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Perspectives of Soliton Physics
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Vortex Motion and Soliton

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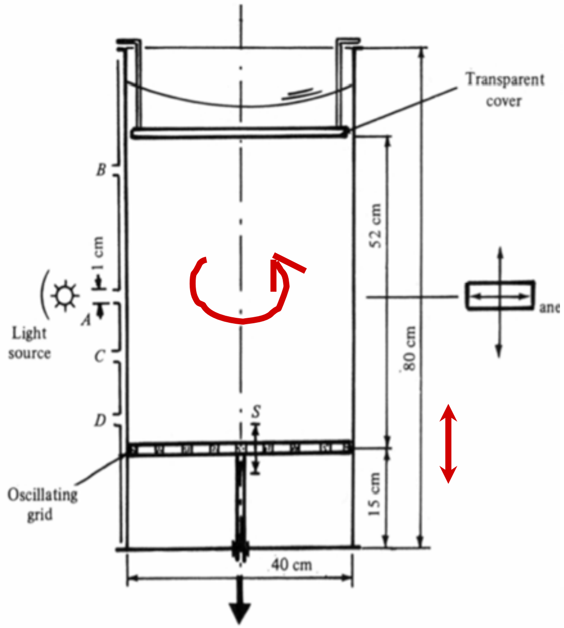
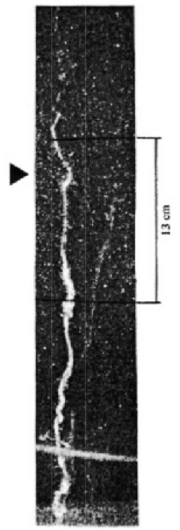
Vortex Soliton





Experiment

Turbulence and waves in a rotating tank

Hopfinger, E.J., Browand, F. & Gagne, Y.

J. Fluid Mech. (1982) **125**, pp 505-534.

Theory

A soliton on a vortex filament

Hasimoto, H.

J.Fluid Mech. (1972) **51**, pp 477-485.

$$\frac{\partial \mathbf{X}}{\partial t} = G \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial^2 \mathbf{X}}{\partial s^2}$$

$\mathbf{X} = (X_f, Y_f, Z_f)$

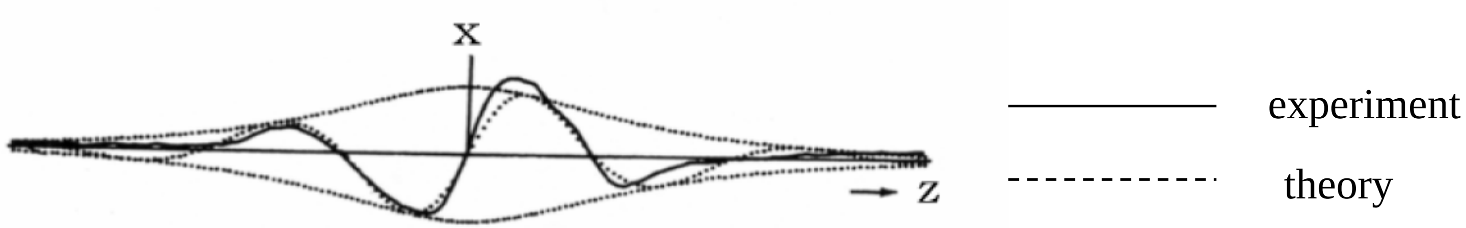
s : arc length

G : self - induction

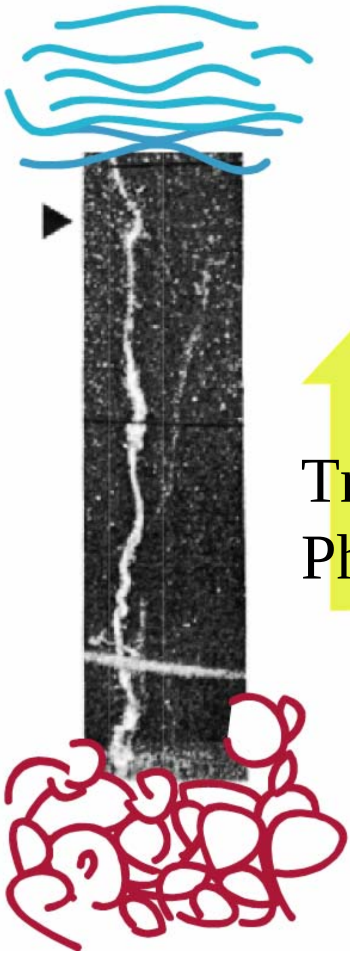
$$X_f(s,t) = \frac{2\mu}{\nu} \operatorname{sech}\left[\nu(s - 2\tau G t)\right] \cos\left[\tau(s - 2\tau G t) + (\nu^2 + \tau^2) G t\right]$$

$$Y_f(s,t) = \frac{2\mu}{\nu} \operatorname{sech}\left[\nu(s - 2\tau G t)\right] \sin\left[\tau(s - 2\tau G t) + (\nu^2 + \tau^2) G t\right]$$

$$Z_f(s,t) = S - \frac{2\mu}{\nu} \tanh\left[\nu(s - 2\tau G t)\right]$$



Transport by a vortex soliton



Laminar region

Transport of Physical quantities

Turbulent region

….. The nonlinear waves transport mass, momentum and energy from the vigorously turbulent region near the grid to the rotation-dominated flow above. ……

(excerpt from the abstract of the J.F.M paper by Hopfinger, Browand & Gagne.)

