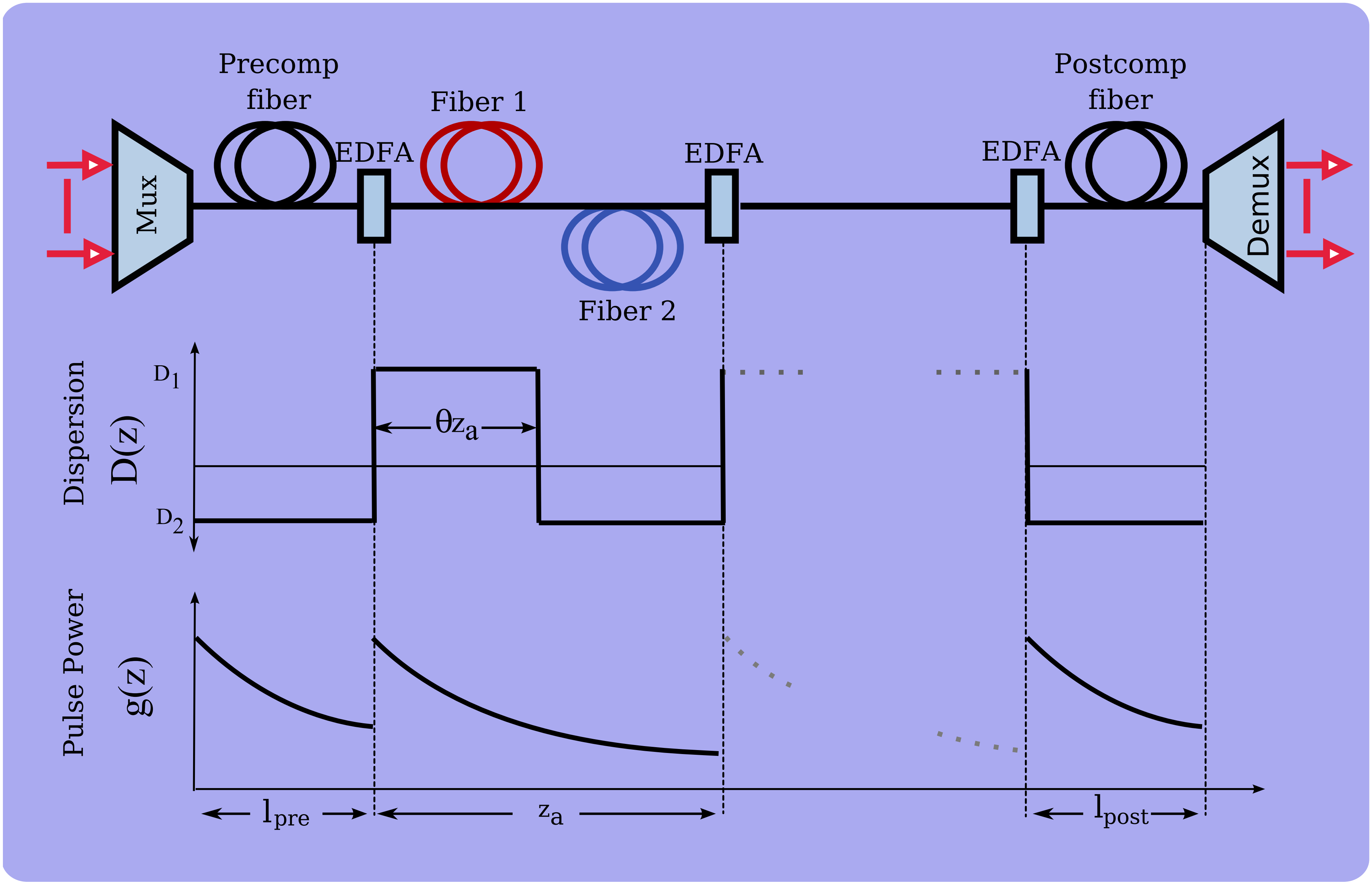
$$iu_z(z,t) + \frac{1}{2}D(z)u_{tt} + g(z)|u|^2u = 0$$

- $D(z) = \langle D \rangle + \frac{1}{z_a}\Delta\left(\frac{z}{z_a}\right)$
- $0 < z_a << 1$,

  $\Delta = \begin{cases} \Delta_1 & \text{in "anomalous"} \\ \Delta_2 & \text{in "normal"} \end{cases}$

  $g(z) = g\left(\frac{z}{z_a}\right)$ given
- $\langle \Delta \rangle = \frac{\int_0^{z_a} \Delta dz}{z_a} = 0$

Dispersion-Managed System-fig's



Asymptotic analysis of P-NLS

$$iu_z(z,t) + \frac{1}{2}\left(\langle D \rangle + \frac{1}{z_a}\Delta\left(\frac{z}{z_a} \right)\right) u_{tt} + g\left(\frac{z}{z_a} \right) |u|^2 u = 0$$

Multiple scales (ref. MJA & G. Biondini Opt.Lett. ’98)

$$u = u(\zeta, Z, t; z_a) : \zeta = \frac{z}{z_a}, Z = z; \quad z_a << 1$$

$$\partial_z u = \frac{1}{z_a}\partial_\zeta u + \partial_Z u; \quad u = u^{(0)} + z_a u^{(1)} + z_a^2 u^{(2)} + ...$$

At $O(\frac{1}{z_a})$ find linear eq.

$$iu_\zeta^{(0)} + \frac{1}{2}\Delta(\zeta) u_{tt}^{(0)} = 0$$

Solve by Fourier Transforms $\Longrightarrow$

Asymptotic analysis of P-NLS (con't)

$$\hat{u}^{(0)}(\omega) \equiv \mathcal{F}\{u^{(0)}(t)\} = \int_{-\infty}^{\infty} u^{(0)}(t) e^{-i\omega t}\, dt$$

Take FT of leading order equation and find solution:

$$i\hat{u}^{(0)}_{\zeta} - \frac{\omega^2}{2}\Delta(\zeta)\hat{u}^{(0)} = 0$$

$$\hat{u}^{(0)} = \hat{U}(Z,\omega) e^{-\frac{i}{2}\omega^2 C(\zeta)} + O(z_a)\ , \quad C(\zeta) \equiv \int_0^{\zeta} \Delta(\zeta')d\zeta'$$

Next order, secularity condition determines $\hat{U}(Z,\omega)$:
the **DMNLS** equation (nonlinear & nonlocal)

DMNLS equation

$$i\hat{U}_Z - \frac{1}{2}\omega^2\langle D\rangle\hat{U} + < g e^{\frac{i}{2}\omega^2 C}\mathcal{F}\{|u^{(0)}|^2 u^{(0)}\} >= 0 \quad (DMNLS)$$

where $< F >= \int_0^1 F d\zeta$. Alternatively $< \cdot >$:

$$< \cdot >= \iint r(\omega_1\omega_2)\hat{U}(\omega+\omega_1)\hat{U}(\omega+\omega_2)\hat{U}^*(\omega+\omega_1+\omega_2)d\omega_1 d\omega_2$$

where $r(x)(2\pi)^2 =< g e^{iCx} >$ (see also Gabitov, Turitsyn).
If $\Delta \to 0$ find NLS eq. in Fourier domain. Lossless case: $g = 1$:

$$r(x) = \frac{\sin(sx)}{(2\pi)^2 sx}; \quad s = \frac{\theta\Delta_1}{2} \quad (map\ strength)$$

DM solitons

DM solitons, ansatz: $\hat{U}(z,\omega) = \hat{f}(\omega)e^{i\lambda^2 z/2}$,

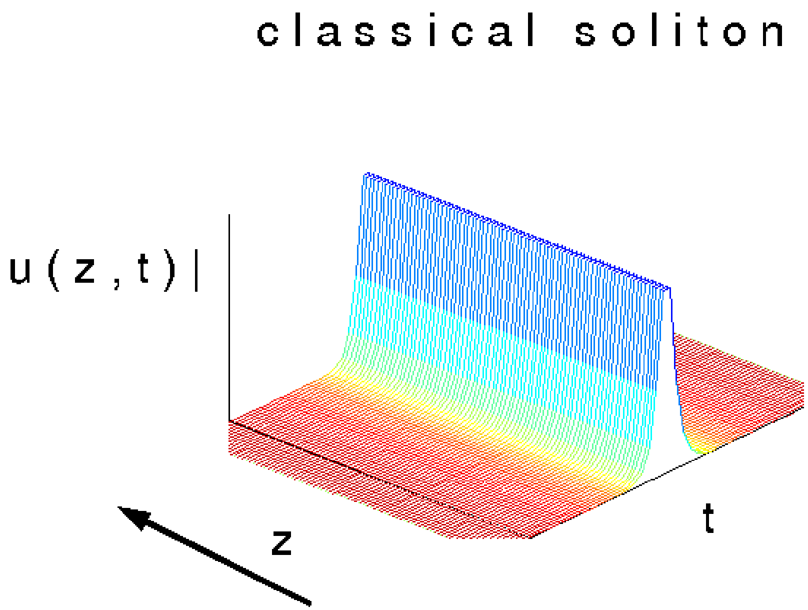
$$-\frac{\lambda^2}{2}\hat{f} - \frac{\omega^2}{2}\langle D\rangle\hat{f} + \iint r(\omega_1\omega_2)\hat{f}\cdot\hat{f}\cdot\hat{f}^*\,d\omega_1 d\omega_2 = 0$$

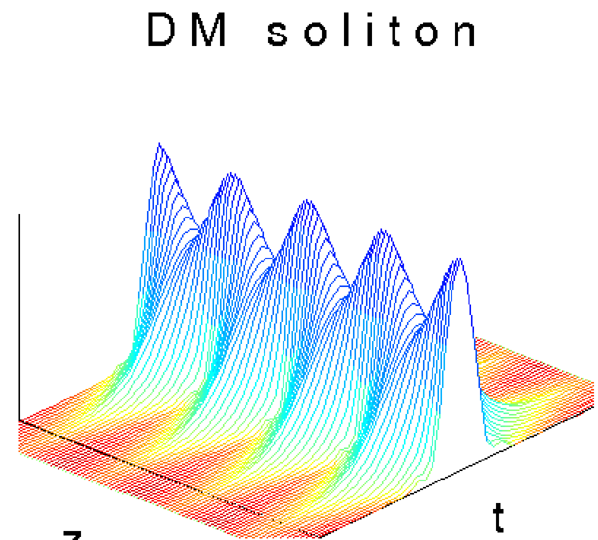
- Nonlinear fixed-point eq. – numerical computations $\langle D\rangle > 0$; existence: Zharnitsky et al (PRE. 2000)
- When $s = \Delta_1 = 0$ recover classical case: $f(t) = \lambda\,\mathrm{sech}(\lambda t)$.
- Can also find:
  1. "Dark-gray" DM solitons: (MJA and Z. Musslimani PRE 2003)
  2. Higher-order DMNLS equation and multi-humped DM solitons (MJA,T. Hirooka,T. Inoue, JOSAB 2002)

DM $\Longrightarrow$ breathing soliton

$$g = 1, r(x) = \frac{\sin(sx)}{(2\pi)^2 sx}$$

$s = 0$ $\qquad\qquad$ $s \neq 0$





Quasi-linear pulses

For $s >> 1, g = 1$ there is an approximate solution to the DMNLS equation when $|\hat{U}|$ depends weakly on s:

$$i\hat{U}_Z - \frac{1}{2}\omega^2\langle D\rangle\hat{U} + \Phi(|\hat{U}|^2)\hat{U} \sim 0 \quad (1)$$

