Back-ultradiscretized Box and Ball system

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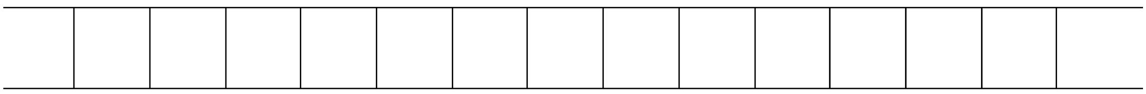
I shall talk about an extended Box and Ball system, in which the number of balls in a box are arbitrary. It could be over the capacity of a box or negative number.
I explain how to express interactions of positive balls with negative ones by introducing $\bigoplus$ operator.

Box and Ball system

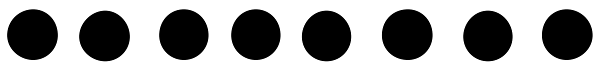
Box and ball system was found by D.Takahashi and J. Satsuma in 1990.

D.Takahashi and J.Satsuma, "A Soliton Cellular Automaton", J.Phys. Soc.Jpn.**59**(1990) 3514-3519.

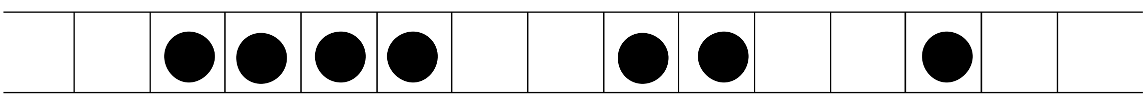
We prepare a one-dimensional array of an infinite number of boxes,



and a finite number of balls.



Each box can contain only one ball.
A distribution of balls in the boxes is given at $t = 0$.



The time evolution of the system from time t to $t+1$ is given by:

1. Move every ball only once.

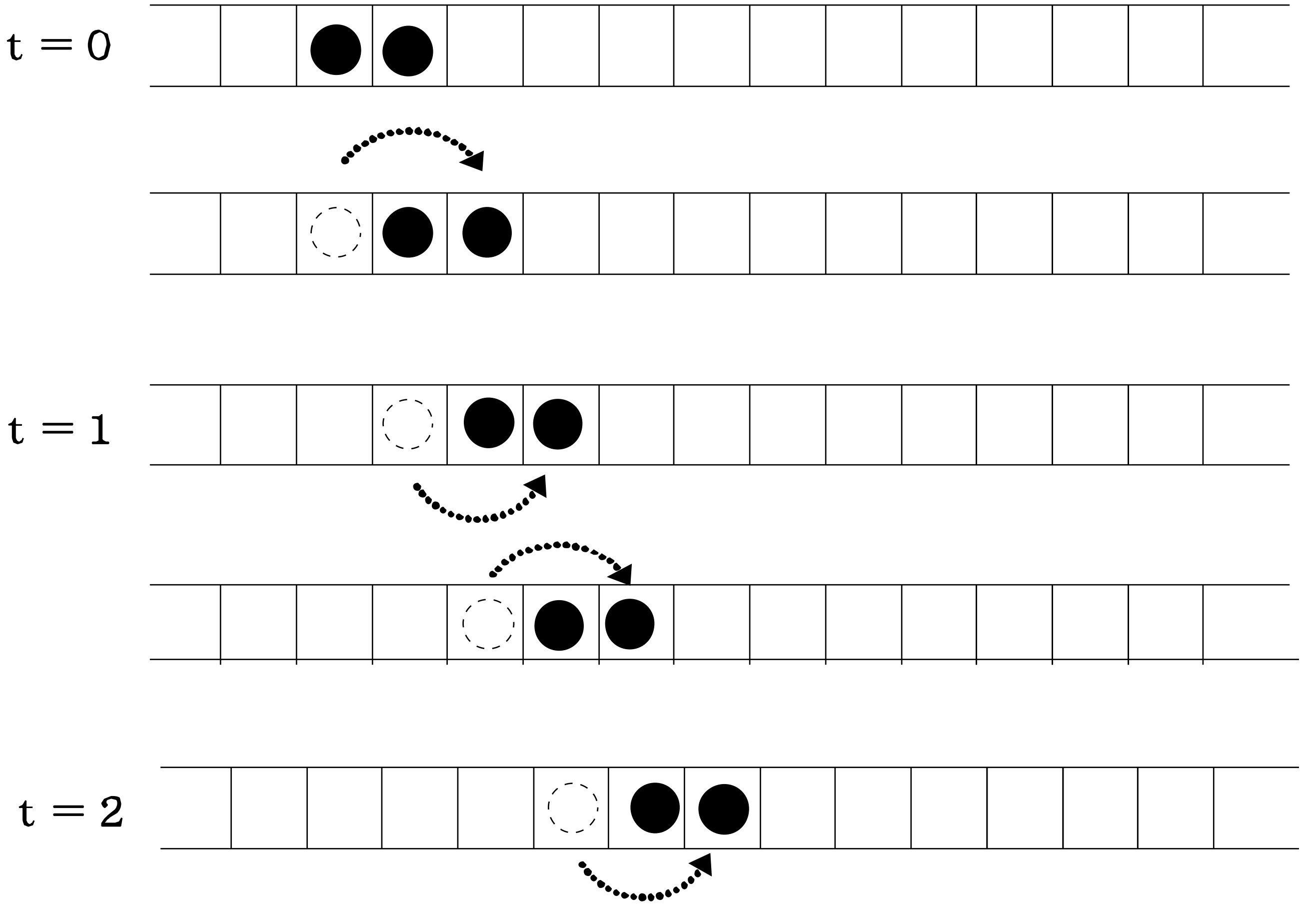
2. First, move the leftmost ball in the given distribution to its nearest right empty box.