↓

Transport properties of waves on a vortex filament
Y. Kimura
Physica D (1989**) 37**, 485-489

Impulse & angular momentum

Hasimoto vortex soliton

$$\frac{\partial \mathbf{X}}{\partial t} = \frac{\Gamma}{4\pi}\log\left(\frac{\mathrm{L}}{\varepsilon}\right)\left(\frac{\partial \mathbf{X}}{\partial s}\times\frac{\partial^2 \mathbf{X}}{\partial s^2}\right)$$ **(Localized Induction Equation)**

G : self-induction constant

→ Determines the time scale

$$X_{\mathrm{f}}(s,\mathrm{t}) = \frac{2\mu}{\nu}\ \mathrm{sech}\left[\nu(s-2\tau G\mathrm{t})\right]\ \cos\left[\tau(s-2\tau G\mathrm{t})+(\nu^2+\tau^2)G\mathrm{t}\right]$$

$$Y_{\mathrm{f}}(s,\mathrm{t}) = \frac{2\mu}{\nu}\ \mathrm{sech}\left[\nu(s-2\tau G\mathrm{t})\right]\ \sin\left[\tau(s-2\tau G\mathrm{t})+(\nu^2+\tau^2)G\mathrm{t}\right]$$

$$Z_{\mathrm{f}}(s,\mathrm{t}) = S - \frac{2\mu}{\nu}\ \tanh\left[\nu(s-2\tau G\mathrm{t})\right]$$

Three parameters: (ν,τ) Shape of the vortex soliton

G Strength of the vortex soliton

linear and angular impulse

linear impulse

$$\mathbf{P} = \frac{1}{2}\int \mathbf{X}(s,t)\times\vec{\omega}\ dV = \frac{\Gamma}{2}\int \mathbf{X}(s,t)\times \mathbf{t}(s,t)\ dV$$

position vec. vorticity $(\vec{\omega} = \Gamma\ \mathbf{t})$ tangential vec.

angular impulse

$$\mathbf{M} = \frac{1}{3}\int \mathbf{X}\times(\mathbf{X}\times\vec{\omega})\ dV = \frac{\Gamma}{3}\int \mathbf{X}\times(\mathbf{X}\times \mathbf{t})\ dV$$

$$\mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 4\Gamma\sigma\nu\tau_0/(\nu^2+\tau_0^2)^2 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 0 \\ 0 \\ -8\Gamma\sigma\nu\tau_0^2/3(\nu^2+\tau_0^2)^3 \end{pmatrix}$$

They are localized around the kink part ! → transported

Sound emitted by a vortex soliton

(Lighthill's theory on vortex sound)

$$p = \frac{M^2 x_i x_j}{4\pi r^3} \dddot{Q}_{ij}(t - Mr)$$

Sound pressure in the far field

Where $$Q_{ij} = \frac{1}{3}\int X_i \times (\mathbf{X} \times \vec{\omega})_j \, dV$$

$$p \propto \frac{1}{r} \operatorname{sech}\frac{\pi\tau_0}{2\nu} \sin 2\theta \sin(\omega t^* - \phi) + \frac{4}{r} \operatorname{cosech}\frac{\pi\tau_0}{\nu} \sin^2\theta \cos 2(\omega t^* - \phi)$$

Two types of rotating quadrapole

Size estimate for the mass transport

$$\left.|\mathbf{M}|\middle/|\mathbf{P}|\right. \approx \frac{\tau_0}{\left(\nu^2 + \tau_0^2\right)} \approx \frac{1}{\tau_0} \quad (\tau_0 >> \nu)$$

$$\approx \frac{1}{\nu} \quad (\tau_0 \sim \nu)$$

For more precise structure and size, we needed to investigate the flow field around a vortex soliton

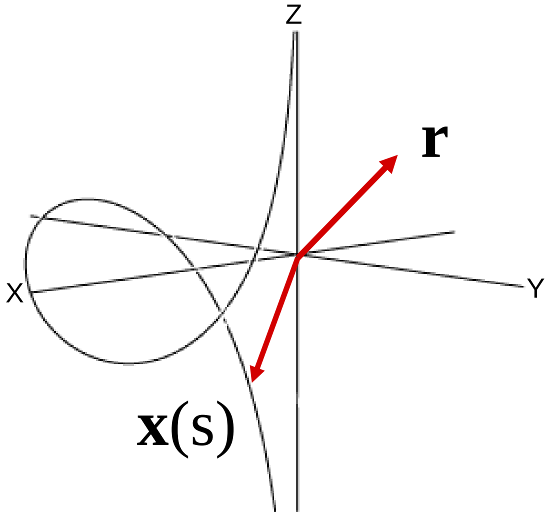
Dynamical system for the particle motion

$$\frac{d\mathbf{r}}{dt} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\mathbf{x}'(s)\times(\mathbf{r}-\mathbf{x}(s))}{|\mathbf{r}-\mathbf{x}(s)|^3}\,ds + \omega\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2\tau G \end{bmatrix}$$