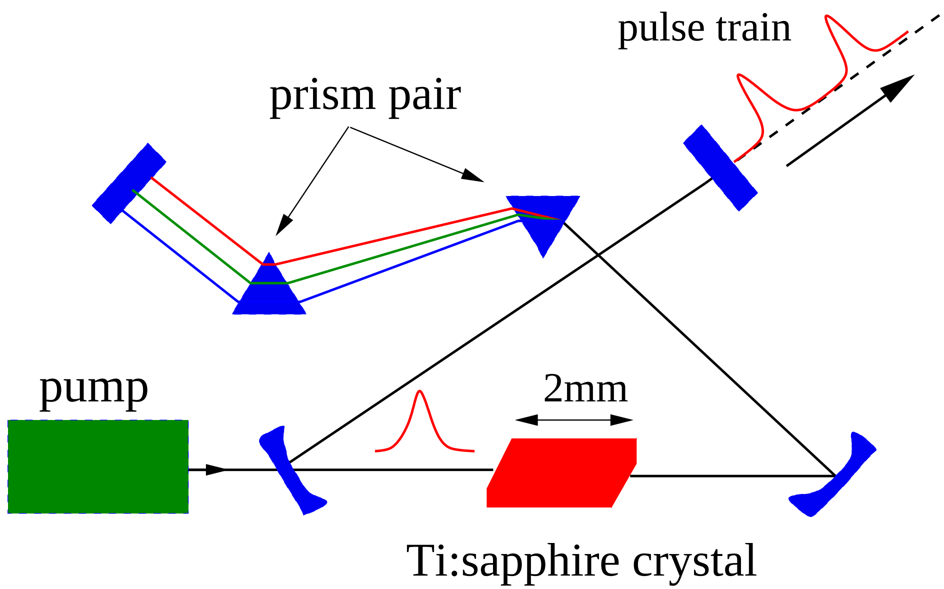
$$\Phi(|\hat{U}|^2) = \frac{1}{2\pi s}[(log\ s - \gamma)|\hat{U}|^2 - \int_{-\infty}^{\infty} f(\omega - \omega')|\hat{U}|^2(\omega')d\omega']$$

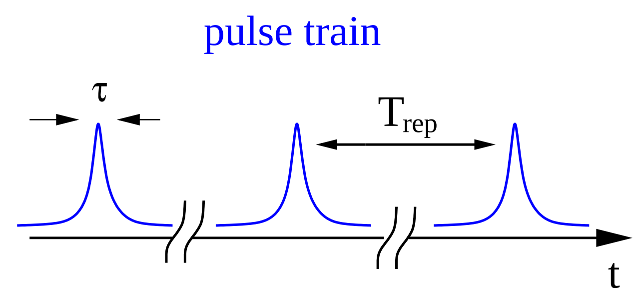
where $f(x) = \frac{1}{\pi}\int log\ t\ e^{i\omega t}dt$. We may solve eq. (1):

$$\hat{U}(\omega, z) = \hat{U}_0 exp[-i < d > \omega^2 z + i\Phi(|\hat{U}_0|^2 z)] \quad (2)$$

where $\hat{U}_0 = \hat{U}(\omega, 0)$; (2) is a quasi-linear mode.

DM -Mode-locked Ti:sapphire lasers

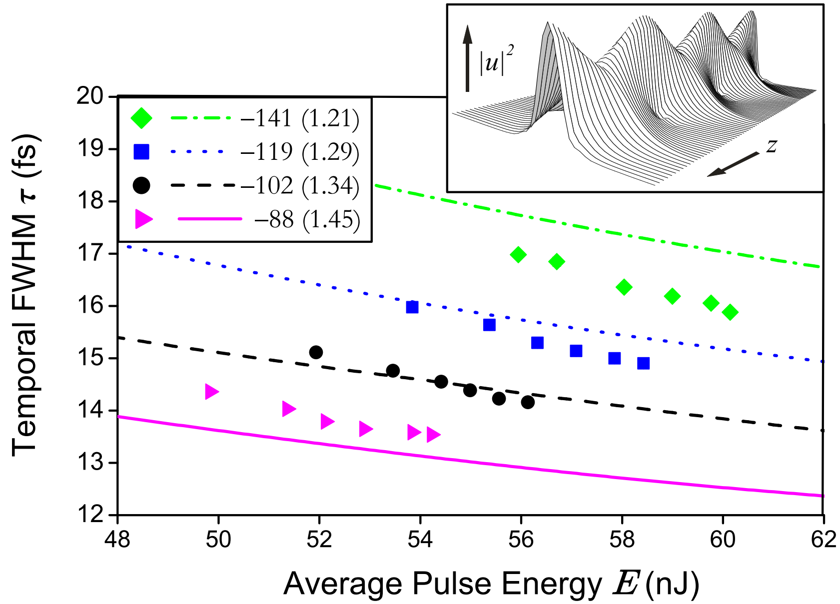




$$\tau = 10\text{fs} = 10^{-14}\,\text{sec}, \quad T_{\text{rep}} = 10\text{ns}, \quad f_{\text{rep}} = \frac{1}{T_{\text{rep}}} = 100\text{MHz}$$

Similar for mode-locked Sr:Forsterite, fiber lasers, ...
Applications: highly stable oscillators; optical clocks...
cf. MJA, B. Ilan, S. Cundiff OL, 2004; Q. Quraishi, S. Cundiff, B. Ilan, MJA, PRL 2005

Theory vs. Ti:sapphire experiments



Curves: DMNLS Symbols: experiments

GDD = $\langle k'' \rangle l_c$ (in fs^2)

- Remarkable agreement; only one fitting parameter for all GDDs
- Pulses are well approximated by DM solitons
- Ref.: Phys. Rev. Lett., 94, 2005

Lattice Modes

$$iu_z + \Delta u + V(x,y)u + |u|^2 u = 0$$

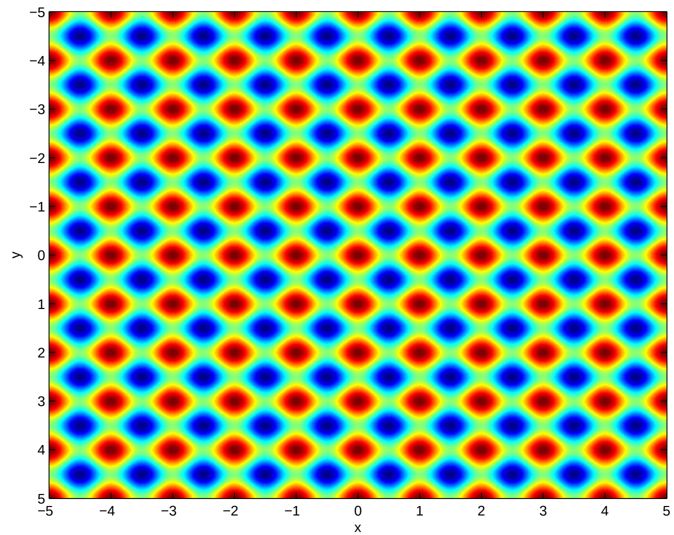
$V(x,y)$ given lattice potential; e.g.

$$V(x,y) = V_0(cos^2\pi x + cos^2\pi y)$$

Recent research: observation and investigation of localized modes– "lattice solitons"; e.g. Segev group: Nature 2003, PRL 2003...

Recently we have found solitary waves on complex backgrounds in cases where $V(x,y)$ have defects, dislocations and Quasi-Crystal (QC) structure (MJA, Ilan, Schonbrun, Piestun, PRE, 2006).

Periodic Lattice



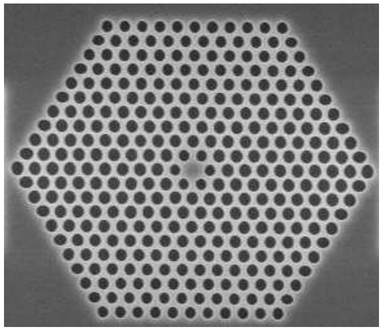
$$V(x, y) = (\cos^2 \pi x + \cos^2 \pi y)$$

Lattices appear frequently in nature:

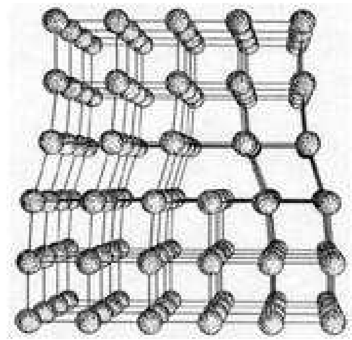
- Optical waves on lattice backgrounds
- Photonic Crystal Fibers
- Bose-Einstein Condensates ....

Most studies consider *periodic* lattices

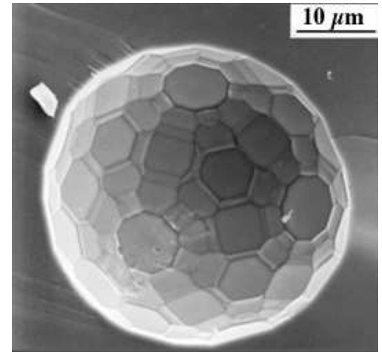
Irregular lattices



vacancy defect



edge−dislocation



quasicrystal

- Also appear widely in Nature
- Point defects (e.g., vacancies)
- Line defects (e.g., edge-dislocations)
- Quasi-crystal structures (e.g. Penrose tiles)

Localized nonlinear modes

Maxwell + Kerr effect $\Longrightarrow$ **NLS** Eq.

$$i\psi_z + \Delta\psi - V(\vec{x})\psi + |\psi|^2\psi = 0$$

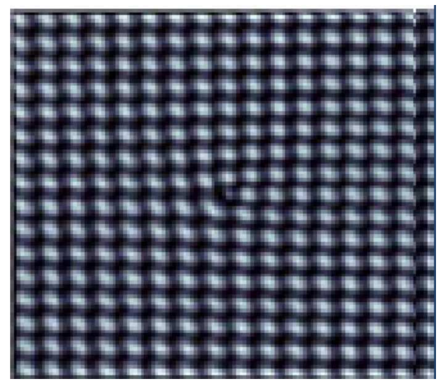
$\psi(\vec{x}, z) = f(\vec{x})e^{-i\mu z} \Longrightarrow [\Delta + \mu - V(\vec{x}) + |f|^2]f = 0$

$f(\vec{x})$ = localized, $P := \iint |f(\vec{x})|^2 \, dxdy < \infty$

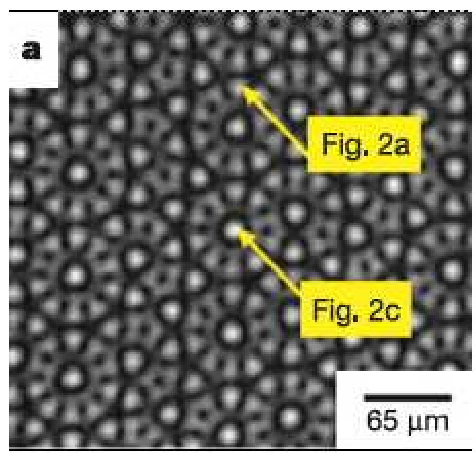
- $V(\vec{x}) \equiv 0$ (pure NLS):
  - 1-D solitons exist when $\mu < 0$
  - can have collapse in (2+1)D (i.e., $||\psi|| \stackrel{z \to Z_c}{\longrightarrow} \infty$)
- periodic and aperiodic $V(\vec{x})$:
  - localized modes (solitons) found numerically
  - some rigorous theory
  - localized modes recently observed in experiments

Irregular photonic lattices

Schonbrun & Piestun (2006): use phase masks to introduce various defects and dislocations:



