The density matrix of a finite segment of the Heisenberg spin chain

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Contents and context

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- Multiple integral formula for the density matrix of a chain segment
- Reduction of the multiple integrals
- Exponential formula for the density matrix
- Summary and outlook

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Our contribution: finite temperature T and magnetic field h; arbitrary finite size

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Related work for $T, h, 1/L = 0$:

- Jimbo et al. 1992, Jimbo & Miwa 1996
- Kitanine, Maillet, Slavnov & Terras 2000, 2002, 2004 (‘Lyon group’)
- Boos, Korepin, Smirnov 2001, 2002, 2003
- Kato, Nishiyama, Sakai, Sato, Shiroishi, Takahashi, Tsuboi from 2003 (‘ISSP group’)
- Boos, Jimbo, Miwa, Smirnov & Takeyama from 2004

- Ground state correlations for $L=4$, $\Delta=1$. $U|gs\rangle = |gs\rangle$, $S^\alpha|gs\rangle = 0$, $\alpha = x,y,z$.

$$|gs\rangle = 2|\downarrow\uparrow\downarrow\uparrow\rangle + 2|\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\uparrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle$$

$\Rightarrow \langle gs|gs\rangle = 12, \quad \langle \sigma_j^\alpha\rangle = 0$ perfectly homogeneous, no structure

$$\sigma_1^z\sigma_2^z\,|gs\rangle = -2|\downarrow\uparrow\downarrow\uparrow\rangle - 2|\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle \qquad \langle\sigma_1^z\sigma_2^z\rangle = -\frac{2}{3}$$

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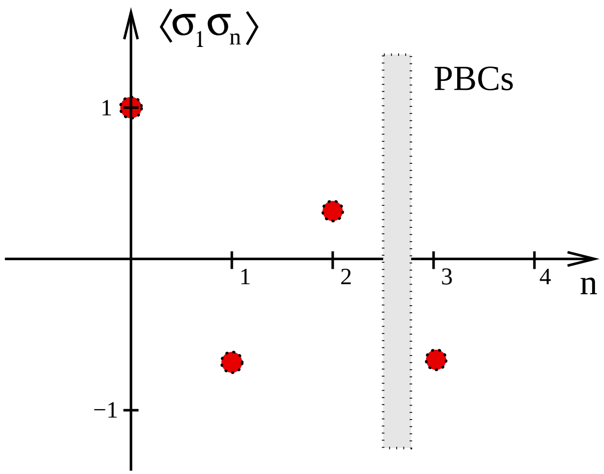

For translationally invariant systems structure reveals itself in the two-point functions

The thermodynamic limit

- One is often interested in $L \longrightarrow \infty$, thermodynamic limit
- Above method can be carried out by hand up to $L = 10$ lattice sites ($2^{10} \sim 10^3$, $S^z = 0$: $\binom{10}{5} = 252$) (Hulthén 1938)
- Using a computer we gain a further factor of 3 or 4 in the system size ($2^{30} = (2^{10})^3 \sim 10^9$).

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- Using a computer we gain a further factor of 3 or 4 in the system size ($2^{30} = (2^{10})^3 \sim 10^9$).
- Analytical solution: $\langle S_1^z S_{n+1}^z \rangle = \frac{1}{4} \langle \sigma_1^z \sigma_{n+1}^z \rangle$ from multiple integral formula and functional equations

	$L = 4$	$L = 6$	$L \longrightarrow \infty$
$n = 1$	$-\frac{1}{6}$	$-\frac{1}{4(\sqrt{13}-2)}$	$\frac{1}{12} - \frac{\ln 2}{3} \simeq -0.148$
$n = 2$	$\frac{1}{12}$	$\frac{1}{4\sqrt{13}}$	$\frac{1}{12} - \frac{4\ln 2}{3} + \frac{3\zeta(3)}{4} \simeq 0{,}061$
$n = 3$	—	$-\frac{29}{4(65+8\sqrt{13})}$	$\frac{1}{12} - 3\ln 2 + \frac{37\zeta(3)}{6} - \frac{14\zeta(3)\ln 2}{3} - \frac{3\zeta^2(3)}{2} - \frac{125\zeta(5)}{24} + \frac{25\zeta(5)\ln 2}{3} \simeq -0.050$
$n = 4$	—	—	$\simeq 0.034652776982$
$n = 5$	—	—	$\simeq -0.030890366647$
$n = 6$	—	—	$\simeq 0.024446738328$
$n = 7$	—	—	$\simeq -0.022498222763$

$n = 1$ Hulthén 1938, $n = 2$ Takahashi 1977, $n = 3, \ldots, 7$ ISSP group 2003-05