Bio-Savart integral rotation translation

Moving frame (translation+rotation) in which the soliton is stationary

cf. A vortex filament moving without change of form
S. Kida, J.Fluid Mech. (1981) **112**, pp397-409



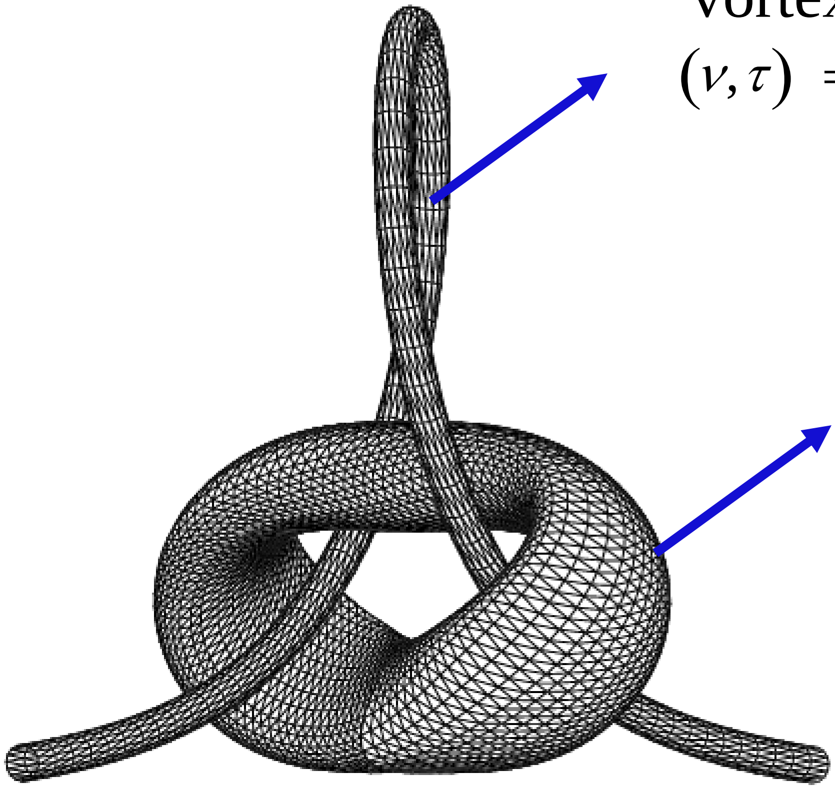
$$\omega = \left(\nu^2 + \tau^2\right) G$$