3. Next, move the leftmost ball which is not moved yet of the new distribution to its nearest right empty box.

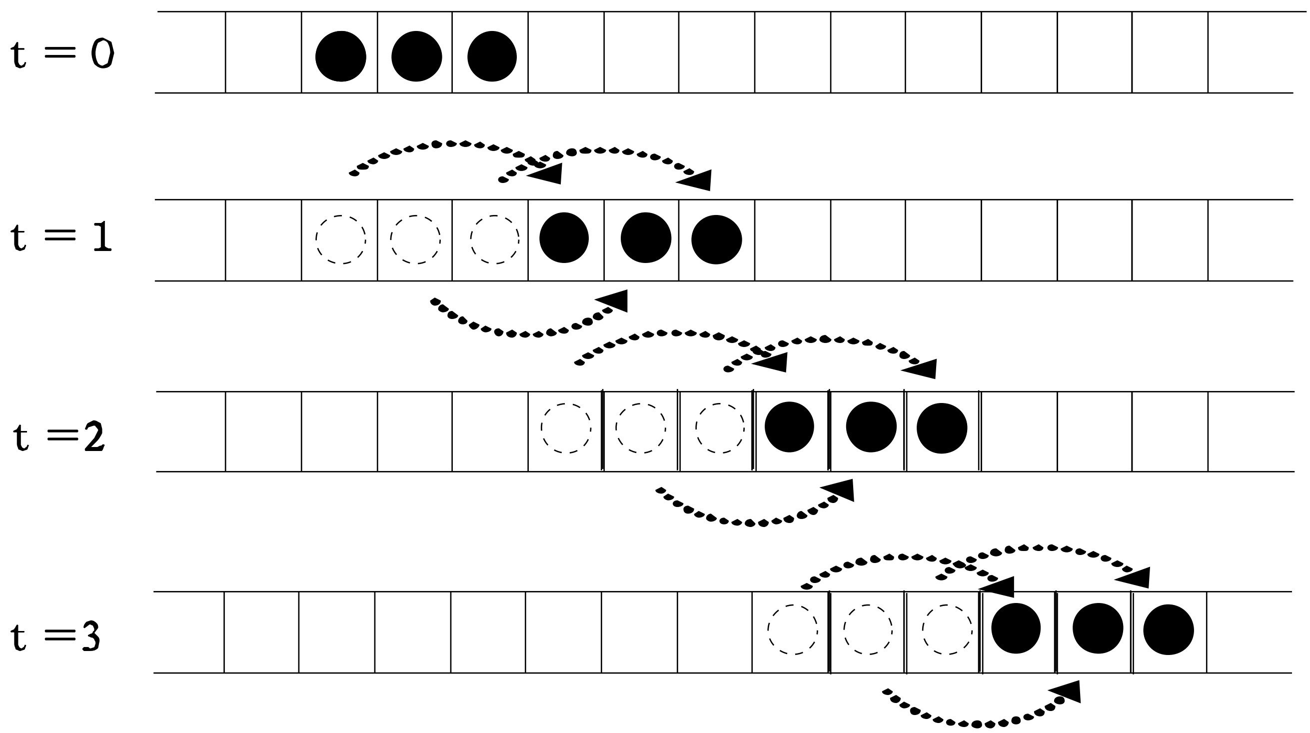
4. Repeat the above procedure until all balls are moved.

These balls behave like "Solitons";

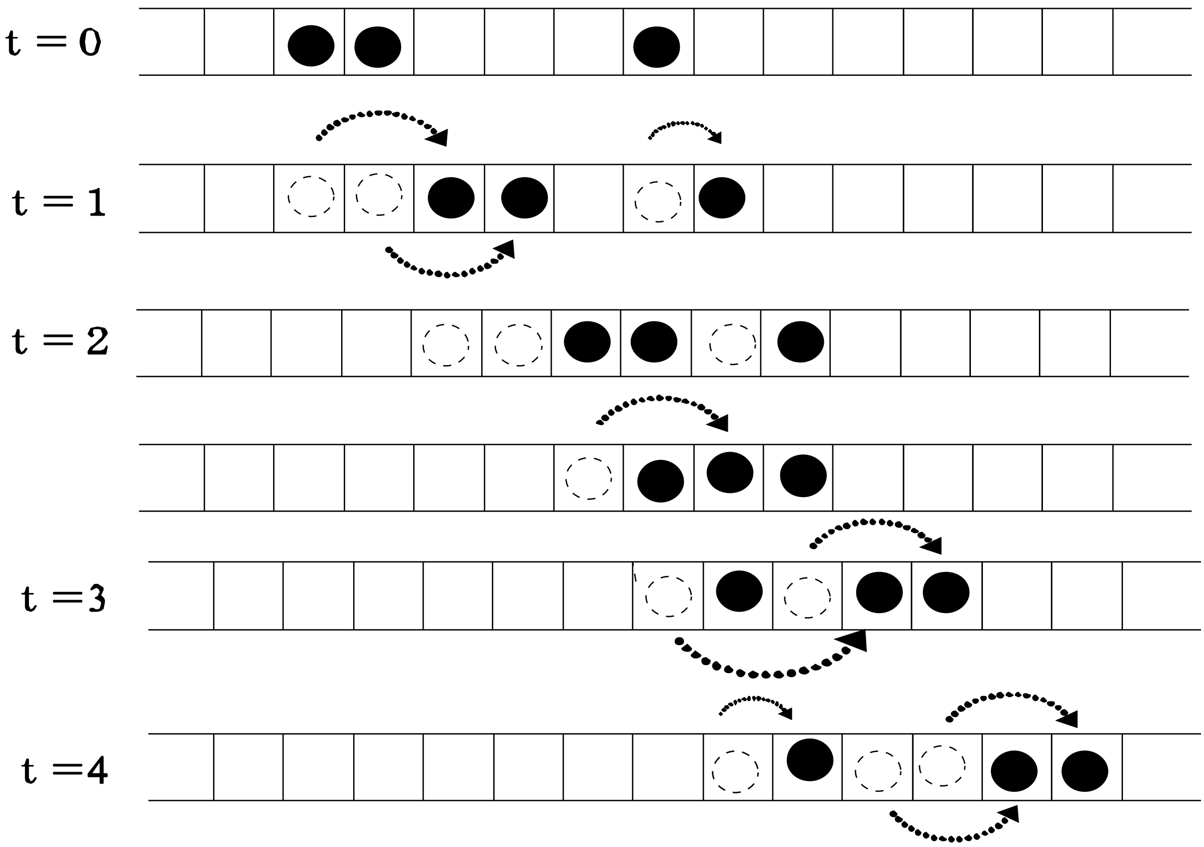
A "soliton" of length 1 moving with speed 1.