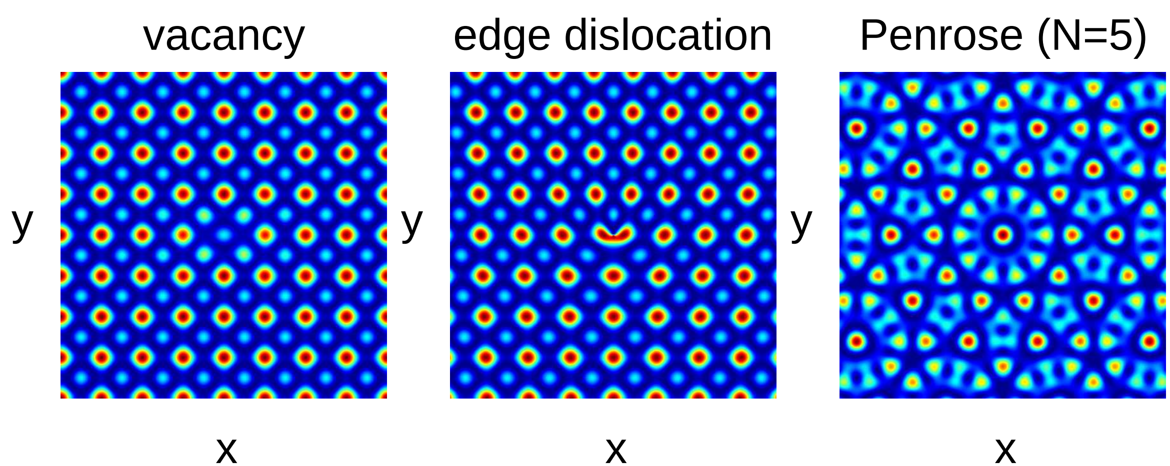
Freedman *et al.* (2006): photonic quasi-crystals:



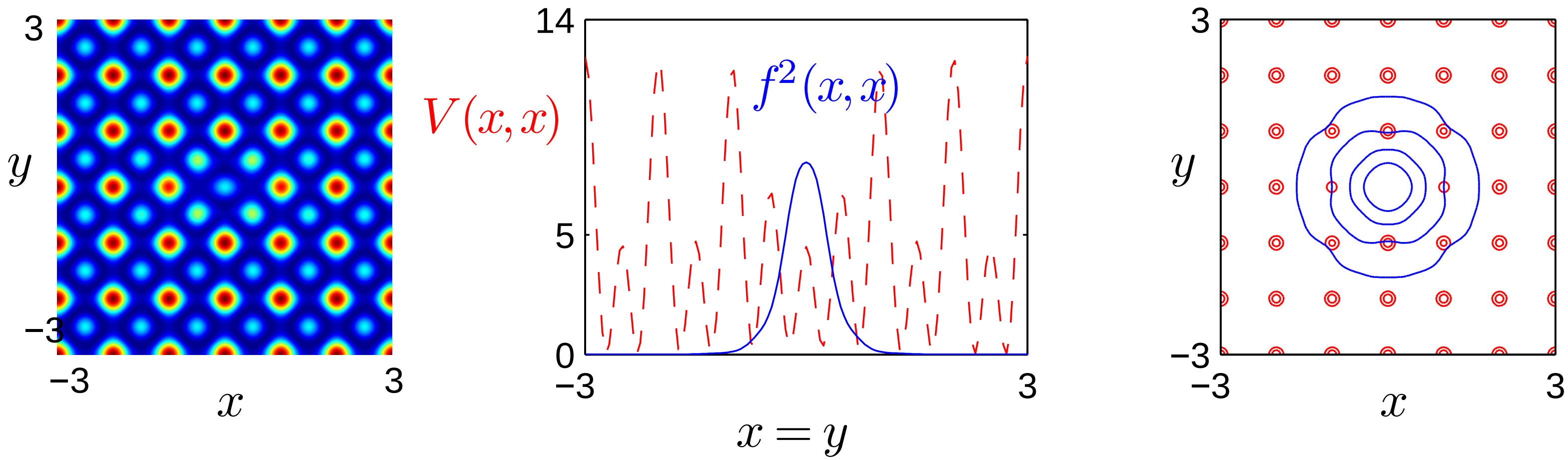
Localized nonlinear modes–"solitons"

Using new numerical methods we can find localized modes on irregular lattice potential backgrounds.

In particular : vacancy; edge dislocation, quasi-crystal backgrounds:



Vacancy modes



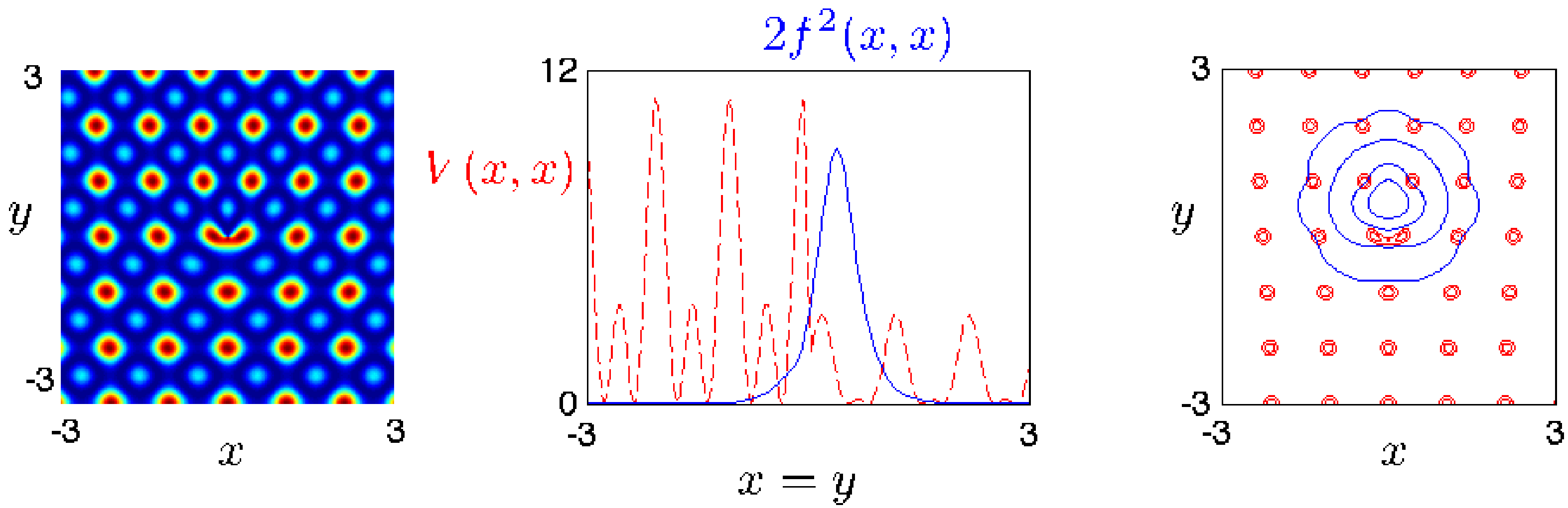
$$V(x,y) = \frac{V_0}{25}\left|2\cos(Kx) + 2\cos(Ky) + e^{i\theta(x,y)}\right|^2$$

$$\theta(x,y) = \tan^{-1}\left(\frac{y-y_0}{x}\right) - \tan^{-1}\left(\frac{y+y_0}{x}\right), \quad y_0 = \frac{\pi}{K}$$

$K = 2\pi, V_0 = 12.5, \mu = 0.5$

Similar to offsite periodic-lattice modes

Edge-dislocation modes

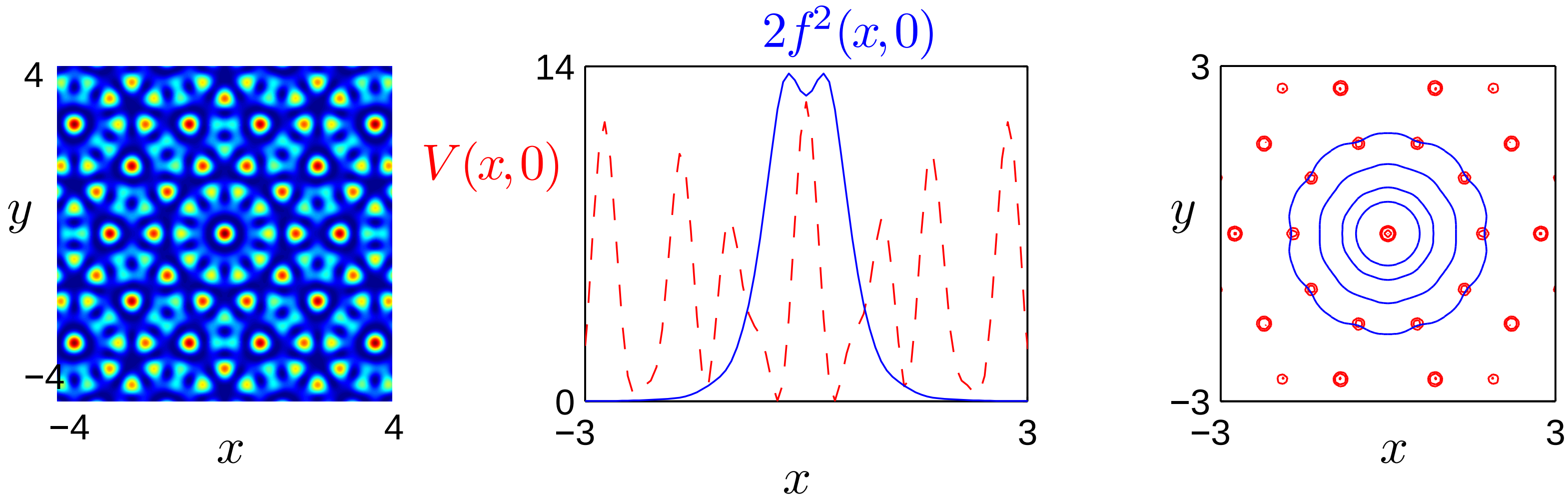


$$V(x,y) = \frac{V_0}{25}\left\{2\cos[Kx+\theta(x,y)]+2\cos(Ky)+1\right\}^2$$

$$\theta(x,y) = \frac{3\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right)$$

$K = 2\pi, V_0 = 12.5, \mu = 0.5$

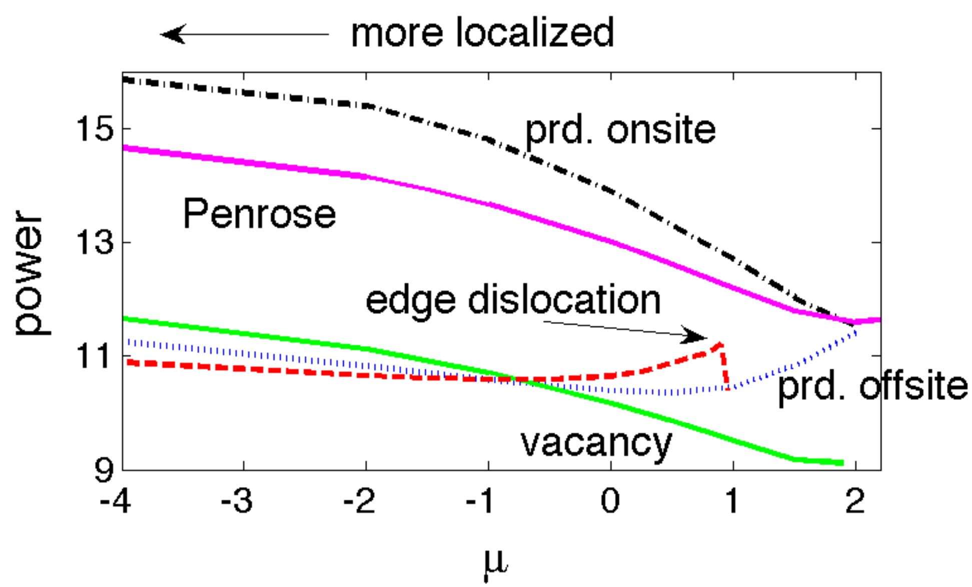
Quasi-crystal modes



$$V(x,y) = \frac{V_0}{25}\left|\sum_{n=0}^{N-1} e^{i\vec{k}_n\cdot\vec{x}}\right|^2, \quad \vec{k}_n = \left(K\cos(\frac{2\pi n}{N}), K\sin(\frac{2\pi n}{N})\right)$$