angular velocity

$\kappa = 2\nu\ \mathrm{sech}(\nu s)$: curvature

τ: torsion

Vortex soliton and torus



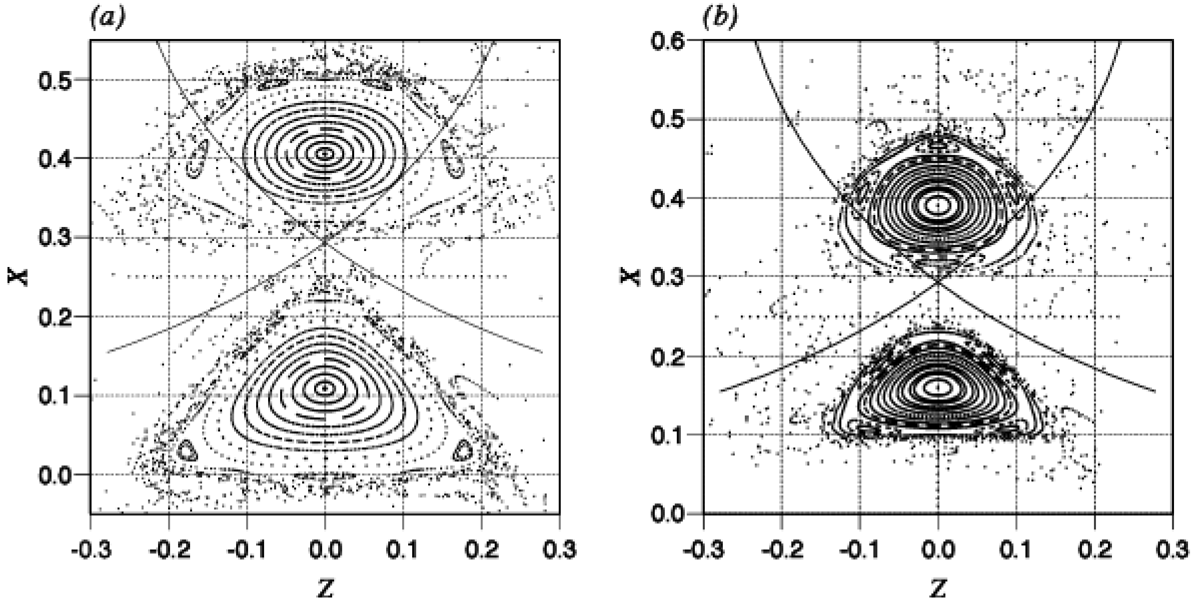
vortex soltion with

$(\nu, \tau) = (1.924, 0.3827), \quad G = 0.1832$

torus formed by a single trajectory of a particle

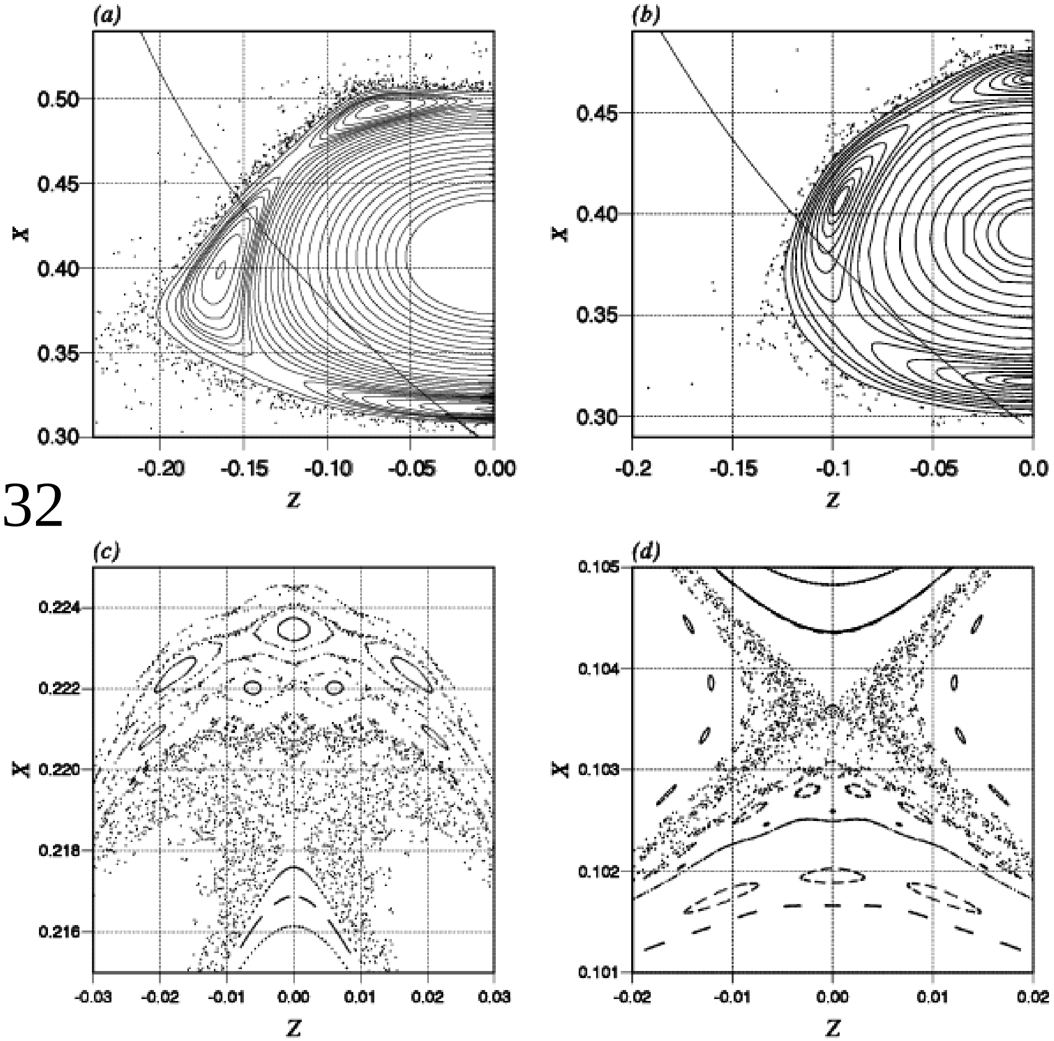
Poincaré sections (intersection of the torus)

$$(\nu, \tau) = (1.924,\ 0.3827)$$



G = 0.1832 G = 0.3113

Poincaré sections (magnified view)