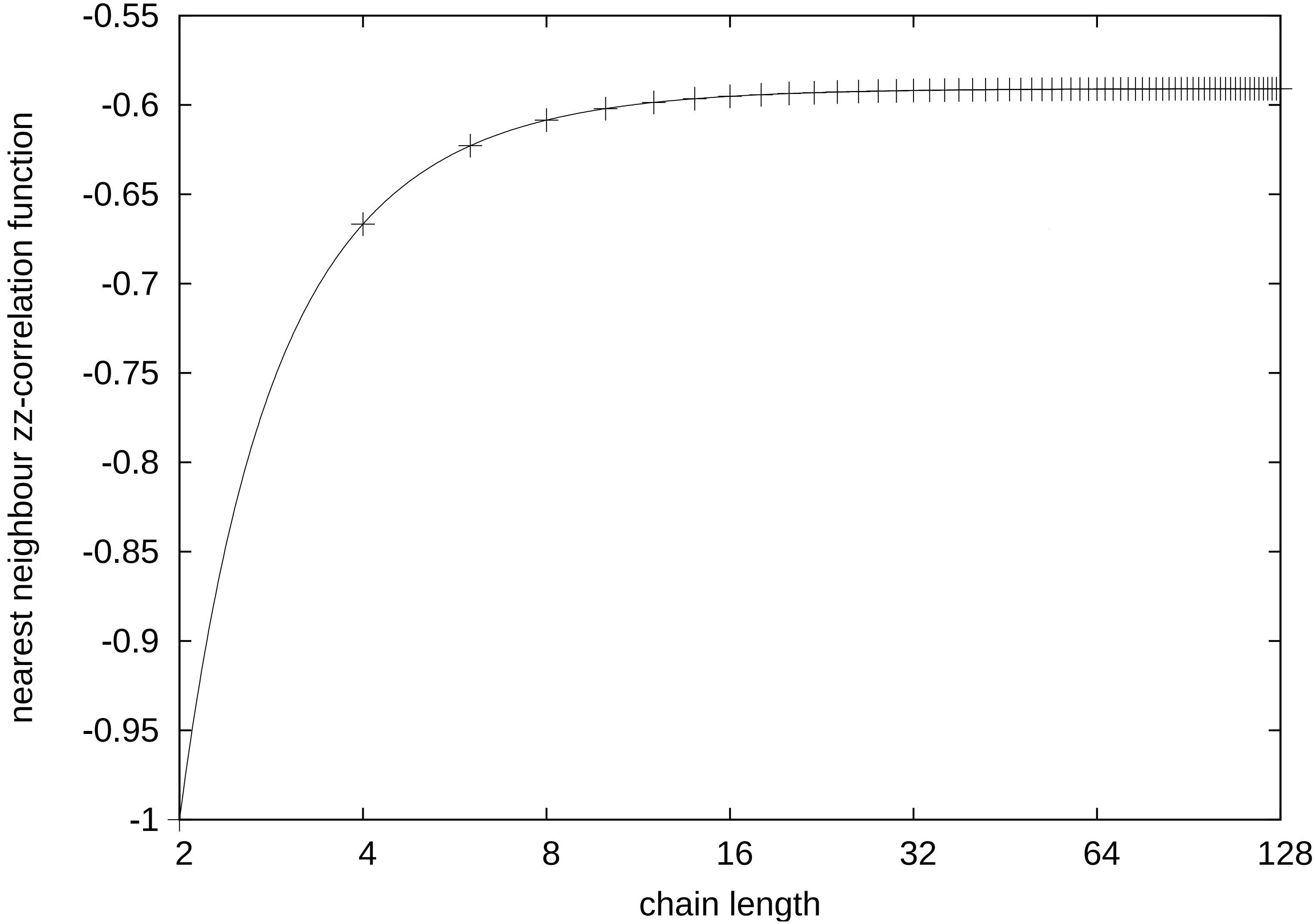
Finite size

L	$\langle\sigma_1^z\sigma_2^z\rangle$	$\langle\sigma_1^z\sigma_3^z\rangle$
2	−1.00000000000000	—
4	−0.66666666666667	0.33333333333333
6	−0.62283903060711	0.27735009811261
8	−0.60851556815620	0.26103720534839
16	−0.59519136338473	0.24696584167998
32	−0.59193864328956	0.24374937989865
64	−0.59113127886152	0.24297329183505
128	−0.59092994011745	0.24278223127753
256	−0.59087965782193	0.24273481483257
512	−0.59086709385781	0.24272300601642
1024	−0.59086395383499	0.24272006021644
∞	−0.59086290741326	0.24271907982574