- $N = 5, K = 2\pi, V_0 = 12.5, \mu = 0.5$
- onsite $\Longrightarrow$ small dimple; as μ increases dimple size increases; no dimple for offsite modes

Power vs. eigenvalue μ



- power lowest with vacancy
- gap “reduced” with edge-dislocation
- periodic-onsite and quasi-crystal modes are “similar”
- $\frac{dP}{d\mu} < 0$ necessary for linear stability (VK); evolution studies carried out

Discrete Optical Solitons

Tight binding approximation; e.g. $|V_0| >> 1$

$$V(x, y) = V_0(\cos^2 \pi x + \cos^2 \pi y)$$

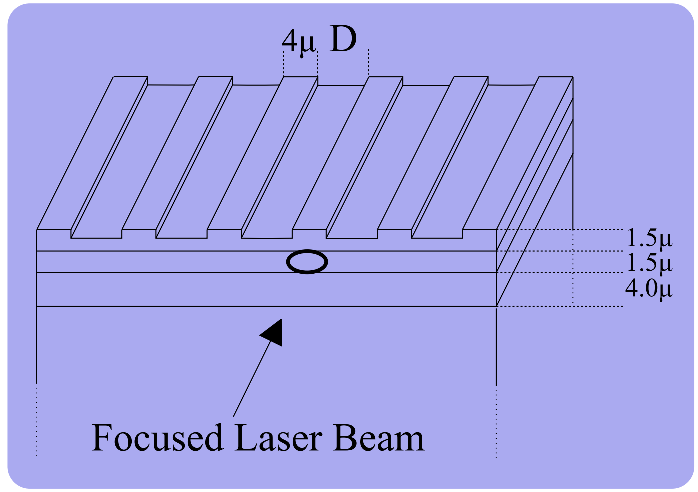
allows lattices to be approximated by discrete equations.

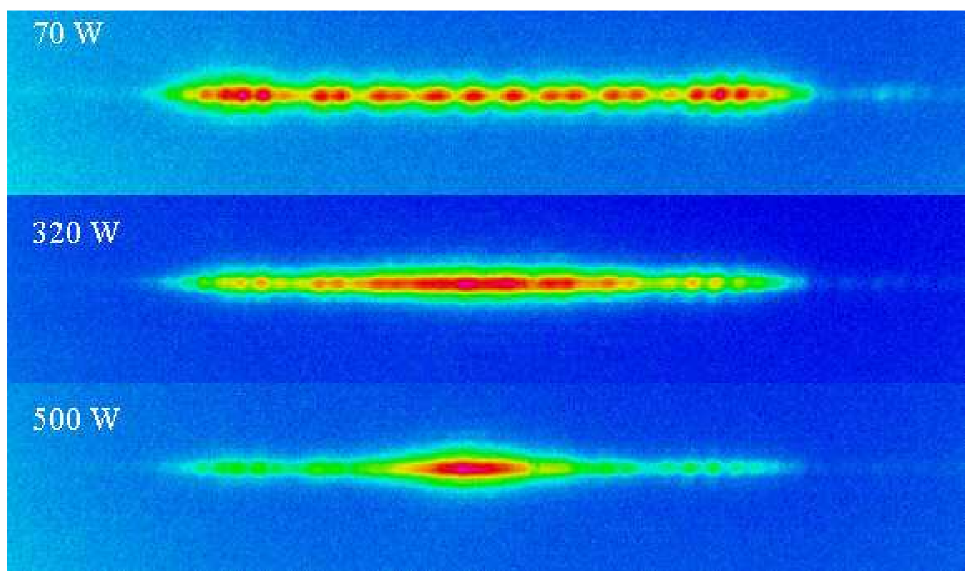
One-dimension, theoretical prediction: D. Christodouledes and R. Joseph (Opt. Lett, 1988) – discrete spatial NLS equation (normalized)

$$i\partial_z u_n + \frac{1}{2h^2}(u_{n+1} + u_{n-1} - 2u_n) + |u_n|^2 u_n = 0$$

Experimental studies: Silberberg group (PRL 1998, 1999)

Discrete Optical Solitons-fig's





Power : a)Low b)Medium c)High

Discrete Diffraction-Management

Diffraction-management-experimental, Silberberg group (PRL, 2000)
Theoretical MJA and Z. Musslimani (PRL, 2001; Physica D, 2003)

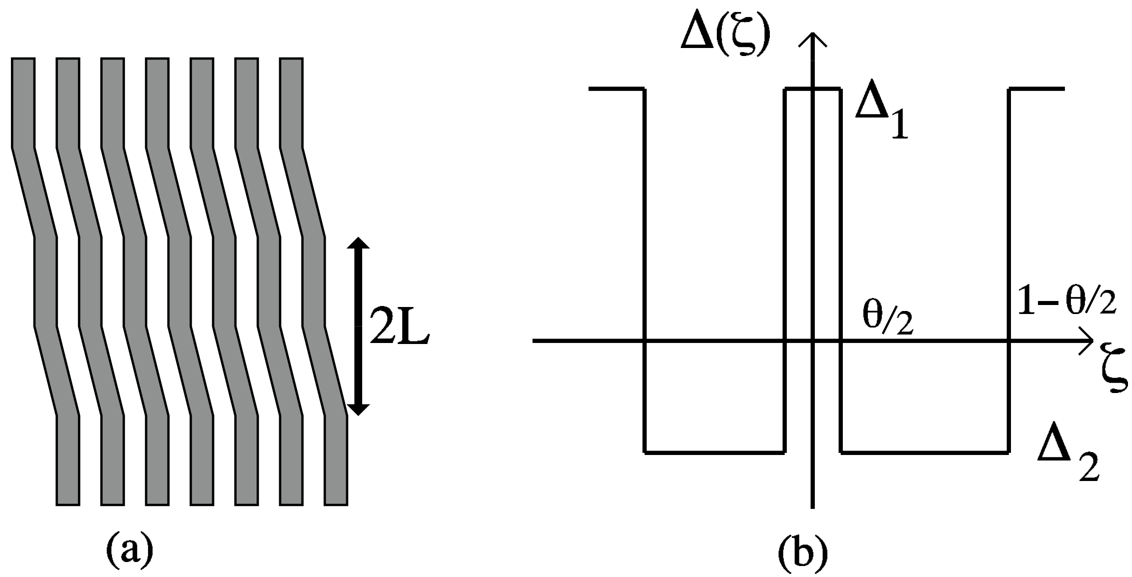
$$i\partial_z u_n + \frac{D(z)}{2}(u_{n+1} + u_{n-1} - 2u_n) + |u_n|^2 u_n = 0$$

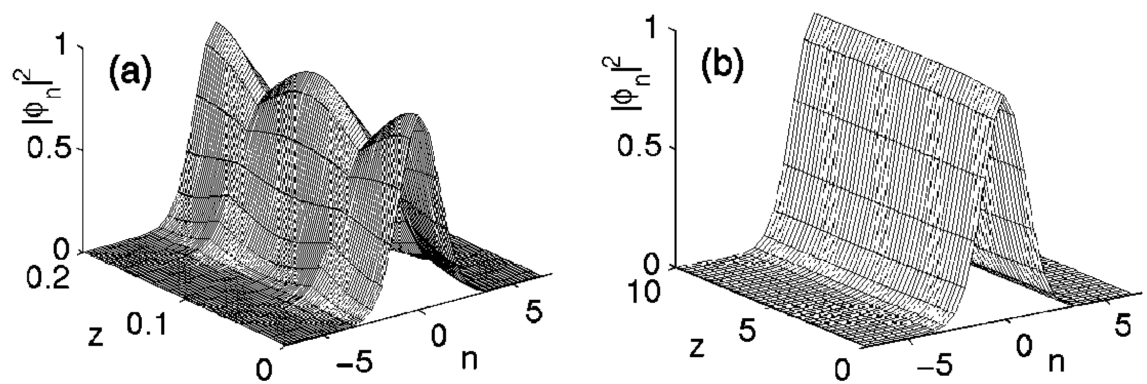
where $D(z) = \langle D \rangle + \frac{1}{\varepsilon}\Delta\left(\frac{z}{\varepsilon}\right)$.

Find discrete analog of DMNLS equation-and discrete DM solitons. As $h \to 0$: PNLS equation

$$iu_z + \frac{D(z)}{2}u_{xx} + |u|^2 u = 0$$

Discrete Diffraction-Management-fig.





Beam propagation over one period (a) and stationary evolution (b) obtained by direct numerical simulation evaluated at each map period.

Ground States-BR/DS Equations

$$iU_z + \frac{1}{2}\Delta U + |U|^2 U - \rho U V_x = 0$$

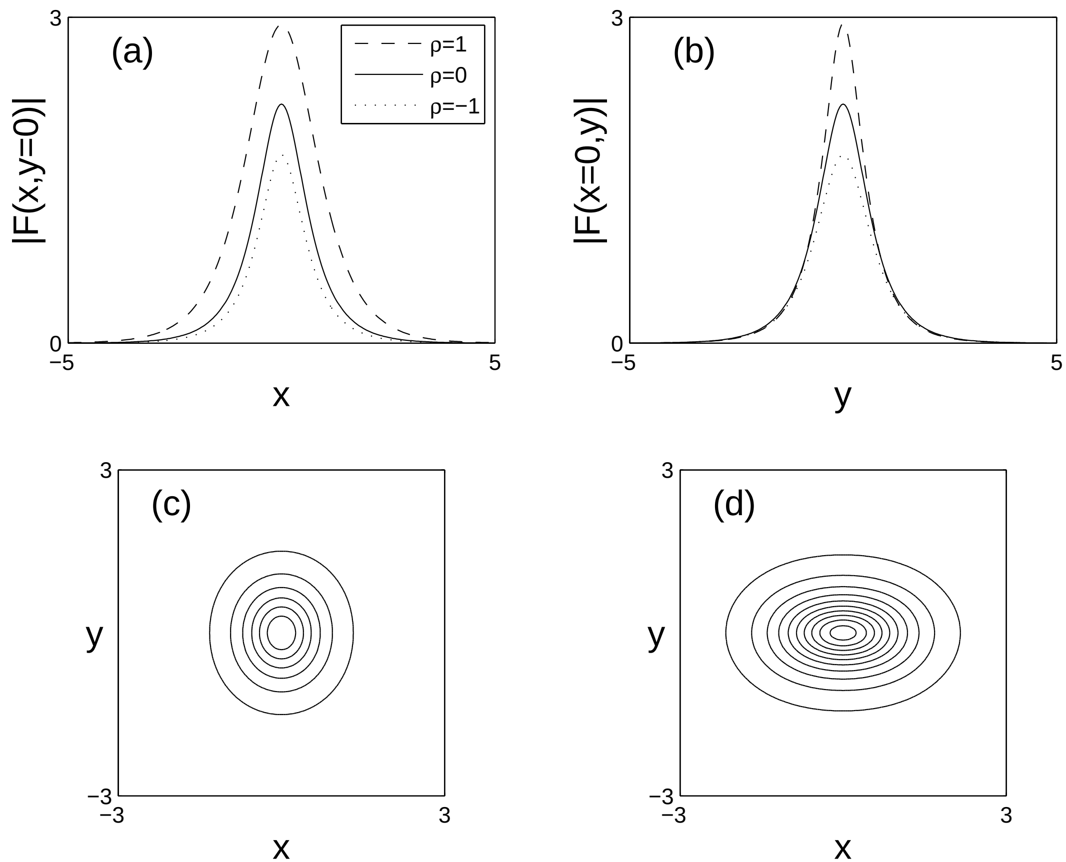
$$V_{xx} + \nu V_{yy} = (|U|^2)_x$$

$\rho < 0$: water waves (Benney-Roskes, 1969, Davey-Stewartson 1974: add surface tension)
$\rho > 0$: $\chi^{(2)}$ NL optics (MJA, G.Biondini, S. Blair, PLA 1996–steady case)
When $\nu > 0$: collapse possible (Virial Thm). Ground states-collapse profiles (MJA, B. Ilan, I. Barkitas, 2005):
$U = F(x,y)e^{i\mu z}, V = G(x,y)$

$$-\mu F + \frac{1}{2}\Delta F + |F|^2 F - \rho F G_x = 0$$

$$G_{xx} + \nu G_{yy} = (|F|^2)_x$$

Ground States-BR/DS (con't)