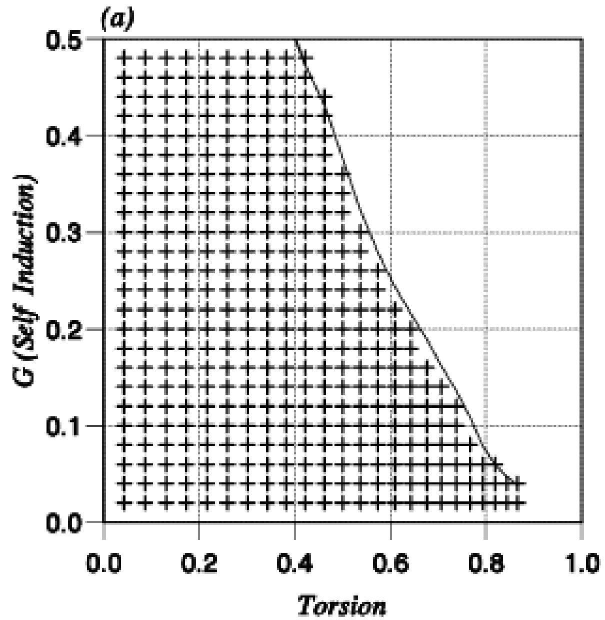


G=0.1832

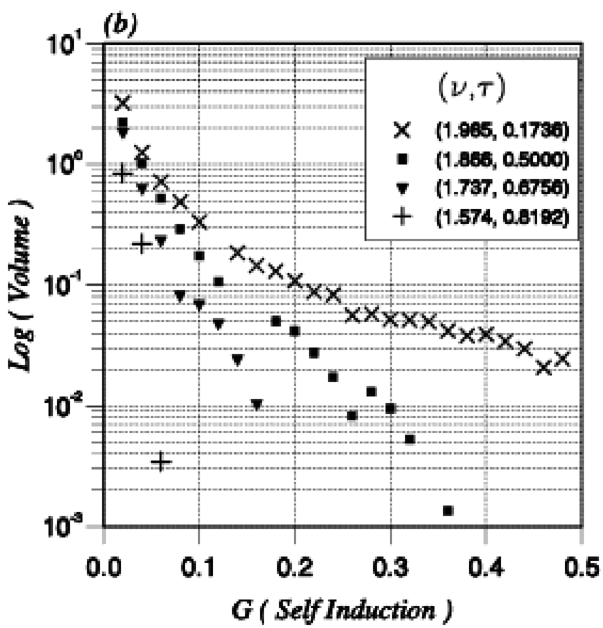
G=0.3113

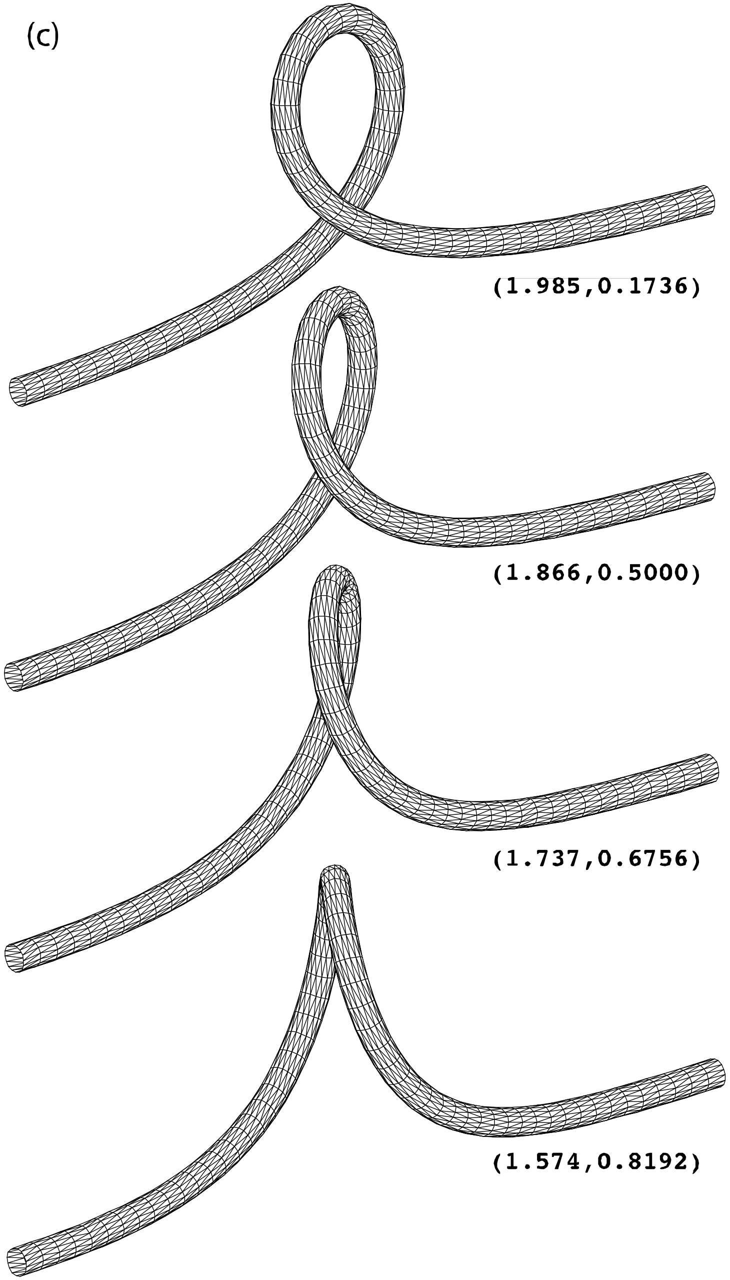
Volume of the torus

Parameters for which a torus is observed