A "soliton" of length 2 moving with speed 2.



A “soliton” of length 3 moving with speed 3.



A collision of two “solitons”.

These “solitons” behave like the well-known solitons:

1. Each “soliton” moves with its own speed.

2. They preserved their identities after colliding with each other.

3. The system has an infite number of conserved quantities.

In 1996, Tokihiro *et al* found that the Box and Ball system is an ultradiscrete equation obtained, through the coordinates and dependent variable transformation,
from the discrete Lotka-Volterra equation

$$\frac{1}{\delta}(u_n^{m+1} - u_n^m) = u_n^m u_{n-1}^m - u_n^{m+1} u_{n+1}^{m+1}.$$

T. Tokihiro, D. Takahashi, J. Matsukidaira and J. Satsuma, "From Soliton Equations to Integrable Cellular Automata through a Limiting Procedure", *Phys. Rev. Lett.* **76** (1996) 3247-3250.

"Ultradiscretization" of equations is realized by transforming discrete equations into the equations described by
the *max-plus algebra*
through a limiting procedure.

For this purpose we use the following limiting procedure:
1) Max operation:

$$\lim_{\epsilon \to +0} \epsilon \log \left(\exp(\frac{X}{\epsilon}) + \exp(\frac{Y}{\epsilon}) + \exp(\frac{Z}{\epsilon}) \right)$$
$$= \max(X, Y, Z),$$

where $\max(X, Y, Z)$ gives the maximum value of X,Y,Z.

2) Plus operation:

$$\lim_{\epsilon \to +0} \epsilon \log \left(\exp(\frac{X}{\epsilon}) \ \exp(\frac{Y}{\epsilon}) \ \exp(\frac{Z}{\epsilon}) \right)$$
$$= X + Y + Z.$$

Tokihiro *et al* have used the following series of transformations.

They start with the discrete Lotka-Volterra equation,

$$\boxed{\frac{1}{\delta}(u_n^{m+1} - u_n^m) = u_n^m u_{n-1}^m - u_n^{m+1} u_{n+1}^{m+1},}$$

which is written as

$$\frac{u_n^{m+1}}{u_n^m} = \frac{1 + \delta u_{n-1}^m}{1 + \delta u_{n+1}^{m+1}}.$$

Let $u_n^m = \exp(U_n^m/\epsilon), \delta = \exp(-1/\epsilon)$.

Then the equation is ultradiscretized,

$$U_n^{m+1} - U_n^m = \max(0, U_{n-1}^m - 1) - \max(0, U_{n+1}^{m+1} - 1).$$

By the coordinates transformation,

$$U^n_{n-m} = V^m_n,$$

we obtain

$$V^{m+1}_{n+1} - V^m_n = \max(0, V^{m+1}_n - 1) - \max(0, V^m_{n+1} - 1),$$

which is transformed into

$$S^{m+1}_{n+1} - S^m_n = \min(0, 1 - S^m_{n+1} + S^{m+1}_n),$$

by the dependent variable transformation

$$V^m_n = S^m_{n+1} - S^{m+1}_n,$$

which is further transformed into

$$\boxed{B_n^{m+1} = \min(1 - B_n^m, \sum_{k=-\infty}^{n-1}(B_k^m - B_k^{m+1})),}$$

by the dependent variable transformation

$$S_n^m = \sum_{k=-\infty}^{n} B_k^m,$$

The last equation of $\boxed{B_n^m}$,which is called "BB equation", describes

the Box and Ball system.

We note that the capacity of the box is unity ($\delta = \exp(-1/\epsilon)$).

Accordingly $\boxed{B_n^m}$, a number of balles in a box is limited to be 0 or 1.

Is there any discrete equation which gives the BB equation *directly* in the ultradiscrete limit without any transformation of variables and coordinates ?

?	$\longrightarrow$	BB equation

We propose the following discrete equation,

$$\frac{1}{u_{n+1}^{m+1}} - \frac{1}{u_n^m} = \delta(u_{n+1}^m - u_n^{m+1})$$

with the boundary condition,

$$\lim_{|n|\to\infty} u_n^m = 1.$$

We call it "BBB equation" (Back-ultradiscretized BB equation).

BBB equation cannot be ultradiscretized in the explicit form because of the difficulty of "negative terms" in the equation.

We use a method described in the following paper in order to overcome the difficulty.