zz-correlators as functions of the system size

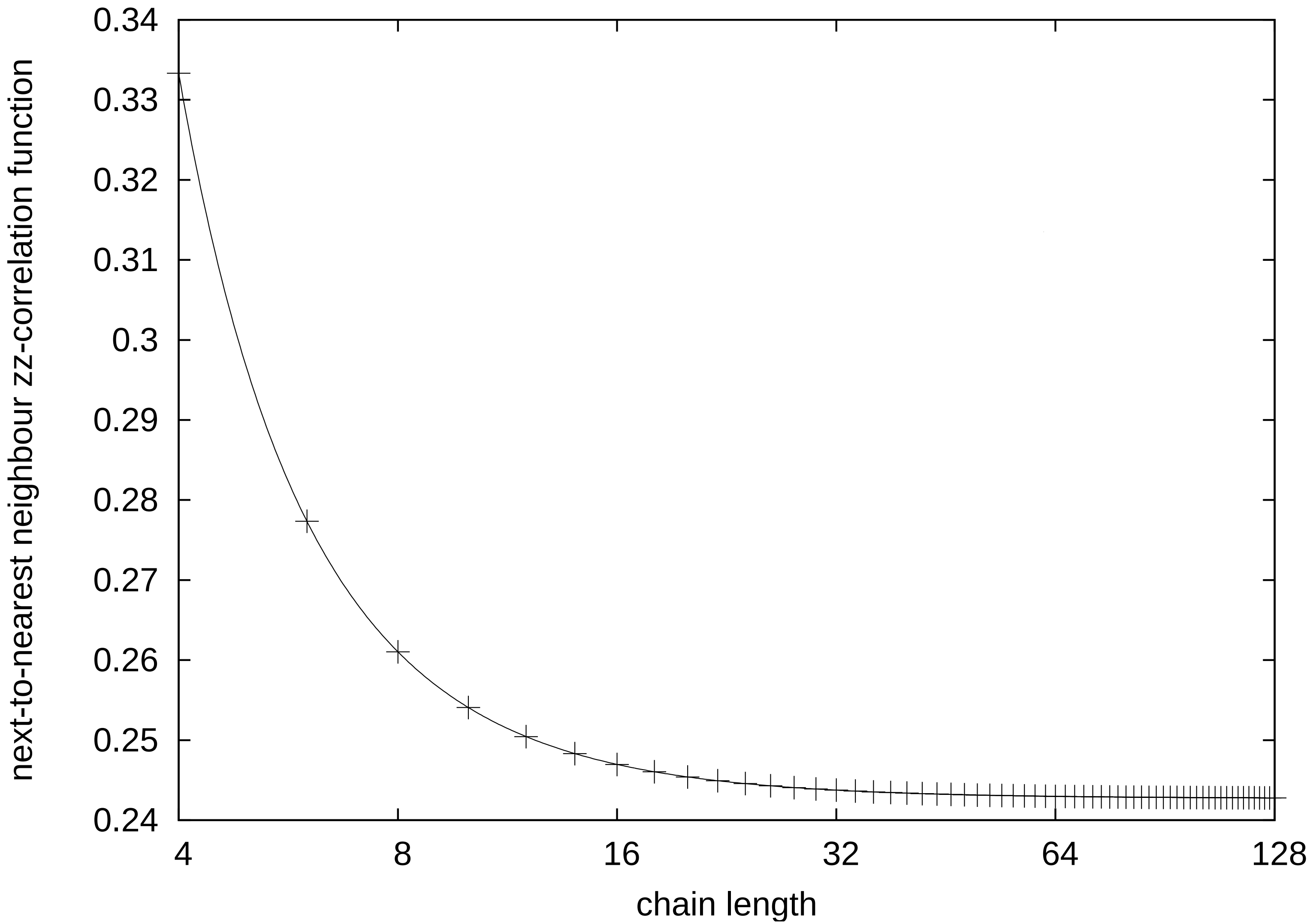
Finite size n = 1

- Finite size data from integral formula (with J. Damerau, N. P. Hasenclever, A. Klümper)



Finite size n = 2

- Finite size data from integral formula (with J. Damerau, N. P. Hasenclever, A. Klümper)



Integrability of the XXZ chain

- All properties of the XXZ chain can be derived from the well-known trigonometric solution

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of the Yang-Baxter equation, where

$$b(\lambda) = \frac{\mathrm{sh}(\lambda)}{\mathrm{sh}(\lambda+\eta)}\,,\; c(\lambda) = \frac{\mathrm{sh}(\eta)}{\mathrm{sh}(\lambda+\eta)}$$

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- This R-matrix generates the XXZ Hamiltonian

$$H_{XXZ} = 2J\,\mathrm{sh}(\eta) \sum_{j=1}^{L} \partial_\lambda (PR)_{j-1,j}(\lambda)\Big|_{\lambda=0}$$

P transposition in $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\Delta = \mathrm{ch}(\eta)$.

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- Setting $\Theta = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\Phi}, \mathrm{e}^{-\mathrm{i}\Phi})$, $\Phi \in [0, 2\pi]$, we fix twisted boundary conditions requiring that

$$\begin{pmatrix} {e_0}^1_1 & {e_0}^1_2 \\ {e_0}^2_1 & {e_0}^2_2 \end{pmatrix} = \Theta \begin{pmatrix} {e_L}^1_1 & {e_L}^1_2 \\ {e_L}^2_1 & {e_L}^2_2 \end{pmatrix} \Theta^{-1}$$

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- L-matrix

$$L_j{}^\alpha_\beta(\lambda) = R^{\alpha\gamma}_{\beta\delta}(\lambda)\, e_j{}^\delta_\gamma$$

Monodromy matrix

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \Theta L_L(\lambda)\dots L_1(\lambda)$$

Yang-Baxter algebra

$$\check{R}(\lambda-\mu)\big(T(\lambda)\otimes T(\mu)\big) = \big(T(\mu)\otimes T(\lambda)\big)\check{R}(\lambda-\mu)$$

where $\check{R} = PR$. Twisted transfer matrix $t(\lambda) = \mathrm{tr}\, T(\lambda)$.

Ground state

- Algebraic Bethe ansatz for eigenstates of $t(\lambda)$

$$|\{\lambda\}\rangle = B\big(\lambda_1 - \tfrac{\eta}{2}\big)\dots B\big(\lambda_N - \tfrac{\eta}{2}\big)|0\rangle$$

$\binom{1}{0}^{\otimes L}$ is the ferromagnetic reference state, and the set of Bethe roots $\{\lambda\} = \{\lambda_j\}_{j=1}^N$ must be determined from the Bethe ansatz equations

$$1 + \frac{e^{-2i\Phi}\,\mathrm{sh}^L(\lambda_j - \frac{\eta}{2})}{\mathrm{sh}^L(\lambda_j + \frac{\eta}{2})}\prod_{k=1}^{N}\frac{\mathrm{sh}(\lambda_j - \lambda_k + \eta)}{\mathrm{sh}(\lambda_j - \lambda_k - \eta)} = 0$$

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- Ground state of the XXZ Hamiltonian is the transfer matrix eigenstate with $\{\lambda_j\}_{j=1}^{L/2}$ the unique real solution of the Bethe ansatz equations for $N = L/2$. It determines a meromorphic auxiliary function

$$\mathfrak{a}(\lambda) = \frac{\mathrm{e}^{-2\mathrm{i}\Phi} \, \mathrm{sh}^L(\lambda - \frac{\eta}{2})}{\mathrm{sh}^L(\lambda + \frac{\eta}{2})} \prod_{k=1}^{L/2} \frac{\mathrm{sh}(\lambda - \lambda_k + \eta)}{\mathrm{sh}(\lambda - \lambda_k - \eta)}$$