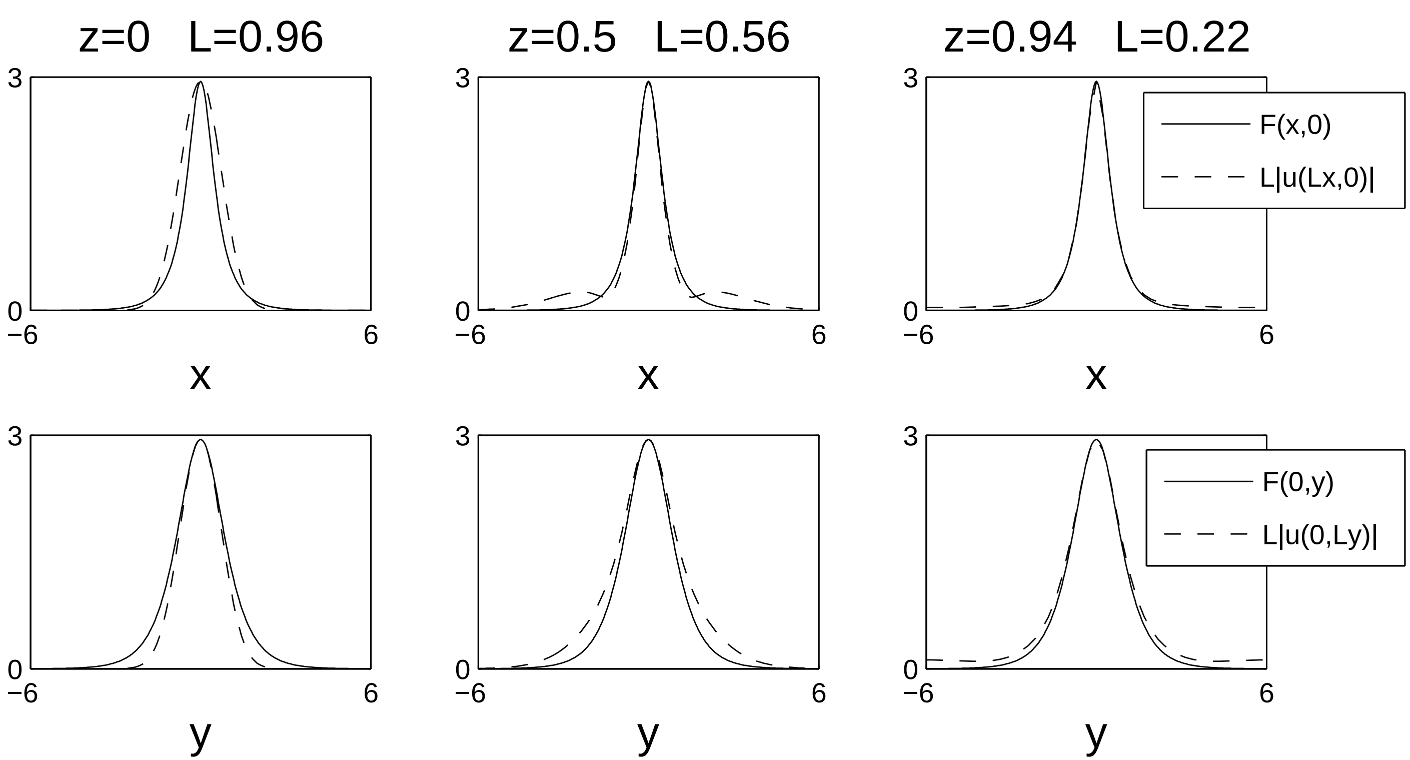


$F(x, y); \nu = 0.5$; Top: "slices"; bottom: contour plots; c) $\rho = -1$, d)$\rho = 1$

Quasi-Self-Similar Collapse

$z \to z_c$, theory: $U \sim \frac{1}{L(z)} F(\frac{x}{L(z)}, \frac{y}{L(z)}), V \sim \frac{1}{L(z)} G(\frac{x}{L(z)}, \frac{y}{L(z)})$

By calculation: $L(z) = \frac{F(0,0)}{U(0,0,z)} \to 0$. Below compare $L(z)|U(Lx, Ly)|$ with $F(x, y)$ in collapse regime along x axis (top) & y axis (bottom) with $(\nu, \rho) = (0.5, 1)$.



Computational Method

$$L_\mu w = N[w]w$$

Take FT (k) find: $P(\mu,k)\hat{w}(k) = \mathcal{F}\left(N[w]w\right)(k)$ hence:

$$\hat{w} = \frac{\mathcal{F}\,(N[w]w)}{P(\mu,k)} \qquad (1)$$

If iterate (1) find divergence. Renormalize: $w = \lambda v$, iterate and determine λ_n:

$$\hat{v}_{n+1} = \frac{\mathcal{F}\,(N[\lambda_n v_n]v_n)}{P(\mu,k)} \equiv \hat{R}(\lambda_n, v_n) \qquad (2)$$

$$||\hat{v}_n||^2 \equiv (\hat{v}_n, \hat{v}_n) = (\hat{v}_n, \hat{R}(\lambda_n, v_n)) \qquad (3)$$

Given $\hat{v}_0$; find λ_0 from (3) and $\hat{v}_1$ from (2). Repeat. Ref. MJA and Z. Musslimani (Opt. Lett., 2005).

Water Waves

Classical equations: Define the domain D by $D = \{-\infty < x, y < \infty, \quad -h < z < \eta(x,y,t), \quad t > 0\}$. The water wave equations satisfy the following system for $\phi(x,y,z,t)$ and $\eta(x,y,t)$:

$$\Delta\phi = 0 \ \text{ in } \ D,$$

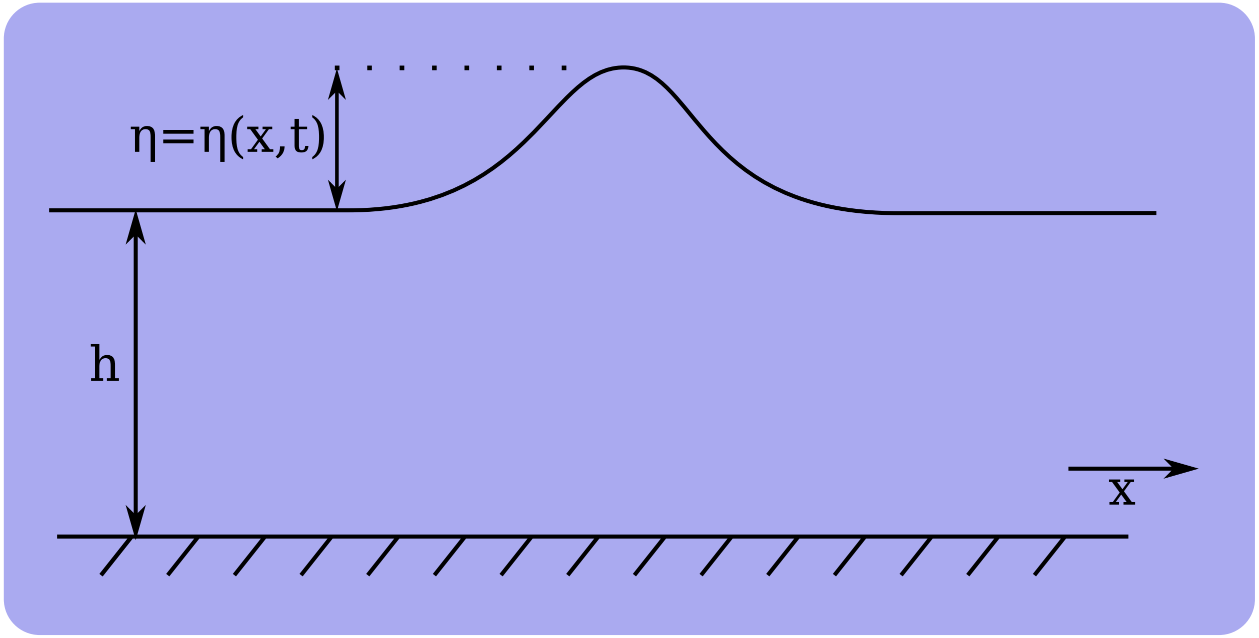
$$\phi_z = 0 \ \text{ on } \ z = -h,$$

$$\eta_t + \nabla\phi \cdot \nabla\eta = \phi_z \ \text{ on } \ z = \eta,$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right), \ \text{ on } \ z = \eta,$$

where g: gravity, σ: surface tension. Recent work with A. Fokas, Z. Musslimani (JFM, 2006) leads to a reformulation. Nonlocal eq. on a *fixed* domain.

Water Waves: fig.



Water Waves-Nonlocal System

We find two equations: one nonlocal equation and one PDE:

$$\iint dxdy e^{ikx+ily}(i\eta_t \cosh[\kappa(\eta+h)]+(\frac{kq_x}{\kappa}+\frac{lq_y}{\kappa})\sinh[\kappa(\eta+h)]) = 0 \qquad (I),$$

$$q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \cdot \nabla \eta)^2}{2(1+|\nabla \eta|^2)} = \sigma \nabla \cdot (\frac{\nabla \eta}{\sqrt{1+|\nabla \eta|^2}}) \qquad (II),$$

where $\kappa^2 = k^2 + l^2, q(x,y,t) = \phi(x,y,\eta(x,y,t))$. See also Zakharov (1968)-and Craig et al (1993,1994,..) small amplitude series for Dirichlet-Neumann map.

WW-Nonlocal System-Remarks

- Can find integral relations by taking $k, l \to 0$. First two:

$$\frac{\partial}{\partial t}\iint dxdy\ \eta(x,y,t) = 0 \quad (Mass)$$

$$\frac{\partial}{\partial t}\iint dxdy(x\eta) = \iint dxdy\ q_x(\eta + h) \quad (COM_x)$$

Center of Mass-x; RHS is related to x-momentum! Also COM -in y-direction.

- May extend formalism to infinite and variable depth.
- Can derive KP, Benney-Luke, Boussinesq, NLS systems.
- Find Lump solutions-numerically.