Satoshi Tujimoto and Ryogo Hirota,
"Ultradiscrete KdV Equation",
J. Phys.Soc.Jpn.**67**,(1998),1809-1810,

We rewrite BBB equation as

$$\frac{1-\delta u_{n+1}^{m} u_{n+1}^{m+1}}{u_{n+1}^{m+1}} = \frac{1-\delta u_{n}^{m} u_{n}^{m+1}}{u_{n}^{m}},$$

which is expressed by

$$u_{n+1}^{m+1} z_{n+1}^{m+1} = u_{n}^{m} z_{n}^{m+1}, \tag{3}$$

introducing an auxiliary dependent varible z_n^m,

$$z_n^m = \frac{1}{1-\delta u_n^{m-1} u_n^{m-1}}. \tag{4}$$

We rewrite Eq.(4),

$$z_n^m = 1 + \delta u_n^{m-1} u_n^m z_n^m,$$

which is expressed using Eq.(3) by

$$z_n^m = 1 + \delta u_n^{m-1} u_{n-1}^{m-1} z_{n-1}^m,$$

Hence BBB equation is transformed into a coupled equations of u_n^m and z_n^m,

$$\begin{cases} z_n^m = 1 + \delta u_n^{m-1} u_{n-1}^{m-1} z_{n-1}^m, \\ u_n^m z_n^m = u_{n-1}^{m-1} z_{n-1}^m, \end{cases}$$

which is ultradiscretized,

$$Z_n^m = \max(0, U_n^{m-1} + U_{n-1}^{m-1} + Z_{n-1}^m - 1), \tag{5}$$

$$U_n^m + Z_n^m = U_{n-1}^{m-1} + Z_{n-1}^m, \tag{6}$$

with the boundary condition $U_{|\infty|}^m = Z_{|\infty|}^m = 0$.

We show that Eq.(5) and Eq.(6) are transformed into the BB equation by eliminating the variable Z_n^m.

Subtracting $U_n^m + Z_n^m$ from Eq.(5) and using Eq.(6),we obtain

$$-U_n^m = \max(-U_n^m - Z_n^m, U_n^{m-1} - 1),$$

which is written as

$$U_n^m = \min(1 - U_n^{m-1}, U_n^m + Z_n^m). \qquad (7)$$

On the other hand we have from Eq.(6),

$$Z_k^m - Z_{k-1}^m = U_{k-1}^{m-1} - U_k^m. \qquad (8)$$

Adding Eq.(8) from $k = -\infty$ to n and using the boundary condition $Z_{-\infty}^m = 0$, we obtain

$$Z_n^m = U_{n-1}^{m-1} - U_n^m + \sum_{k=-\infty}^{n-1} (U_{k-1}^{m-1} - U_k^m)$$

$$= -U_n^m + \sum_{k=-\infty}^{n-1} (U_k^{m-1} - U_k^m). \qquad (9)$$

Substituting Eq.(9) into Eq.(7) we obtain

$$U_n^m = \max(1 - U_n^{m-1}, \sum_{k=-\infty}^{n-1} (U_k^{m-1} - U_k^m)) \tag{10}$$

which is the BB equation.

Thus we have obtained the BB equation without a transformation of variables and coordinates.

BBB equation

$$\frac{1}{u_{n+1}^{m+1}} - \frac{1}{u_n^m} = \delta(u_{n+1}^m - u_n^{m+1})$$

is transformed into two types of bilinear equation,

$$f_{n-1}^m f_{n+1}^{m+1} + \delta f_{n+1}^m f_{n-1}^{m+1} - (1+\delta) f_n^m f_n^{m+1} = 0, \text{(i)}$$

and

$$f_n^{m-1} f_{n+1}^{m+1} - \delta f_n^{m+1} f_{n+1}^m - (1-\delta) f_n^m f_{n+1}^m = 0, \text{(ii)}$$

by the dependent variable transformation

$$u_n^m = \frac{f_{n+1}^m f_n^{m+1}}{f_n^m f_{n+1}^{m+1}}. \qquad (11)$$

Note that Eqs.(i) and (ii) are interchanged by the transformation,

$$m \quad \rightleftarrows \quad n \quad \text{and} \quad \delta \to -\delta$$

BBB equation

$$\frac{1}{u_{n+1}^{m+1}} - \frac{1}{u_n^m} = \delta(u_{n+1}^m - u_n^{m+1})$$

is invariant under the transformations,

$$m \quad \rightleftarrows \quad n \quad \text{and} \quad \delta \to -\delta, \quad \text{(a)}$$

and

$$u_n^m \to \frac{1}{\delta u_n^m} \quad \text{and} \quad n \to -n \quad \text{(b)}.$$

We show (b). Transforming $u_n^m \to \frac{1}{\delta u_n^m}$ into BBB equation we have

$$\delta(u_{n+1}^{m+1} - u_n^m) = (\frac{1}{u_{n+1}^m} - \frac{1}{u_n^{m+1}}),$$

which is written as

$$\delta(u_{n-1}^{m+1} - u_n^m) = (\frac{1}{u_{n-1}^m} - \frac{1}{u_n^{m+1}}),$$

by transforming $n \to -n$. Shifting n by 1 we obtain BBB equation,

$$\frac{1}{u_{n+1}^{m+1}} - \frac{1}{u_n^m} = \delta(u_{n+1}^m - u_n^{m+1}).$$

Mapping of BBB equation under periodic boundary condition.