Volume of the torus for various parameters



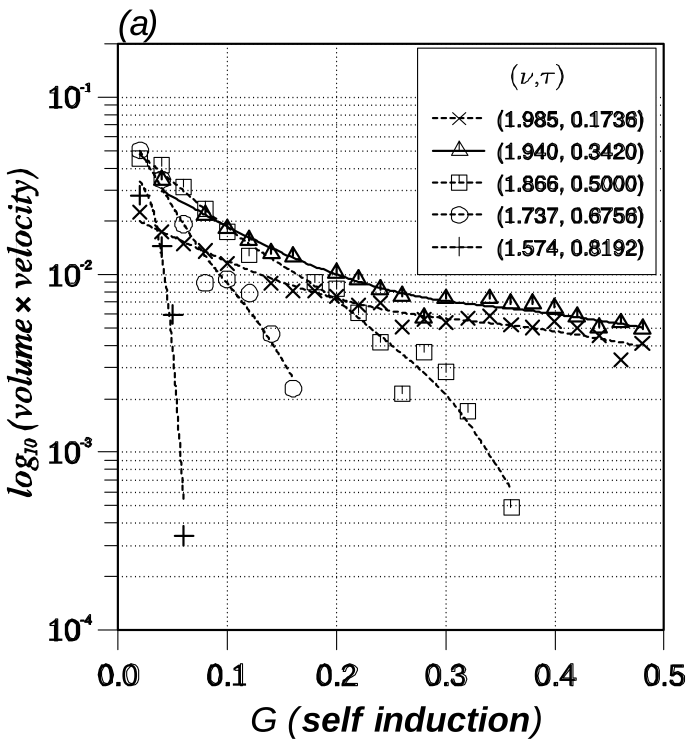


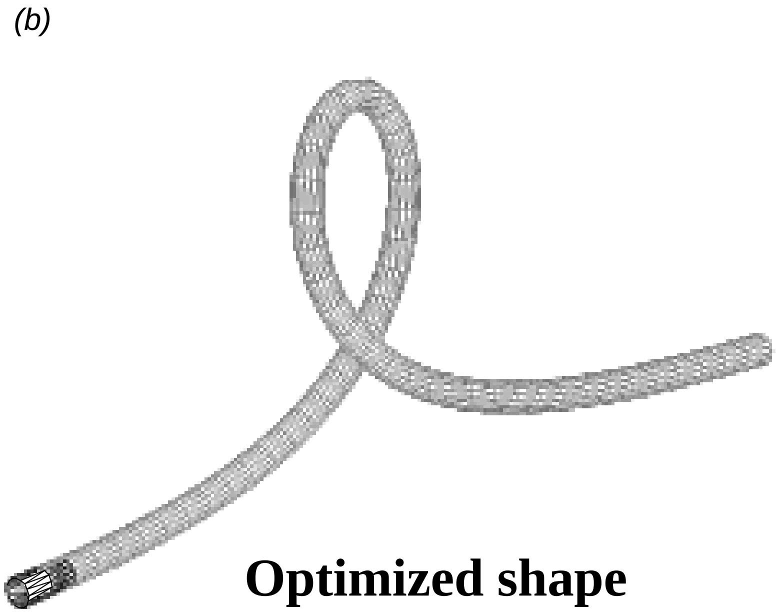
Optimized Shape for the Maximum Transport