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- In terms of this function the ground state eigenvalue $\Lambda_0(\lambda)$ becomes

$$\Lambda_0(\lambda) = \big(1 + \mathfrak{a}(\lambda + \tfrac{\eta}{2})\big)\, e^{i\Phi} \prod_{j=1}^{L/2} \frac{\mathrm{sh}(\lambda - \lambda_j - \frac{\eta}{2})}{\mathrm{sh}(\lambda - \lambda_j + \frac{\eta}{2})}$$

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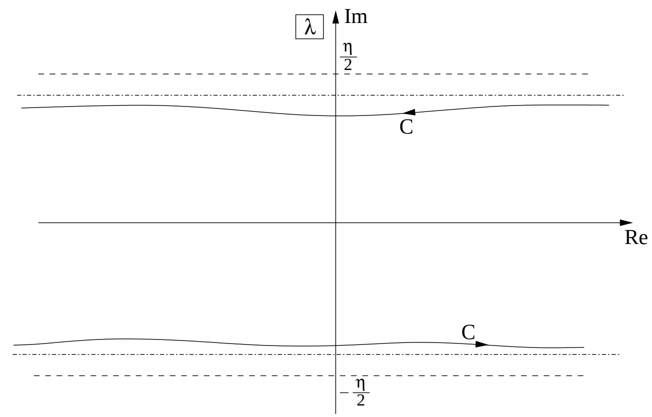
- Bethe ansatz equations imply that $\Lambda_0(\lambda)$ is regular at the points $\lambda_j - \frac{\eta}{2}$, $j = 1, \dots, L/2$. Extensive numerical studies support the conjecture that $\Lambda_0(\lambda)$ is non-zero inside a strip $-|\eta| \le \mathrm{Im}\,\lambda \le 0$. Then $1 + \mathfrak{a}(\lambda)$ is analytic inside the strip $-\frac{|\eta|}{2} < \mathrm{Im}\,\lambda \le \frac{|\eta|}{2}$ and its only zeros in this strip are the Bethe roots. This together with the obvious analytic and asymptotic properties of $\mathfrak{a}(\lambda)$ is enough to set up a set of functional equations for the second logarithmic derivatives of $\mathfrak{a}(\lambda)$ and $1 + \mathfrak{a}(\lambda)$ which together with their known asymptotics determine $\mathfrak{a}(\lambda)$ uniquely.

Auxiliary function

- Non-linear integral equation

$$\ln \mathfrak{a}(\lambda) = -2i\Phi + L\eta + L\ln\left(\frac{\operatorname{sh}(\lambda - \frac{\eta}{2})}{\operatorname{sh}(\lambda + \frac{\eta}{2})}\right) - \int_{\mathcal{C}} \frac{d\omega}{2\pi} K_\eta(\lambda - \omega)\ln(1 + \mathfrak{a}(\omega))$$



The canonical contour (for the critical regime) $\mathcal{C}$ surrounds the real axis in counterclockwise manner inside the strip $-\frac{|\eta|}{2} < \operatorname{Im}\lambda < \frac{|\eta|}{2}$

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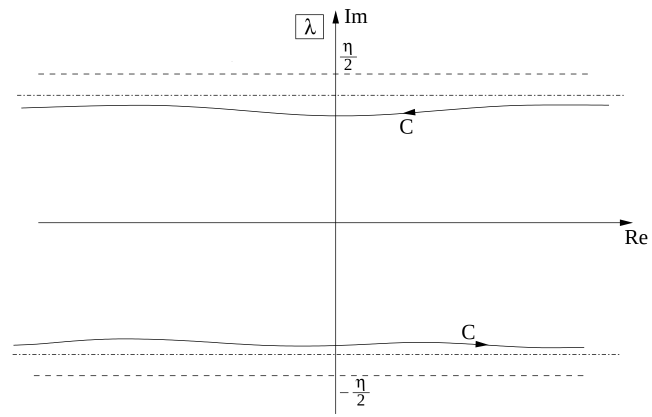
- The kernel $K_\eta(\lambda)$ is defined as

$$K_\eta(\lambda) = \frac{\mathrm{sh}(2\eta)}{i\,\mathrm{sh}(\lambda-\eta)\,\mathrm{sh}(\lambda+\eta)}$$

The ground state eigenvalue $\Lambda_0(\lambda)$ can be expressed as an integral over $\mathfrak{a}(\lambda)$,

$$\ln \Lambda_0(\lambda) = i\Phi + \frac{L(i\pi-\eta)}{2} + \int_{\mathcal{C}} \frac{d\omega}{2\pi} K_{\frac{\eta}{2}}(\lambda-\omega)\ln(1+\mathfrak{a}(\omega))$$

This determines the ground state energy and the eigenvalues of the higher conserved quantities as a function of L.



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Density matrix of a finite segment

- The density matrix is a means to describe a sub-system as part of a larger system in thermodynamic equilibrium in terms of the degrees of freedom of the sub-system.

$$\rho_L = \frac{\mathrm{e}^{-\frac{H}{T}}}{\mathrm{tr}\, \mathrm{e}^{-\frac{H}{T}}}$$

  statistical operator for chain at temperature T. Then the density matrix of the sub-system consisting of the first m lattice sites is

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- If we follow the common convention and use the same symbols for the local operators ${e_j}^{\alpha}_{\beta}$ and for their restriction to the first m sites, we find the expression

$$\begin{aligned} {D_L}^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_m}(T) &= \operatorname{tr}_{1\dots m} {e_1}^{\alpha_1}_{\beta_1} \dots {e_m}^{\alpha_m}_{\beta_m} D_L(T) \\ &= \langle {e_1}^{\alpha_1}_{\beta_1} \dots {e_m}^{\alpha_m}_{\beta_m} \rangle_T \end{aligned}$$

for the matrix elements of the density matrix.