We consider BBB equation

$$z_n^m = 1 + \delta u_n^{m-1} u_{n-1}^{m-1} z_{n-1}^m, \qquad (12)$$

$$u_n^m z_n^m = u_{n-1}^{m-1} z_{n-1}^m, \qquad (13)$$

under periodic boundary condition,

$$u_{n+N}^m = u_n^m, \quad z_{n+N}^m = z_n^m.$$

Let $a_n = \delta u_n^{m-1} u_{n-1}^{m-1}$. Then Eq.(12) reads

$z_n - a_n z_{n-1} = 1$, for $n = 0, 1, 2, \cdots, N-1$,

where we put $z_n^m = z_n$ for short,

Equation(12) is expressed by a matrix form,

$$\mathbf{A}\mathbf{Z} = \mathbf{C}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & -a_0 \\ -a_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -a_2 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -a_{N-1} & 1 \end{pmatrix},$$

$$\mathbf{Z} = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \dots \\ z_{N-2} \\ z_{N-1} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \dots \\ 1 \\ 1 \end{pmatrix}.$$

Solutions to the linear equation are given by

$$z_n = \frac{1}{\Delta}\left[1 + \sum_{k=1}^{N-1}\left(\prod_{s=0}^{k-1} a_{n-s}\right)\right],$$

$$\text{for } n = 0, 1, 2, \cdots, N-1,$$

where $\Delta = 1 - \prod_{s=0}^{N-1} a_s$.

Substituting z_n into Eq.(13),we obtain an expression for u_n^m,

$$u_n^m = u_{n-1}^{m-1} \frac{\left[1 + \sum_{k=1}^{N-1}\left(\prod_{s=0}^{k-1} a_{n-1-s}^m\right)\right]}{\left[1 + \sum_{k=1}^{N-1}\left(\prod_{s=0}^{k-1} a_{n-s}^m\right)\right]}, \quad (14)$$

($a_n^m = \delta u_n^{m-1} u_{n-1}^{m-1}$), which gives an explicit mapping of u_n^m under periodic boundary conditions.

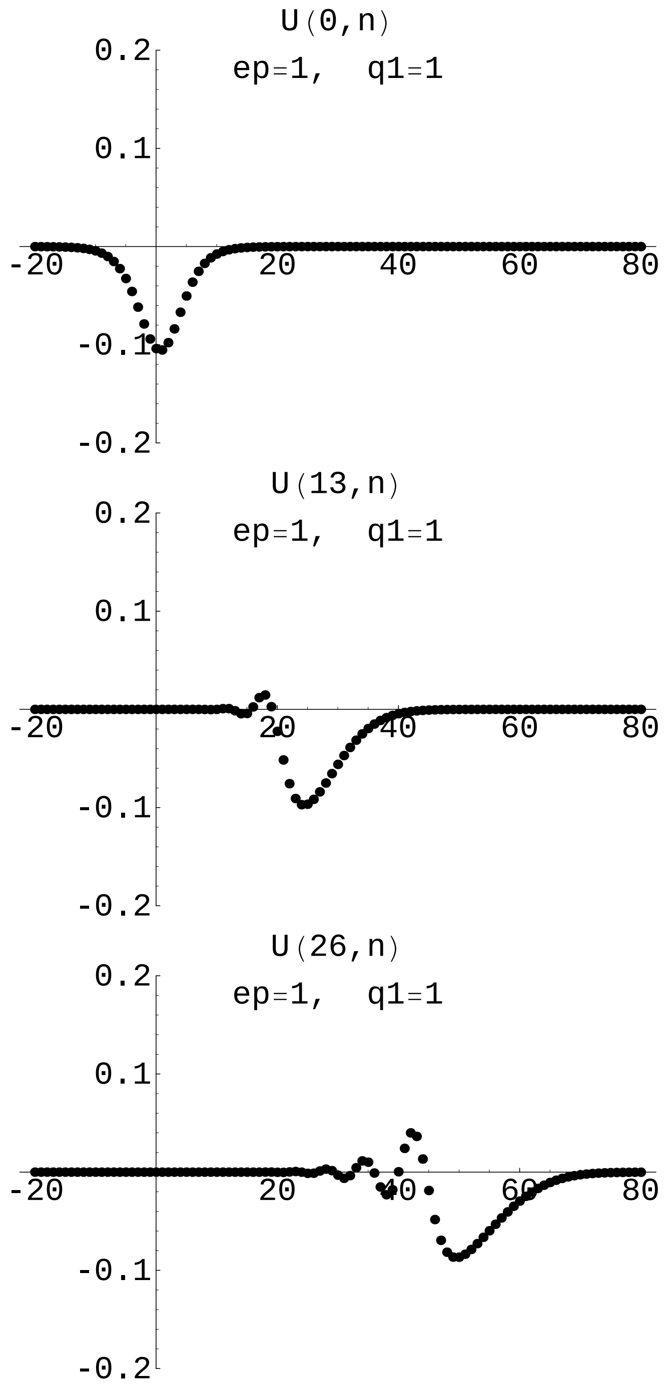
Mapping of ultradiscrete BBB equation under periodic boundary condition.