Planar soliton carries large volume but its speed is slow

→ There is an optimized shape for the maximum rate of transport

volume × velocity

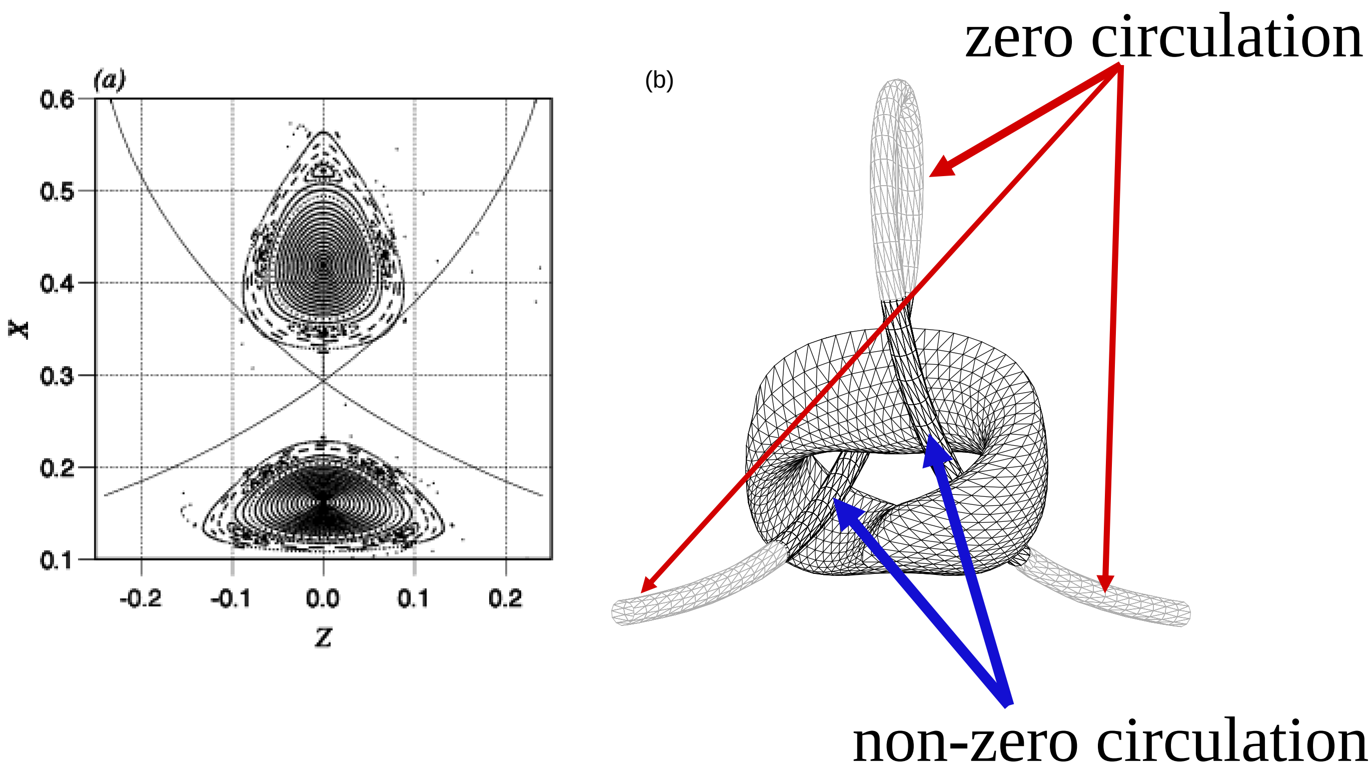




Optimized shape

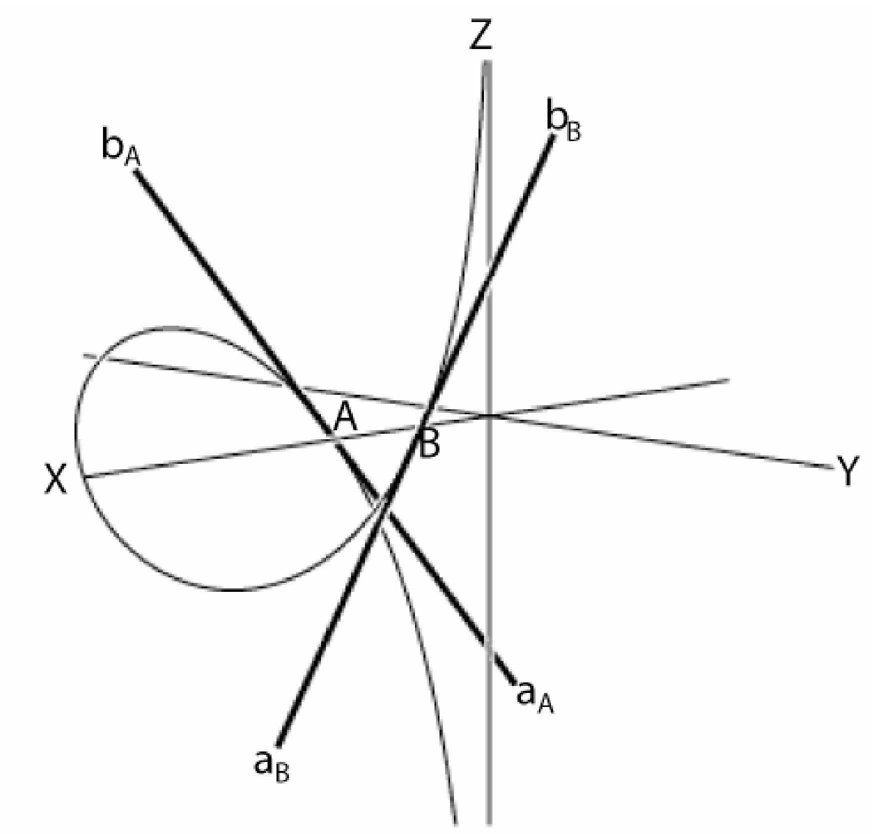
(ν = 1.940, τ = 0.3420)

Essential mechanism for producing torus



A truncated vortex soliton still produces a torus and islands !

Chopsticks model



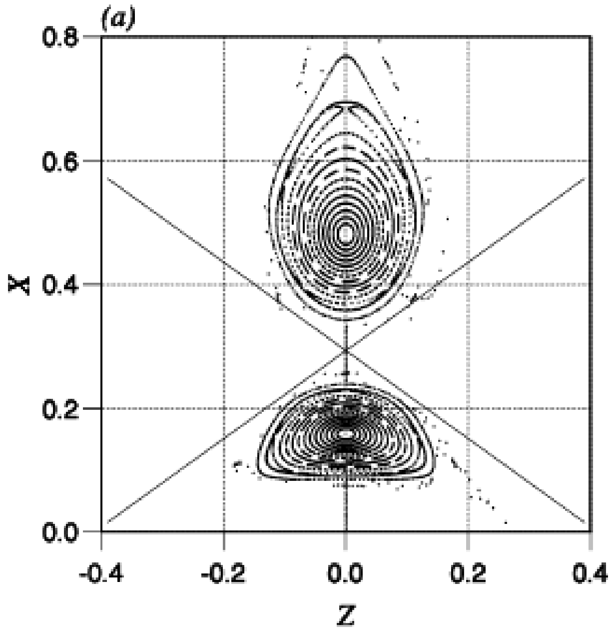
$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2\pi}\frac{1}{\left|\mathbf{t}_A \times (\mathbf{x}-\mathbf{a}_A)\right|^2}\begin{pmatrix} \sin\phi\ \sin\theta\ z - \cos\theta\ (y+y_0) \\ \cos\theta\ (x-x_0) - \cos\phi\ \sin\theta\ z \\ \cos\phi\ \sin\theta\ (y+y_0) - \sin\phi\ \sin\theta\ (x-x_0) \end{pmatrix}$$