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for the matrix elements of the density matrix.

- Here we are interested in the unique (normalized) ground state $|\Psi_0\rangle$ of the system of finite even length. In the limit $T \to 0+$ the statistical operator ρ_L converges to the projector $|\Psi_0\rangle\langle\Psi_0|$ onto the ground state, and the formula for the density matrix elements turns into

$$\begin{aligned} {D_L}^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_m} &= \lim_{T\to 0+} {D_L}^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_m}(T) \\ &= \langle \Psi_0 | {e_1}^{\alpha_1}_{\beta_1} \dots {e_m}^{\alpha_m}_{\beta_m} | \Psi_0 \rangle \end{aligned}$$

Density matrix of a finite segment

- We use a trick suggested in [KMT99] in order to express this entirely in terms of data related to the monodromy matrix $T(\lambda)$

$$e_j{}^\alpha_\beta = t^{j-1}(0) T^\alpha_\beta(0) t^{-j}(0)$$

It follows that

$$D_L{}^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_m} = \langle \Psi_0 | T^{\alpha_1}_{\beta_1}(0) \dots T^{\alpha_m}_{\beta_m}(0) t^{-m}(0) | \Psi_0 \rangle$$

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- In order to apply the techniques developed in [GKS04] for the finite-temperature case we regularize the expression by introducing inhomogeneity parameters ξ_j, $j = 1, \dots, m$, in the following way. Define an 'inhomogeneous density matrix'

$$D_L{}^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_m}(\xi_1, \dots, \xi_m) = \frac{\langle \{\lambda\} | T^{\alpha_1}_{\beta_1}(\xi_1 - \frac{\eta}{2}) \dots T^{\alpha_m}_{\beta_m}(\xi_m - \frac{\eta}{2}) | \{\lambda\} \rangle}{\langle \{\lambda\} | \{\lambda\} \rangle \prod_{j=1}^m \Lambda_0(\xi_j - \frac{\eta}{2})}$$

where $|\{\lambda\}\rangle$ is the (unnormalized) Bethe ansatz ground state. Then

$$D_L{}^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_m} = \lim_{\xi_1, \dots, \xi_m \to \frac{\eta}{2}} D_L{}^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_m}(\xi_1, \dots, \xi_m)$$

Integral representation

- The inhomogeneous density matrix element for the ground state of the chain of finite length

$$D_L{}^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_m}(\xi_1,\dots,\xi_m) = \left[\prod_{j=1}^{|\alpha^+|}\int_{\mathcal{C}}\frac{d\omega_j}{2\pi i(1+\mathfrak{a}(\omega_j))}\prod_{k=1}^{x_j-1}\mathrm{sh}(\omega_j-\xi_k-\eta)\prod_{k=x_j+1}^{m}\mathrm{sh}(\omega_j-\xi_k)\right]$$
$$\left[\prod_{j=|\alpha^+|+1}^{m}\int_{\mathcal{C}}\frac{d\omega_j}{2\pi i(1+\overline{\mathfrak{a}}(\omega_j))}\prod_{k=1}^{y_j-1}\mathrm{sh}(\omega_j-\xi_k+\eta)\prod_{k=y_j+1}^{m}\mathrm{sh}(\omega_j-\xi_k)\right]$$
$$\frac{\det(-G(\omega_j,\xi_k))}{\prod_{1\le j<k\le m}\mathrm{sh}(\xi_k-\xi_j)\,\mathrm{sh}(\omega_j-\omega_k-\eta)}$$

where $\overline{\mathfrak{a}} = 1/\mathfrak{a}$ and where the function $G(\omega,\xi)$ has to be calculated from the linear integral equation

$$G(\lambda,\xi) = \frac{\mathrm{sh}(\eta)}{\mathrm{sh}(\lambda-\xi)\,\mathrm{sh}(\lambda-\xi-\eta)} + \int_{\mathcal{C}}\frac{d\omega\, G(\omega,\xi)}{2\pi(1+\mathfrak{a}(\omega))}K_\eta(\lambda-\omega)$$