Equation (14) is easily ultradiscretized,

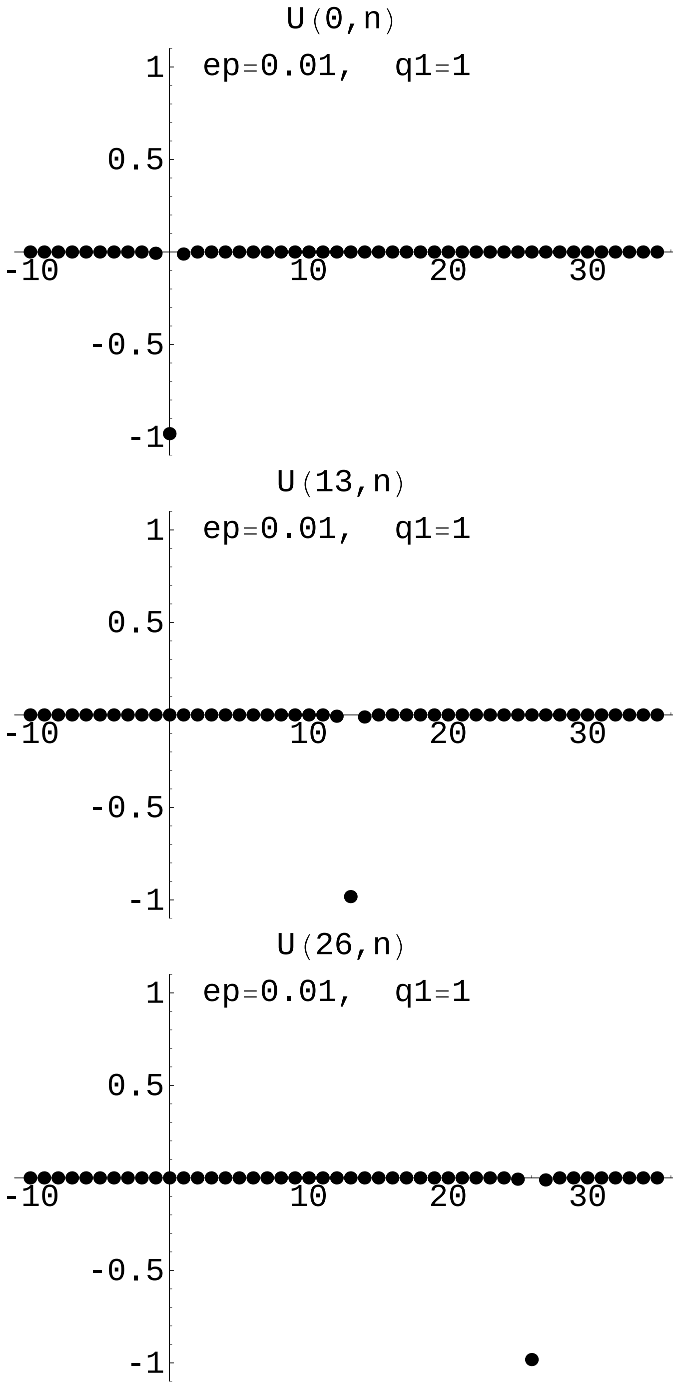
$$U_n^m = U_{n-1}^{m-1} + \max_{k=0}^{N-1}\left(\sum_{s=0}^{k-1} A_{n-1-s}^m\right) - \max_{k=0}^{N-1}\left(\sum_{s=0}^{k-1} A_{n-s}^m\right),$$

where $A_n^m = U_n^{m-1} + U_{n-1}^{m-1} - 1$.

We remark that ultradiscrete BBB system is equal to the Box and Ball system.
However we consider that the number of balls in a box in ultradiscrete BBB system is arbitrary.
It could be over the capacity of the box or a nagative number.



Numerical solutions to BBB equation shows a hole is not a stable entity. It is stable only in the ultradiscrete limit, $\epsilon = 0$.


U(0,n)
ep=0.01, q1=1
U(13,n)
ep=0.01, q1=1
U(26,n)
ep=0.01, q1=1
1
0.5
-0.5
-1
-10
10
20
30

Time development of ultradiscrete BBB system under periodic boundary condition

Initial state in an extended Box and Ball system:" 3 balls" in a box.

```
{0,0, 3,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

{0,0,-2,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0}

{0,0,0,-2,0,0,0,0,1,1,1,1,1,0,0,0,0,0,0}

{0,0,0,0,-2,0,0,0,0,0,0,0,0,1,1,1,1,1,0}

{1,1,1,1,0,-2,0,0,0,0,0,0,0,0,0,0,0,0,1}

{0,0,0,0,1,3,-1,0,0,0,0,0,0,0,0,0,0,0,0}

{0,0,0,0,0,-2,2,1,1,1,0,0,0,0,0,0,0,0,0}

{0,0,0,0,0,0,-2,0,0,0,1,1,1,1,1,0,0,0,0}

{1,0,0,0,0,0,0,-2,0,0,0,0,0,0,0,1,1,1,1}
```

{0,1,1,1,1,1,0,0,-2,0,0,0,0,0,0,0,0,0,0}

{0,0,0,0,0,0,1,1,3,-2,0,0,0,0,0,0,0,0,0}

{0,0,0,0,0,0,0,0,-2,3,1,1,0,0,0,0,0,0,0}

{0,0,0,0,0,0,0,0,0,-2,0,0,1,1,1,1,1,0,0}

{1,1,1,0,0,0,0,0,0,0,-2,0,0,0,0,0,0,1,1}

{0,0,0,1,1,1,1,1,0,0,0,-2,0,0,0,0,0,0,0}

{0,0,0,0,0,0,0,0,1,1,1,2,-2,0,0,0,0,0,0}

{0,0,0,0,0,0,0,0,0,0,0,-1,3,1,0,0,0,0,0}

{0,0,0,0,0,0,0,0,0,0,0,0,-2,0,1,1,1,1,1}

```
{1,1,1,1,1,0,0,0,0,0,0,0,0,-2,0,0,0,0,0}

{0,0,0,0,0,1,1,1,1,1,0,0,0,0,-2,0,0,0,0}

{0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,-2,0,0,0}

{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0, 3,0,0,0}

{1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,1,1,1}
```

Numerical simulation shows that the fundamental interaction is that of a soliton of length 5 with a hole of depth 2. We show later the interaction is expressed by $\tau(1,2)$,