WW-Asymptotic Systems

Nondimensional: $\epsilon = \frac{a}{h}, \mu = \frac{h}{l_x}, \gamma = \frac{l_x}{l_y}, \epsilon, \mu, \gamma << 1$. Find Benney-Luke (1964) system (nmlz'd surface tension: $\tilde{\sigma}$):

$$q_{tt} - \tilde{\Delta} q + \tilde{\sigma}\mu^2\tilde{\Delta}^2 q + \varepsilon(\partial_t|\tilde{\nabla} q|^2 + q_t\tilde{\Delta} q) = 0 \; (BL)$$

where $\tilde{\Delta} = \partial_x^2 + \gamma^2\partial_y^2 \qquad |\tilde{\nabla} q|^2 = \left(q_x^2 + \gamma^2 q_y^2\right)$.

If $\epsilon = \mu^2 = \gamma^2$ then BL yields KP equation. Let:
$\xi = x - t, T = \epsilon t/2, w = q_\xi$:

$$\partial_\xi(w_T - \tilde{\sigma} w_{\xi\xi\xi} + 3(ww_\xi)) + w_{yy} = 0$$

KP conservation law corresponding to COM:

$$\partial_T \iint \xi w \, d\xi dy = 3 \iint w^2 d\xi dy$$

KP Equation and Lumps

KP equation in standard form (using $w = -2sgn(\tilde{\sigma})|\tilde{\sigma}|^{1/2}u$ etc.)

$$\partial_x(u_t + 6uu_x + u_{xxx}) - 3sgn(\tilde{\sigma})u_{yy} = 0$$

$\tilde{\sigma} > 0$ lumps; $\tilde{\sigma}$ normalized surface tension.

The 1-Lump solution is given by:

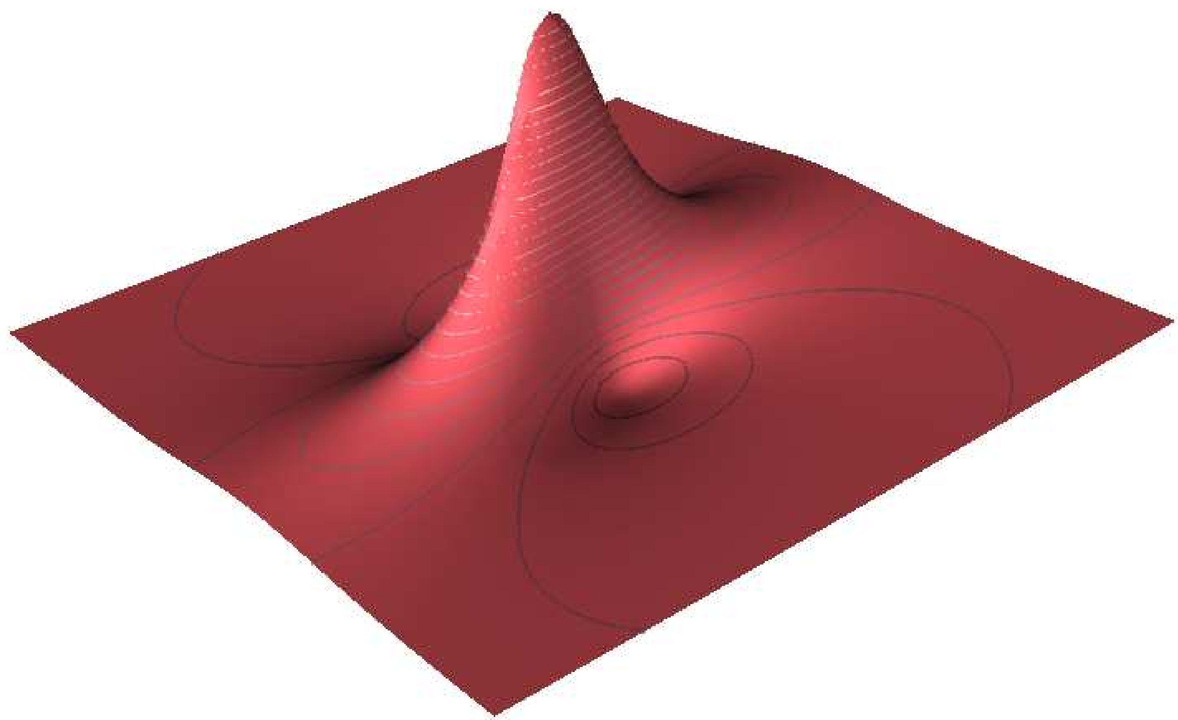
$$u = 16\frac{-4(x' - 2k_R y')^2 + 16k_I^2 y'^2 + \frac{1}{k_I^2}}{[4(x' - 2k_R y')^2 + 16k_I^2 y'^2 + \frac{1}{k_I^2}]^2},$$

where $x' = x - c_x t, y' = y - c_y t, c_x = 12(k_R^2 + k_I^2), c_y = 12k_R;$

$$u(0,0) = \frac{4}{3}(c_x - \frac{c_y^2}{12}) > 0$$

.

Lump Solution of KP



BL Equation and Lumps

$q = q(x - v_x t, y - v_y t), v_x = 1 - \epsilon c_x; v_y = c_y.$
Below: $c_x = 3, c_y = 0;$

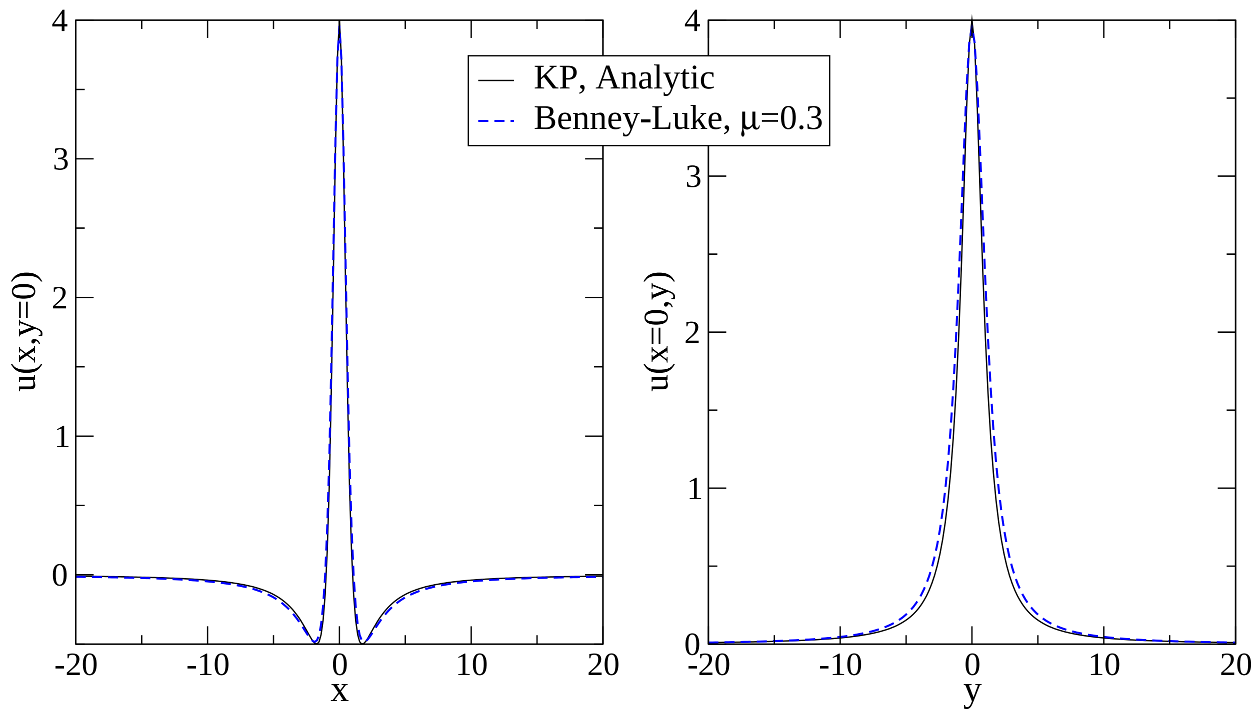


Figure 1: Wave profiles for the KP and BL eq.

BL Eq.and Lumps-con't

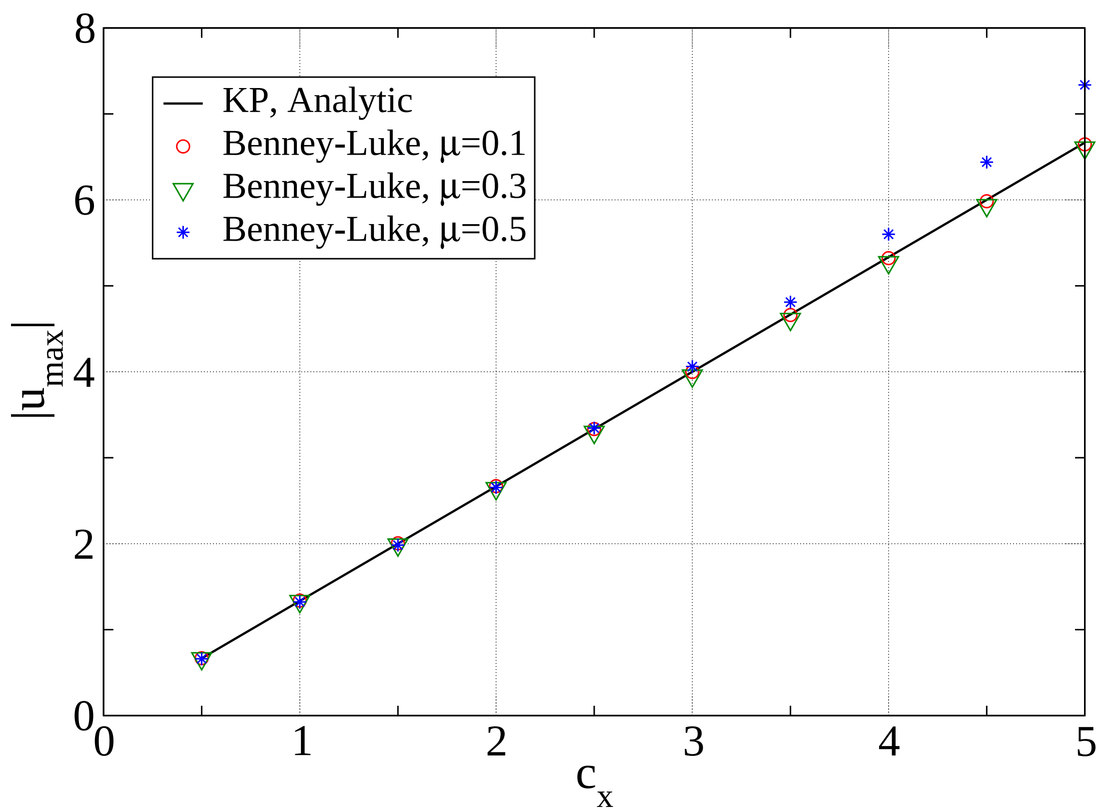


Figure 2: $u(0,0) = u_{max}$ vs. c_x for various values of μ. Fig. shows that KP is a good approx. to the BL equation in this range of parameters.