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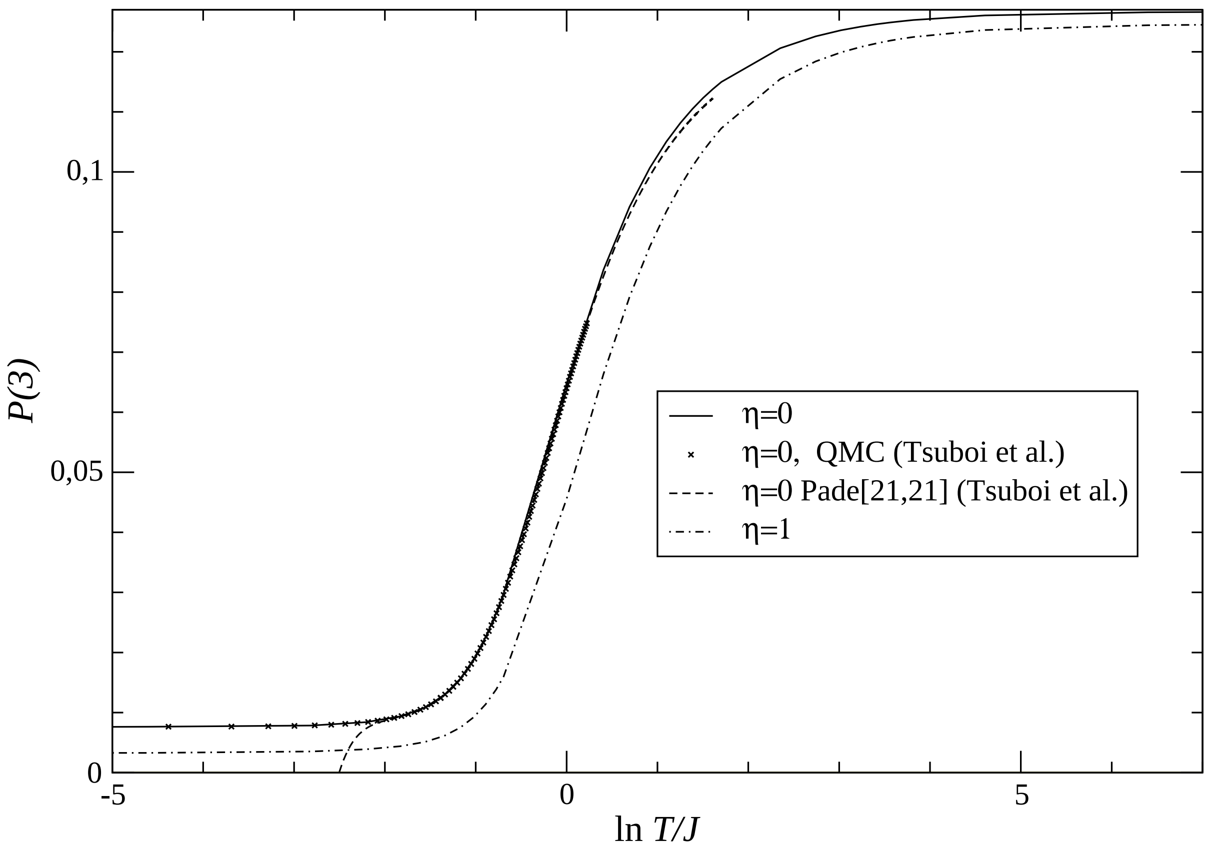
- For the chain at finite T with external magnetic field h replace the auxiliary function with the solution of

$$\ln\mathfrak{a}(\lambda) = -\frac{h}{T} - \frac{2J\,\mathrm{sh}^2(\eta)}{T\,\mathrm{sh}(\lambda)\,\mathrm{sh}(\lambda+\eta)} - \int_{\mathcal{C}}\frac{d\omega}{2\pi}K_\eta(\lambda-\omega)\ln(1+\mathfrak{a}(\omega))$$

Multiple integrals numerically

- Numerical evaluation of multiple integrals for $m = 3$, temperature case



Reduction of integrals

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- Were considered rather useless for practical calculations, since the numerical costs grow exponentially with the number of integrations for multiple integral
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- What is behind?
  Theorem (H. E. Boos et al. 05). The inhomogeneous correlation functions of the isotropic Heisenberg chain at $T, h = 0$ depend on a single transcendental function

$$\psi_0(\xi_1, \xi_2) = 2\mathrm{i}\,\partial_x \ln \left[\frac{\Gamma\left(\frac{1}{2} + \frac{\mathrm{i}x}{2}\right)\Gamma\left(1 - \frac{\mathrm{i}x}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{\mathrm{i}x}{2}\right)\Gamma\left(1 + \frac{\mathrm{i}x}{2}\right)} \right]_{x=\xi_1-\xi_2}$$

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  which is proportional to the two-spinon scattering phase (for XXZ two functions, for XYZ three).
- Why do ζ-functions appear?

$$\psi_0(\xi_1, \xi_2) = 4 \sum_{k=0}^{\infty} (-1)^k (\xi_1 - \xi_2)^{2k} \zeta_a(2k+1), \qquad \zeta_a(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^z} = \begin{cases} (1 - 2^{1-z})\zeta(z) & \text{if } z \neq 1 \\ \ln 2 & \text{if } z = 1 \end{cases}$$

Reduction of integrals

Consider the inhomogeneous emptiness formation probability for the XXX chain

$$D^{11}_{11}(\xi_1,\xi_2)(\xi_1-\xi_2) = \int_C \frac{d\omega_1}{2\pi(1+\mathfrak{a}(\omega_1))} \int_C \frac{d\omega_2}{2\pi(1+\mathfrak{a}(\omega_2))} \det\big(G(\omega_j,\xi_k)\big) \underbrace{\frac{(\omega_1-\xi_1-\mathrm{i})(\omega_2-\xi_2)}{\omega_1-\omega_2-\mathrm{i}}}_{=:r(\omega_1,\omega_2)}$$

$$= \frac{1}{2}\int_C \frac{d\omega_1}{2\pi(1+\mathfrak{a}(\omega_1))} \int_C \frac{d\omega_2}{2\pi(1+\mathfrak{a}(\omega_2))} \det\big(G(\omega_j,\xi_k)\big) \big(r(\omega_1,\omega_2)-r(\omega_2,\omega_1)\big)$$

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Here

$$r(\omega_1,\omega_2)-r(\omega_2,\omega_1) = \frac{(\omega_1-\xi_1-\mathrm{i})(\omega_2-\xi_2)}{\omega_1-\omega_2-\mathrm{i}} + \frac{(\omega_2-\xi_1-\mathrm{i})(\omega_1-\xi_2)}{\omega_1-\omega_2+\mathrm{i}} = \frac{P(\omega_1,\omega_2)}{1+(\omega_1-\omega_2)^2}$$