N-soliton solution to BBB equation

We have the bilinear form of BBB equation,

$$f_{n-1}^{m} f_{n+1}^{m+1} + \delta f_{n+1}^{m} f_{n-1}^{m+1} - (1+\delta) f_{n}^{m} f_{n}^{m+1} = 0. \text{(i)}$$

2-soliton solution is expressed by

$$\begin{aligned} f_n^m(2) = 1 + \exp(s_1) + \exp(s_2) \\ + \exp(s_1 + s_2 + c_{1,2}), \end{aligned}$$

where

$$\begin{aligned} s_j &= -(p_n + q_j m + a_j), \\ q_j &= \frac{1 + q_j \delta}{q_j + \delta} \quad \text{for } j = 1, 2, \\ c_{1,2} &= \frac{(q_1 - q_2)^2}{(q_1 q_2 - 1)^2}. \end{aligned}$$

In general N-soliton is expressed by

$$f_n^m(N) = \sum \exp[\sum_{i=1}^{N} \mu_i s_i + \sum_{i<j}^{(N)} \mu_i \mu_j c_{i,j}]$$

where the first $\sum$ means a summation over all possible combinations of $\mu_i = 0, 1, \mu_2 = 0, 1, \ldots, \mu_N = 0, 1$, and $\sum_{i<j}^{(N)}$ means a summation over all possible pairs (i, j) chosen from the set $\{1, 2, 3, \ldots, N\}$,with the condition that $i < j$.

N-soliton solution to ultradiscretized BBB equation

Let the ultradiscrete form of f_n^m be τ_n^m and

$$
\begin{aligned}
s_i &= -(p_i n + q_i m + a_i), \\
p_i &= \max(0, q_i - 1) - \max(q_i, -1) \\
&\qquad\qquad \text{for } i = 1, 2, \\
c_{1,2} &= |q_1 - q_2| - |q_1 + q_2|,
\end{aligned}
$$

q_i and a_i for $i = 1, 2$ are integers.

Then 2-soliton solution in the ultradiscrete limit is expressed by

$$\tau(2,0) = \max(0, s_1, s_2, s_1 + s_2 + c_{1,2}),$$

We may express $\tau(2,0)$ as

$$\tau(2,0) = \mathrm{Max}_\mu(\mu_1 s_1 + \mu_2 s_2 + \mu_1 \mu_2 c_{1,2},),$$

where Max_μ means a maximum element of a set of all possible combinations under $\mu_1 = 0, 1, \mu_2 = 0, 1$.

In general ultradiscrete N-soliton solution is expressed by

$$\tau(N,0) = \mathrm{Max}_\mu(\sum_{i=1}^{N} \mu_i s_i + \sum_{i<j}^{(N)} \mu_i \mu_j c_{i,j},)$$

where Max_μ means a maximum element of a set of all possible combinations under $\mu_1 = 0,1, \mu_2 = 0,1, \ldots, \mu_N = 0,1$.

Hole solution to ultradiscrete BBB equation

Numerical solutions to BBB equation show that "holes" are not stable entities. They are stable only in the ultradiscrete limit.

τ-functions of ultradiscrete 1-hole solution is expressed by

$$\tau(0,1) = \min(0, h_1),$$

where

$$h_1 = m - n + b_1,$$

where given parameter b_1 is integer.

N-hole solution

τ-function of ultradiscrete N-hole solution is given by

$$
\begin{aligned}
\tau(0,N) = \sum_{i=1}^{N} \min(0,h_i) \\
&= \min(0,\nu_1,\nu_2,\ldots,\nu_N \\
&\quad ,\nu_1+\nu_2,\nu_1+\nu_3,\ldots,\nu_1+\nu_N \\
&\quad ,\ldots,\ldots, \\
&\quad ,\nu_1+\nu_2+\ldots+\nu_N) \\
&= \mathrm{Min}_\nu(\sum_{i=1}^{N} \nu_i h_i,),
\end{aligned}
$$

where

$$h_i = m - n + b_i, \quad \text{for } i = 1,2,\cdots,N,$$

where Min_ν means a minimum element of a set of all combinations under $\nu_1 = 0,1, \nu_2 = 0,1,\cdots,\nu_N = 0,1$.

M-soliton plus N-hole solution

In order to describe interaction of ultradiscrete solitons with holes, we introduce a new operator $\bigoplus$ acting on special forms of $s_i (i = 1, 2, \cdots, M)$ and of $h_j (j = 1, 2, \cdots, N)$, defined by

$$(\sum_{i=1}^{M} s_i + \sum_{i<j}^{(M)} c_{i,j}) \bigoplus (\sum_{k=1}^{N} h_k)$$
$$= \sum_{i=1}^{M} s_i + \sum_{i<j}^{(M)} c_{i,j} + \sum_{k=1}^{N} h_k + 2MN,$$

for natural numbers M and N.

We have for examples,

$$s_i \bigoplus h_j = s_i + h_j + 2,$$

and

$$(s_1 + s_2 + c_{1,2}) \bigoplus (h_1 + h_2 + h_3)$$
$$= s_1 + s_2 + c_{1,2} + h_1 + h_2 + h_3 + 12.$$

In general we have

$$
\begin{aligned}
&(\sum_{i=1}^{M} \mu_i s_i + \sum_{i<j}^{(M)} \mu_i \mu_j c_{i,j}) \bigoplus (\sum_{k=1}^{N} \nu_k h_k) \\
&\quad = \sum_{i=1}^{M} \mu_i s_i + \sum_{i<j}^{(M)} \mu_i \mu_j c_{i,j} + \sum_{k=1}^{N} \nu_k h_k \\
&\quad +2(\sum_{i=1}^{M} \mu_i)(\sum_{k=1}^{N} \nu_k),
\end{aligned}
$$

for all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \cdots, \mu_M = 0, 1$ and
$\nu_1 = 0, 1, \nu_2 = 0, 1, \cdots, \nu_N = 0, 1$.