$$+ \frac{1}{2\pi}\frac{1}{\left|\mathbf{t}_B \times (\mathbf{x}-\mathbf{a}_B)\right|^2}\begin{pmatrix} \sin\phi\ \sin\theta\ z - \cos\theta\ (y-y_0) \\ \cos\theta\ (x-x_0) + \cos\phi\ \sin\theta\ z \\ -\cos\phi\ \sin\theta\ (y-y_0) - \sin\phi\ \sin\theta\ (x-x_0) \end{pmatrix}$$

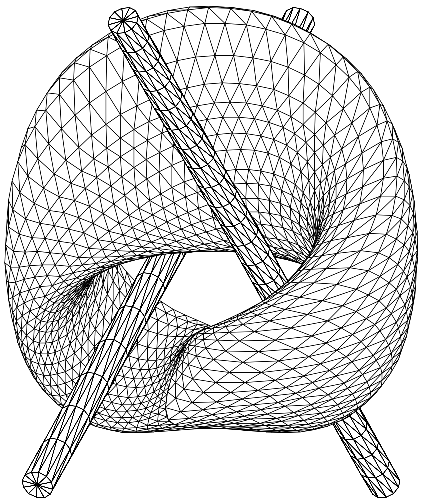
$$+ (\nu^2+\tau^2)G\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} - 2\tau G\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Replacing the vortex soliton with two straight vortex sticks tangent to the vortex soliton

Chopsticks model (3D view and Poincaré section)

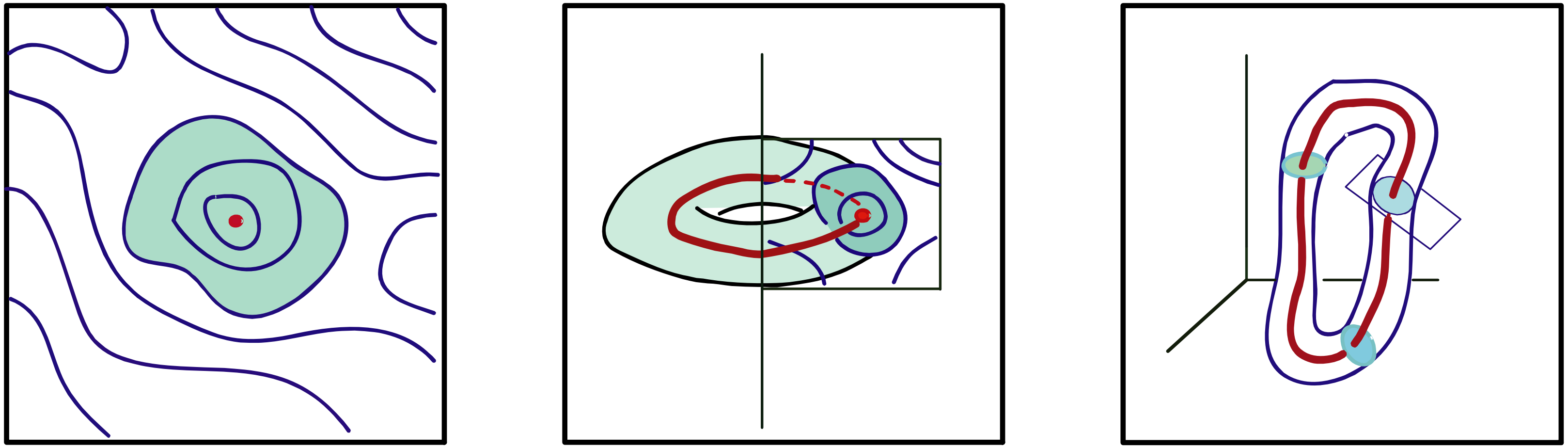


(b)



Chopsticks still produce a torus and the hyperbolic structure !

Closedness of streamlines in steady~slowly varying flows



2D

axisymmetric

3D

fixed point → periodic orbit

Poincare-Hopf theorem
(or index-theorem)

No systematic method
to detect one

Conclusions

- A vortex soliton can carry fluid particles inside a domain with a finite volume that makes a knot with the loop of the vortex soliton for a wide range of the parameter values.
- The configuration of two vortex sticks in 3D space seems to be the essential for producing the torus.

Open problems

- Stability of a vortex soliton.
- Effect of diffusion on the particle motion and the torus.
- Integrability and the torus stracture of the chopsticks model.
- Similarity between the electromagnetic problem.

Thank you very much for your generous support and continuous encouragement, Wadati sensei!