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The polynomial $P(\omega_1,\omega_2)$ can be decomposed in such a way that

$$\frac{P(\omega_1,\omega_2)}{1+(\omega_1-\omega_2)^2} = -\frac{2}{3}(\omega_1-\omega_2) + \frac{p(\omega_1)-p(\omega_2)}{1+(\omega_1-\omega_2)^2}$$

where

$$p(\omega) = \frac{2}{3}\omega^3 - (\xi_1+\xi_2+\mathrm{i})\omega^2 + \big[\mathrm{i}(\xi_1+\xi_2+\tfrac{\mathrm{i}}{3}) + 2\xi_1\xi_2\big]\omega$$

Reduction of integrals

Then

$$D_{11}^{11}(\xi_1,\xi_2)(\xi_1-\xi_2) = \frac{1}{4}\sum_{P\in\mathfrak{S}^2} \mathrm{sign}(P)\int_C \frac{d\omega_1\, G(\omega_1,\xi_{P1})}{\pi(1+\mathfrak{a}(\omega_1))}\int_C \frac{d\omega_2\, G(\omega_2,\xi_{P2})}{\pi(1+\mathfrak{a}(\omega_2))}\left[-\frac{2}{3}\omega_1 + \frac{p(\omega_1)}{1+(\omega_1-\omega_2)^2}\right]$$

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This can be reduced by means of the integral equation for $G(\omega,\xi)$,

$$\int_C\frac{d\omega_2\,G(\omega_2,\xi_{P2})}{\pi(1+\mathfrak{a}(\omega_2))}\frac{1}{1+(\omega_1-\omega_2)^2}=G(\omega_1,\xi_{P2})+\frac{1}{(\omega_1-\xi_{P2})(\omega_1-\xi_{P2}-\mathrm{i})}$$

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Finally (H. E. Boos, FG, A. Klümper, J. Suzuki 06)

$$\begin{aligned}D^{11}_{11}(\xi_1,\xi_2)(\xi_1-\xi_2)=\sum_{P\in\mathfrak{S}^2}\mathrm{sign}(P)\bigg[&\frac{1}{12}(3\xi_{P1}-\xi_{P2}+\mathrm{i})\phi_1(\xi_{P1})\\&+\frac{1}{6}\phi_2(\xi_{P2})-\frac{1}{6}\phi_1(\xi_{P1})\phi_2(\xi_{P2})-\frac{1}{24}(\xi_{P1}-\xi_{P2})(1+(\xi_{P1}-\xi_{P2})^2)\psi(\xi_{P1},\xi_{P2})\bigg]\end{aligned}$$

where

$$\psi(\xi_1,\xi_2)=\int_C\frac{d\omega}{\pi(1+\mathfrak{a}(\omega))}\frac{G(\omega,\xi_1)}{(\omega-\xi_2)(\omega-\xi_2-\mathrm{i})},\qquad \phi_j(\xi)=\int_C\frac{d\omega\,\omega^{j-1}G(\omega,\xi)}{\pi(1+\mathfrak{a}(\omega))}$$

Reduction of integrals

The functions ψ generalizes the 'two-spinon scattering phase' to finite temperature and magnetic field

$$\lim_{T\to 0}\lim_{h\to 0}\psi(\xi_1,\xi_2)=\psi_0(\xi_1,\xi_2)$$

Instead of $\psi(\xi_1,\xi_2)$ we shall rather use the closely related expression

$$\gamma(\xi_1,\xi_2)=\left[1+(\xi_1-\xi_2)^2\right]\psi(\xi_1,\xi_2)-1$$

in terms of which our final formulae look neater. We also define $\lim_{h\to 0}\gamma(\xi_1,\xi_2)=:\gamma_0(\xi_1,\xi_2)$.

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In the same limit the moments $\phi_j(\xi)$ become polynomials of order $j-1$ in ξ,

$$\lim_{T\to 0}\lim_{h\to 0}\phi_j(\xi) = \phi_j^{(0)}(\xi) = (-\mathrm{i}\partial_k)^{j-1}\,\frac{2\mathrm{e}^{\mathrm{i}k\xi}}{1+\mathrm{e}^k}\bigg|_{k=0}$$

Using these polynomials we define the 'normalized moments' $\varphi_j(\xi) = \phi_j(\xi) - \phi_j^{(0)}(\xi)$ which vanish for $T,h\to 0$. We further introduce the symmetric combinations

$$\Delta_n(\xi_1,\dots,\xi_n) = \frac{\det(\varphi_j(\xi_k))\big|_{j,k=1,\dots,n}}{\prod_{1\le j<k\le n}\xi_{kj}}$$

with the shorthand notation $\xi_{kj} = \xi_k - \xi_j$.

Reduction of integrals

Emptiness formation probability for $m = 3$.

$$D^{111}_{111}(\xi_1,\xi_2,\xi_3) = \frac{1}{24} + \frac{1+5\xi_{12}\xi_{13}}{40\xi_{12}\xi_{13}}\Delta_1(\xi_1) + \frac{1+2\xi_{13}\xi_{23}}{24\xi_{13}\xi_{23}}\Delta_2(\xi_1,\xi_2) + \frac{1}{60}\Delta_3(\xi_1,\xi_2,\xi_3)$$
$$+ \frac{1-\xi_{13}\xi_{23}}{24\xi_{13}\xi_{23}}\,\gamma(\xi_1,\xi_2) - \frac{3+2\xi_{12}^2+5\xi_{13}\xi_{23}}{120\xi_{13}\xi_{23}}\,\gamma(\xi_1,\xi_2)\Delta_1(\xi_3) + \text{cyclic permutations.}$$