For other cases we have

$$h_j \bigoplus h_k = h_j + h_k,$$

$$x \bigoplus y = x + y.$$

The operator $\bigoplus$ satifies the basic rules of the max-plus algebra,

$$\begin{aligned}\max(x,y)\bigoplus z &= z\bigoplus \max(x,y)\\ &= \max(z\bigoplus x, z\bigoplus y),\\ \min(x,y)\bigoplus z &= z\bigoplus \min(x,y)\\ &= \min(z\bigoplus x, z\bigoplus y).\end{aligned}$$

We find for examples,

$$\begin{aligned}\max(0,s_i)\bigoplus h_j &= \max(0\bigoplus h_j, s_i\bigoplus h_j)\\ &= \max(h_j, s_i+h_j+2)\end{aligned}$$

and

$$\begin{aligned}&\max(0,s_i)\bigoplus \min(0,h_j)\\ &= \max(0\bigoplus \min(0,h_j), s_i\bigoplus \min(0,h_j))\\ &= \max(\min(0\bigoplus 0, 0\bigoplus h_j), \min(s_i\bigoplus 0, s_i\bigoplus h_j))\\ &= \max(\min(0,h_j), \min(s_i, s_i+h_j+2)).\end{aligned}$$

We assume that $\tau(1,1)$ describing an interaction of a soliton with a hole in the ultradiscrete limit is expressed by,

$$
\begin{aligned}
\tau(1,1) &= \tau(1,0) \bigoplus \tau(0,1) \\
&= \max(0, s_i) \bigoplus \min(0, h_j) \\
&= \max(0 \bigoplus \min(0, h_j), s_i \bigoplus \min(0, h_j)) \\
&= \max(\min(0 \bigoplus 0, 0 \bigoplus h_j) \\
&\quad , \min(s_i \bigoplus 0, s_i \bigoplus h_j)) \\
&= \max(\min(0, h_j), \min(s_i, s_i + h_j + 2)) \\
&= \mathrm{Max}_\mu(\min(\mu_i s_i, \mu_i s_i + h_j + 2\mu_i),) \\
&= \mathrm{Max}_\mu(\mathrm{Min}_\nu(u_i s_i + \nu_j h_j + 2\mu_i \nu_j,),).
\end{aligned}
$$

Hence

$$\tau(1,1) = \mathrm{Max}_\mu(\mathrm{Min}_\nu(u_i s_i + \nu_j h_j + 2\mu_i \nu_j,),).$$

Similarly we have τ-function,$\tau(1,2)$ describing an interaction of 1-soliton with 2-hole is expressed by,

$$\begin{aligned}
&\tau(1,2) = \tau(1,0)\bigoplus\tau(0,2)\\
&= \max(0, s_1)\bigoplus \min(0, h_1, h_2, h_1 + h_2)\\
&= \max(0\bigoplus \min(0, h_1, h_2, h_1 + h_2)\\
&\quad , s_1\bigoplus \min(0, h_1, h_2, h_1 + h_2)),\\
&= \max(\min(0, h_1, h_2, h_1 + h_2)\\
&\quad , (\min(s_1, s_1\bigoplus h_1, s_1\bigoplus h_2, s_1\bigoplus h_1 + h_2))\\
&= \max(\min(0, h_1, h_2, h_1 + h_2)\\
&\quad , \min(s_1, s_1 + h_1 + 2, s_1 + h_2 + 2\\
&\qquad\qquad , s_1 + h_1 + h_2 + 4).
\end{aligned}$$

Hence

$$\begin{aligned}
\tau(1,2) = \mathrm{Max}_\mu(\mathrm{Min}_\nu(\mu_1 s_1 + \nu_1 h_1 + \nu_1 h_2\\
+2\mu_1(\nu_1 + \nu_2),),).
\end{aligned}$$

In general, we conjecture that $\tau(M,N)$ describing interactions of M-soliton with N-hole in the ultradiscrete limit is expressed by

$$
\begin{aligned}
\tau(M,N) &= \tau(M,0) \bigoplus \tau(0,N) \\
&= \mathrm{Max}_{\mu}\Big(\sum_{i=1}^{M} \mu_i s_i + \sum_{i<j}^{(M)} \mu_i \mu_j c_{i,j},\Big) \\
&\quad \bigoplus \mathrm{Min}_{\nu}\Big(\sum_{i=1}^{N} \nu_i h_i,\Big) \\
&= \mathrm{Max}_{\mu}\Big(\mathrm{Min}_{\nu}\Big(\\
&\quad \sum_{i=1}^{M} \mu_i s_i + \sum_{i<j}^{(M)} \mu_i \mu_j c_{i,j} + \sum_{k=1}^{N} \nu_k h_k \\
&\quad +2\Big(\sum_{i=1}^{M} \mu_i\Big)\Big(\sum_{k=1}^{N} \nu_i\Big),\Big),\Big).
\end{aligned}
$$

summary

1. A back-ultradiscretization of the Box and Ball system is obtained ("BBB equation).

   Explicit mapping of BBB equation is obtained under periodic boundary condition.

2. An extended box and ball system is obtained by ultradiscretizing BBB equation, in which the number of balls in a box is arbitrary, It could be over the capacity of a box or negative number.

3. Collisions of positive balls with negative-balls are investigated in detail and an explicit form describing the collisions are obtained.