Lumps from nonlocal WW Equations

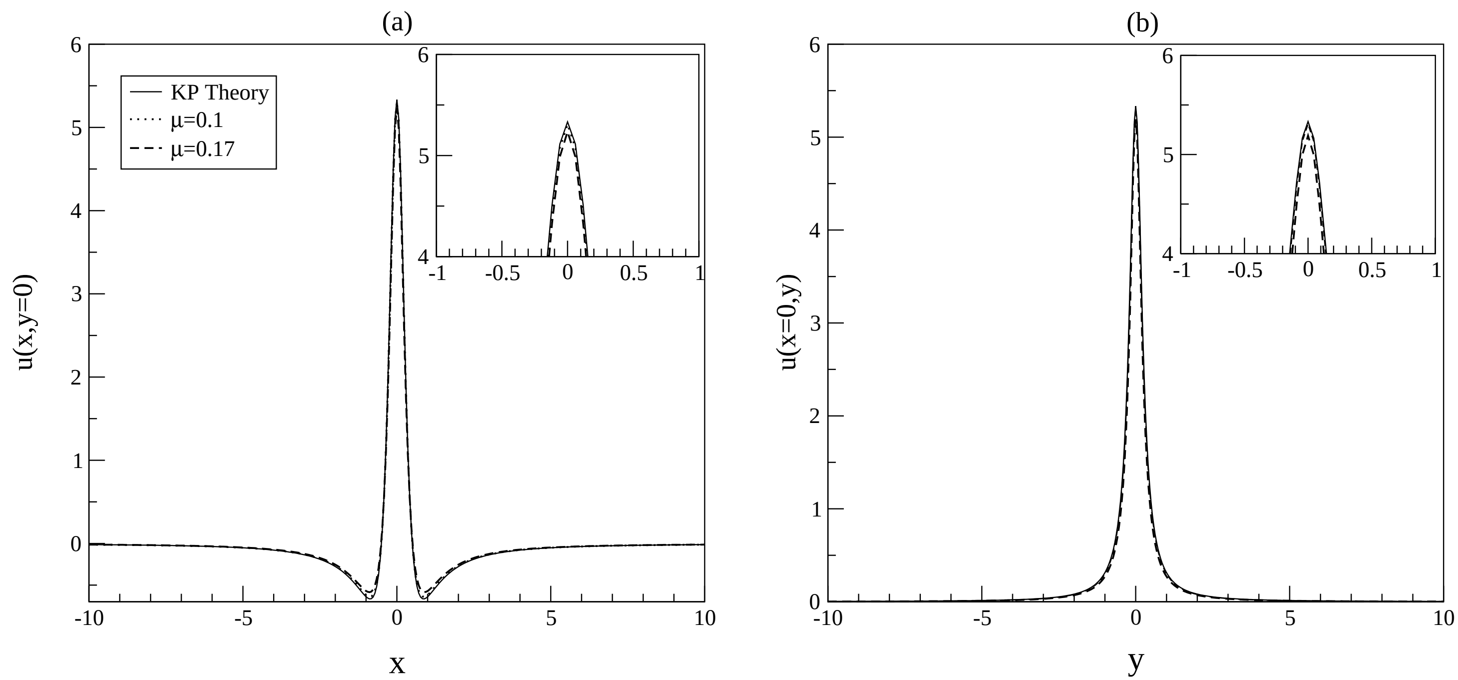


Figure 3: Wave profiles for the full WW equations, $c_x = 4.0, c_y = 0$ Benney-Luke/KP equations are good approx. to the full WW eq. in this range of parameters.

Lumps-WW Equations-figs con't

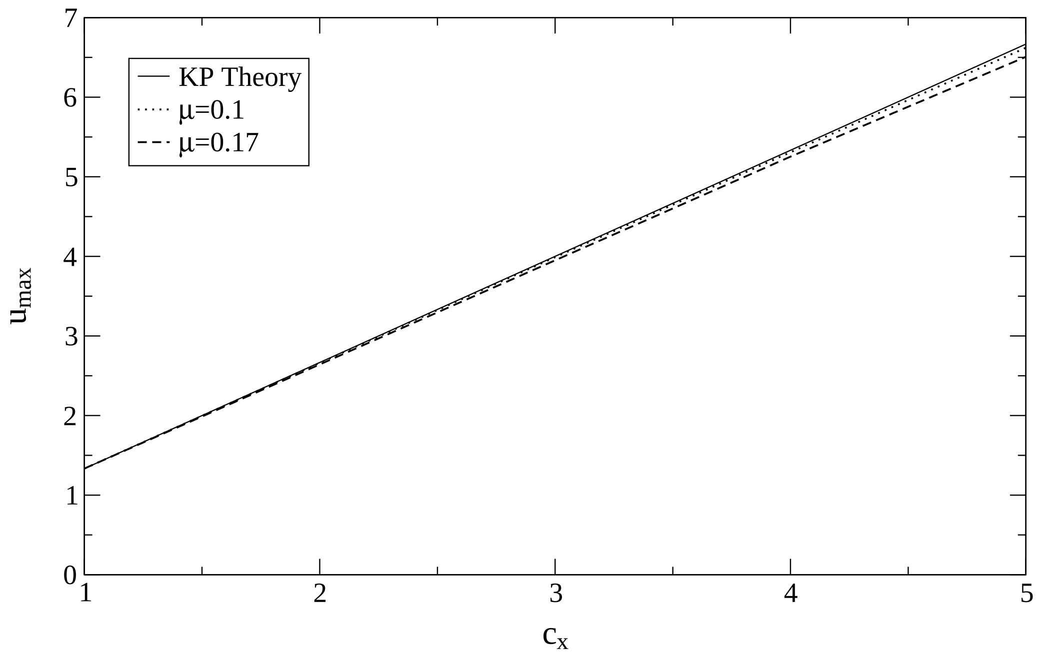


Figure 4: u_{max} vs. c_x the full WW equations for various values of μ. Fig. also shows that the Benney-Luke/KP equations are good approximations to full WW eq. in this range of parameters.

Intermediate Long Wave Models

ILW:

$$u_t + \frac{u_x}{\delta} + 2u^p u_x + T u_{xx} = 0, \quad Tu = \frac{1}{2\delta} \int_{-\infty}^{\infty} \coth[\frac{\pi(\xi - x)}{2\delta}] u(\xi) d\xi$$

Two fluid model: $p = 1$; height top fluid : h_1, bottom h_2, characteristic wavelength λ; $\quad \delta = (h_1 + h_2)/\lambda$

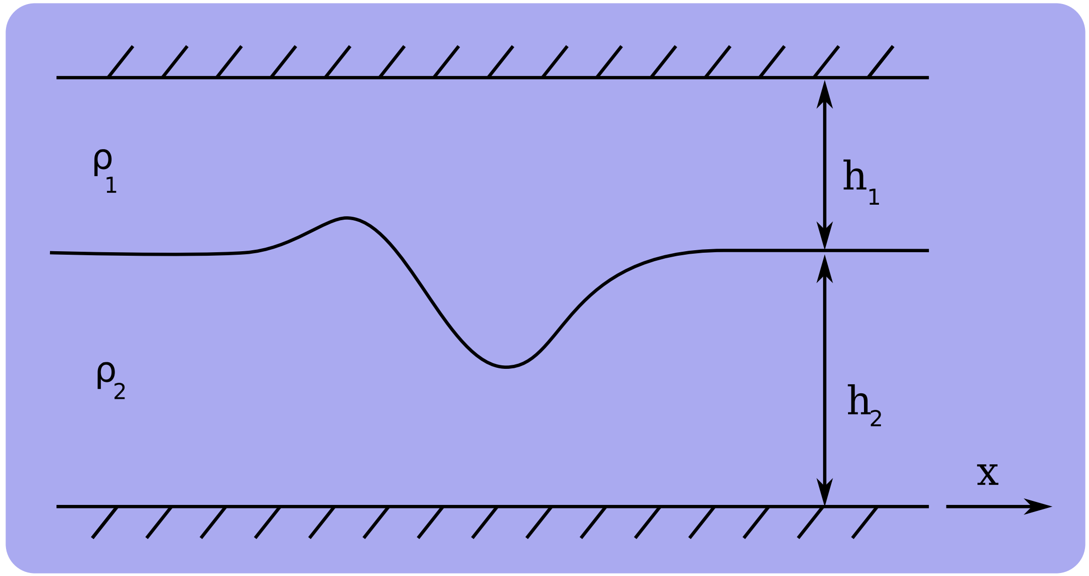
$\delta \to 0 : u_t + 2u^p u_x + \frac{\delta}{3} u_{xxx} = 0,$ HKdV

$\delta \to \infty : u_t + 2u^p u_x + H u_{xx} = 0, \ Hu = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x} d\xi,$ HBO

Solitary waves

1. Compare with exact soliton solution:
   $p = 1, \delta = 0.1, \delta = 1, \delta = 50$
   $u_s = \frac{k \sin(k\delta)}{\cos(k\delta) + \cosh k(x - x_0 - ct)}, c = \frac{1}{\delta} - k \cot(k\delta)$
2. $p = 2,$ no exact solution known.