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In the limit of vanishing magnetic field $\Delta_j \to 0$, and our result reduces to

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The two-point functions for $m = 3$.

$$\langle\sigma_1^z\sigma_3^z\rangle_{T,h} = \tfrac{2}{3}\Delta_2(0,0) - \tfrac{1}{3}\gamma(0,0) - \tfrac{1}{6}(\Delta_2)_{xx}(0,0) + \tfrac{1}{3}(\Delta_2)_{xy}(0,0) - \tfrac{1}{6}\gamma_{xx}(0,0) + \tfrac{1}{3}\gamma_{xy}(0,0)$$

$$\langle\sigma_1^x\sigma_3^x\rangle_{T,h} = -\tfrac{1}{3}\Delta_2(0,0) - \tfrac{1}{3}\gamma(0,0) + \tfrac{1}{12}(\Delta_2)_{xx}(0,0) - \tfrac{1}{6}(\Delta_2)_{xy}(0,0) - \tfrac{1}{6}\gamma_{xx}(0,0) + \tfrac{1}{3}\gamma_{xy}(0,0)$$

Exponential formula

Conjecture. The density matrix of a finite sub-chain of length m of the infinite XXX Heisenberg chain at finite T (for $h = 0$) is determined by the vector

$$h_m(\lambda_1, \dots, \lambda_m) = \frac{1}{2^m} \mathrm{e}^{\Omega_m^T(\lambda_1, \dots, \lambda_m)} \mathbf{s}_m \,, \qquad \mathbf{s}_m = \prod_{j=1}^m s_{j,\bar{j}},$$

$$\Omega_m^T(\lambda_1, \dots, \lambda_m) = \frac{(-1)^{(m-1)}}{4} \int\int \frac{d\mu_1}{2\pi \mathrm{i}} \frac{d\mu_2}{2\pi \mathrm{i}} \frac{\gamma_0(\mathrm{i}\mu_1, \mathrm{i}\mu_2)(\mu_1 - \mu_2)}{[1 - (\mu_1 - \mu_2)^2]^2}$$
$$\times \operatorname{tr}_{\mu_{1,2},2,2} \Big\{ T\big(\tfrac{\mu_1+\mu_2}{2}; \lambda_1, \dots, \lambda_m\big) \otimes \big[T(\mu_1; \lambda_1, \dots, \lambda_m) \otimes T(\mu_2; \lambda_1, \dots, \lambda_m) \mathcal{P}^- \big] \Big\},$$

through

$$h_m^{\varepsilon_1, \dots, \varepsilon_m, \bar{\varepsilon}_m, \dots, \bar{\varepsilon}_1}(\lambda_1, \dots, \lambda_m) = D^{(3-\varepsilon_1)/2, \dots, (3-\varepsilon_m)/2}_{(3+\bar{\varepsilon}_1)/2, \dots, (3+\bar{\varepsilon}_m)/2}(\xi_1, \dots, \xi_m) \cdot \prod_{j=1}^m (-\bar{\varepsilon}_j)\,,$$

where $\lambda_j = -\mathrm{i}\xi_j$ for $j = 1, \dots, m$. By the integral over μ_1, μ_2 it is meant to take the residues at the poles $\lambda_1, \dots, \lambda_m$ of the integrand.

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$$\times \operatorname{tr}_{\mu_{1,2},2,2}\Big\{T\big(\tfrac{\mu_1+\mu_2}{2};\lambda_1,\dots,\lambda_m\big)\otimes\big[T(\mu_1;\lambda_1,\dots,\lambda_m)\otimes T(\mu_2;\lambda_1,\dots,\lambda_m)\mathcal{P}^-\big]\Big\},$$

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where $\lambda_j = -i\xi_j$ for $j=1,\dots,m$. By the integral over μ_1, μ_2 it is meant to take the residues at the poles $\lambda_1,\dots,\lambda_m$ of the integrand.

In the zero temperature limit this was proved in 2005 by H. E. Boos et al. by means of the reduced qKZ equation.

Summary and further results

- Summary
  (i) Multiple integral formula for density matrix (finite T, h, thermodynamic limit performed analytically, or ground state for finite L and twist Φ)
  (ii) reduction (separation) of integrals for XXX even for finite T, h (finite L, Φ)
  (iii) Finite temperature (finite length) exponential formula ($h = 0$) for XXX

Summary and further results

- Summary
  (i) Multiple integral formula for density matrix (finite T, h, thermodynamic limit performed analytically, or ground state for finite L and twist Φ)
  (ii) reduction (separation) of integrals for XXX even for finite T, h (finite L, Φ)
  (iii) Finite temperature (finite length) exponential formula ($h = 0$) for XXX
- Further results
  (i) Multiple integral formulae for two-point function (resummation of density matrix elements) (FG, N. P. Hasenclever and A. Seel 05)
  (ii) Limiting cases, Ising and XX, analytically from the integrals (FG and A. Seel 06)
  (iii) Open XXZ chain with boundary fields (FG, M. Bortz and H. Frahm 05)
  (iv) High order high temperature expansions (Z. Tsuboi and M. Shiroishi 05)
  (v) Exponential formula for XXZ with T and h finite (H. E. Boos, FG, A. Klümper, J. Suzuki unpublished)
  (vi) Inclusion of a disorder parameter and reformulation of the exponential formula in the spirit of CFT (H. E. Boos, M. Jimbo, T. Miwa, F. A. Smirnov, Y. Takeyama)
  (vii) ‘Phenomenological disorder parameter’ for the finite temperature case (H. E. Boos, FG, A. Klümper, J. Suzuki unpublished)