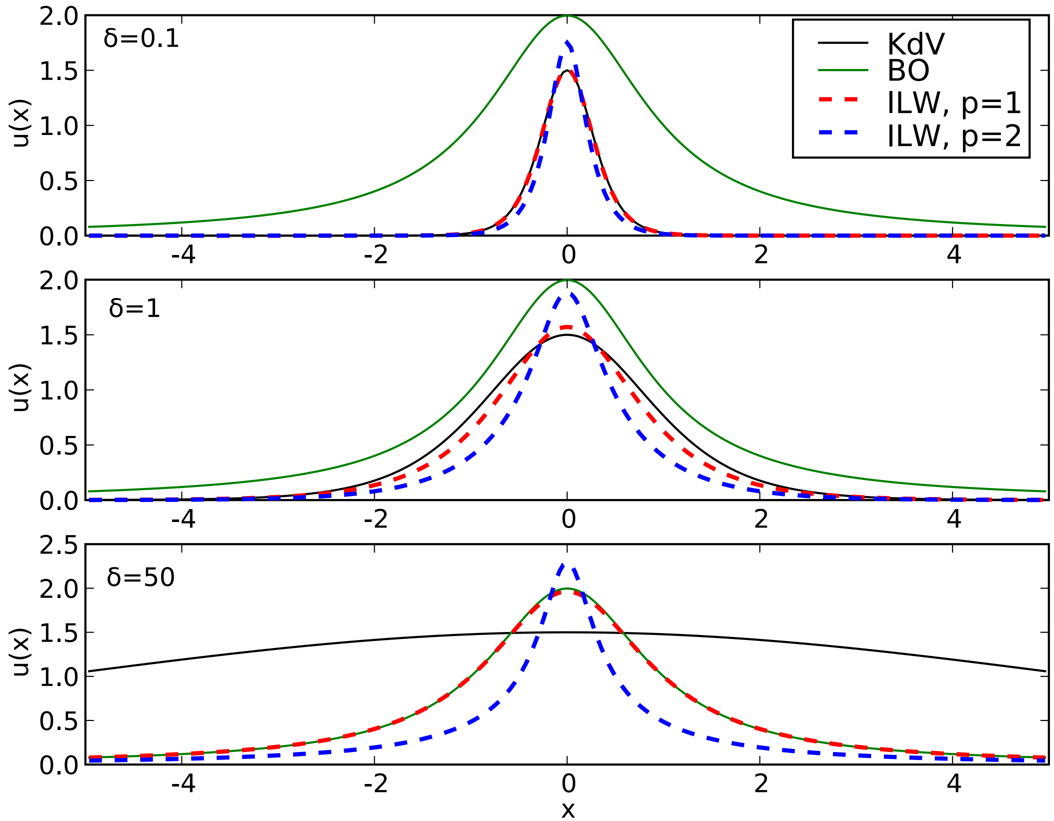
Intermediate Long Wave Fig



ILW -1D Solitary Waves

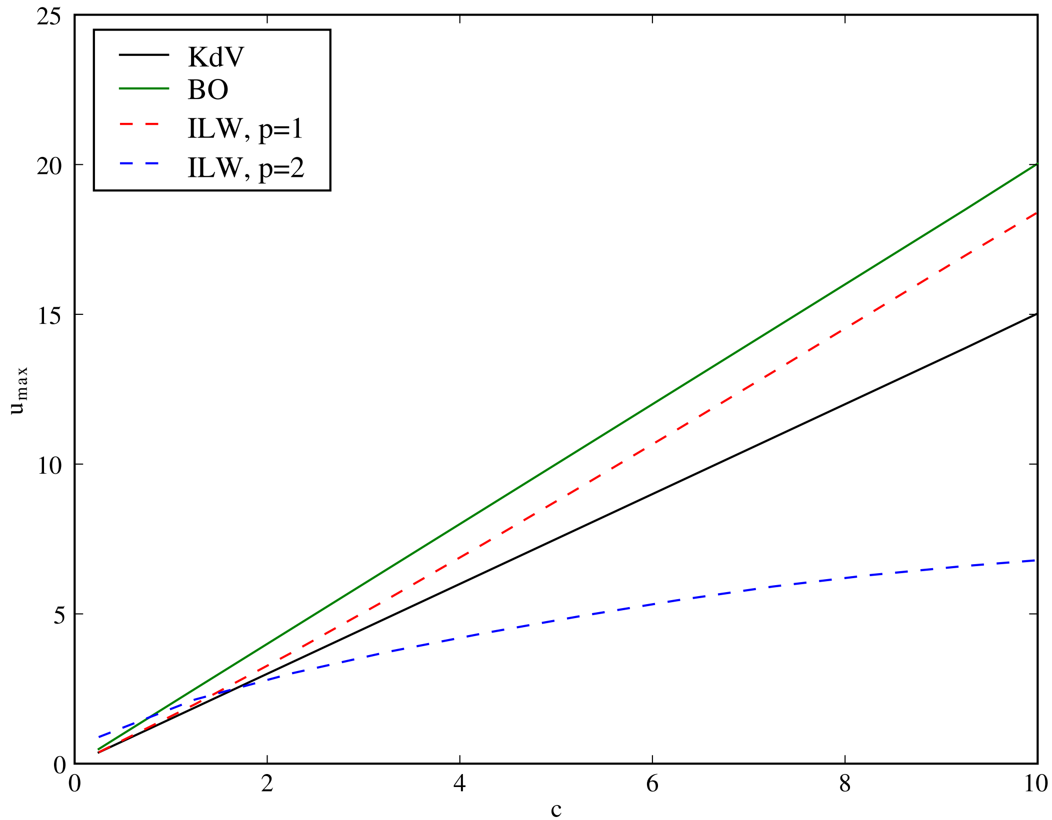
$$p = 1,\ \delta \to 0 \qquad \Rightarrow u_s \sim \frac{k^2\delta}{2}\,\mathrm{sech}^2\left[\frac{k}{2}\left(x - x_0 - \frac{k^2\delta}{3}t\right)\right]$$

$$p = 1,\ \delta \to \infty, k\delta = \pi - \frac{\gamma}{\delta} + \cdots \qquad \Rightarrow u_s \sim \frac{2\gamma}{1+[\gamma(x-x_0-\gamma t)]^2}$$



Solitary waves for ILW $p = 1, 2$, KdV and BO; $c = 1$

ILW: 1D – con't



Speed-amplitude relations for ILW $p = 1, 2$, KdV and BO

ILW models-2D

2D-ILW:

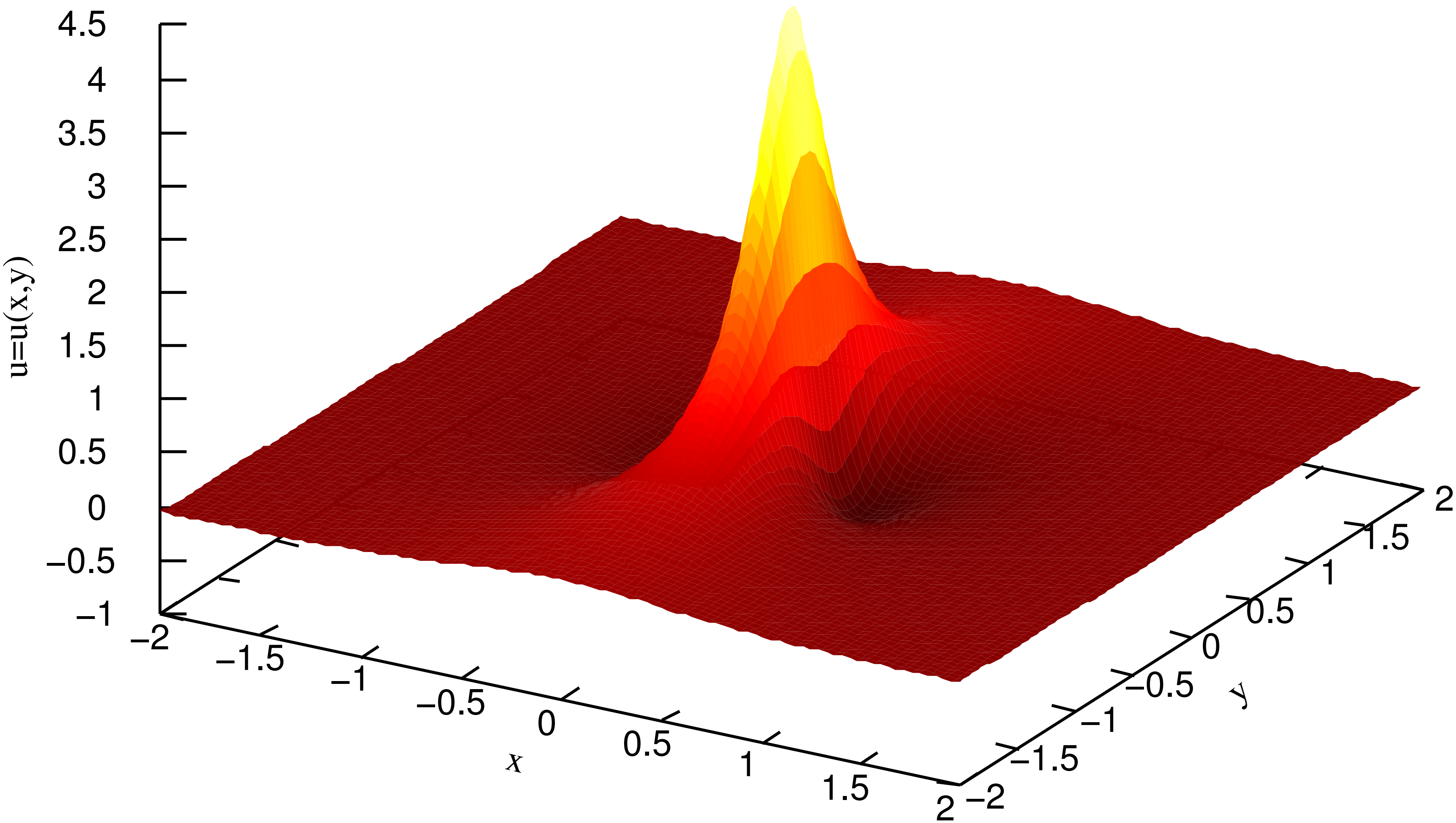
$$\partial_x(u_t + \frac{u_x}{\delta} + 2u^p u_x + Tu_{xx}) - \alpha u_{yy} = 0$$

$\alpha = \pm\frac{\delta}{1+\delta}$

$\delta \to 0$: HKP; $p = 1$: KP; $p \geq 2, \alpha > 0$, wave collapse (Wang, MJA, Segur, 1994)

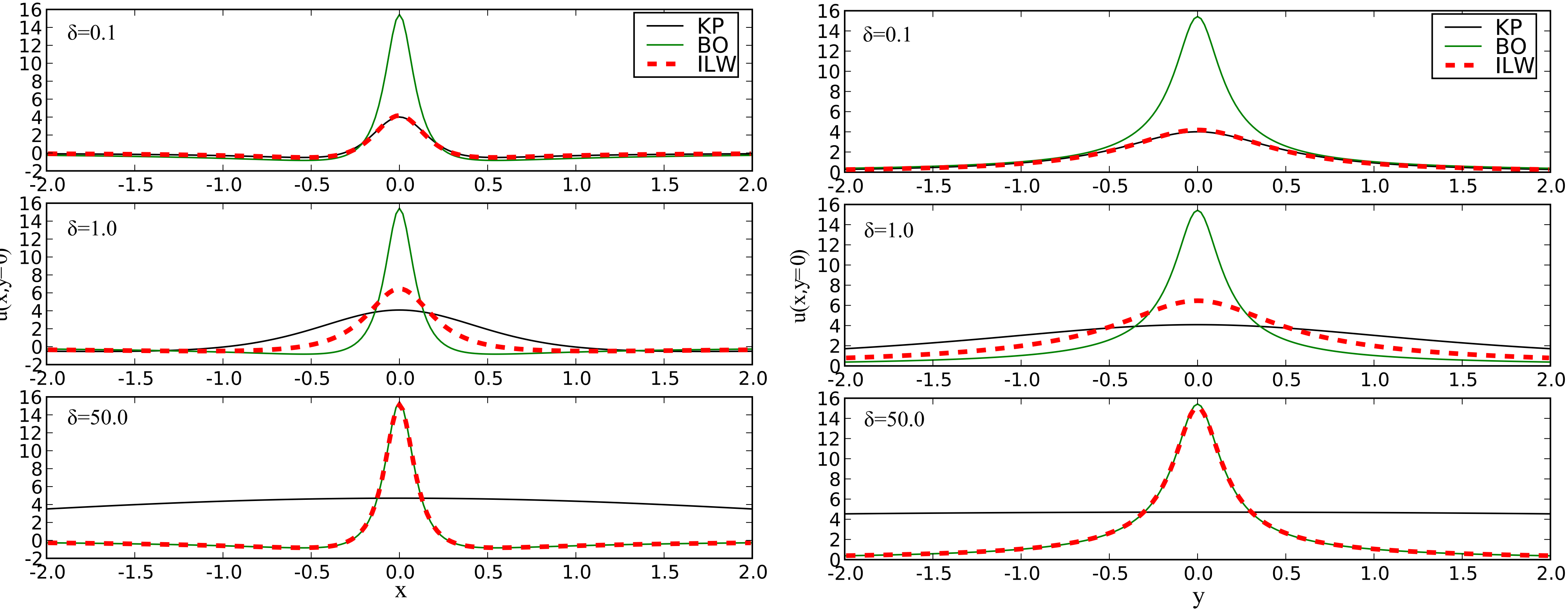
$\delta \to \infty$: 2DHBO ($p = 1$, MJA, HS, 1980)

ILW Models: 2D figures



2D ILW, $\delta = 1.0$, $c = 1.0$, $p = 1$

ILW Models: 2D figures



Conclusion

- Solitary waves -important solutions in NL optics and water waves
- NL optics
  –DMNLS eq. arises in fiber optic communications, ultra-short pulse propagation in M-L lasers. Asymptotic theory $\Longrightarrow$ DMNLS eq. and DM "solitons"
  –Lattice and discrete modes on periodic and complex backgrounds obtained
  –Ground states of collapse profiles $\chi^{(2)}$ NL optics/WW
- Water waves–reformulated as a nonlocal system; integral relations, asymptotic reductions and lumps.
- Intermediate long wave equations
- Numerical method: "spectral renormalization". Use method to find solitary waves for nonlinear: PDE's, differential-difference & singular integro-differential eq.