Order and Chaos in Hamiltonian Dynamical Systems

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Abstract

In this talk we introduce several systems which have ordered structure such as clusters by its own dynamics. The systems consists of particles with long-range interaction, just like many-body systems in astrophysics. Since the systems are Hamiltonian systems, the spatial structures thus formed are not “attractors” or asymptotic states we observe in the infinite future. Rather the states are observed in transiency or in the course of itinerancy among several quasi-stationary states.

1 Part I : Introduction

(my) basic, very primitive question:

“What the world is made of?”

[photo of the Andromeda galaxy]

[picture of a hemoglobin molecule]

- **Various structures (order) exist in every scale.**
- **The world is not uniform.**

2 statistical description

methods to describe structure

most successful one: (equilibrium) statistical mechanics (+ phase transition theory)

3 basis of statistical mechanics

$$\boxed{\text{total system}} = \boxed{\text{system to be observed}}$$
$$+ \boxed{\text{a large system called ``heat bath''}}$$

- **total system is closed, conserved**
- **principle of equal weight for total system $\Rightarrow$ canonical prob. distribution for the system to be observed**
- **(note: equal weight $\Rightarrow$ "equipartition of energy" $\langle \frac{p^2}{2m} \rangle = \frac{k_B T}{2}$ for the system to be observed .)**

3.1 ergodicity : a mathematical way to define "equal weight"

def. **time average = phase space average**

$$\bar{A} = \langle A \rangle \quad (\textbf{for any } A) \tag{1}$$

equivalent def. **invariant set is the whole phase space itself (or empty set).**

(regardless of initial condition one can visit any state.)

Ergodic behavior comes from chaotic motion of particles.

What is chaotic motion?

acknowledgement : T. Kawai (Nagoya Univ.), making of the excellent pendulum

Anyway, assumption of ergodicity seems reasonable, but, sometimes, ergodicity fails to be satisfied...

4 demonstration 1: standard map

standard map : a typical nonlinear dynamical system

$$(\mathbf{x}, \mathbf{p}) \mapsto (\mathbf{x}', \mathbf{p}')$$

$$\mathbf{p}' = \mathbf{p} + \frac{K}{2\pi} \sin(2\pi \mathbf{x})$$

$$\mathbf{x}' = \mathbf{x} + \mathbf{p}' \ (mod\, 1) \ (K > 0)$$

visiting phase point in very much non-uniform way ($1/f$-type fluctuation, sticking/stagnant motion)

5 discrete standard map : investigating non-uniformity of phase space

discretized standard map (Rannou(1984), TK(2007))

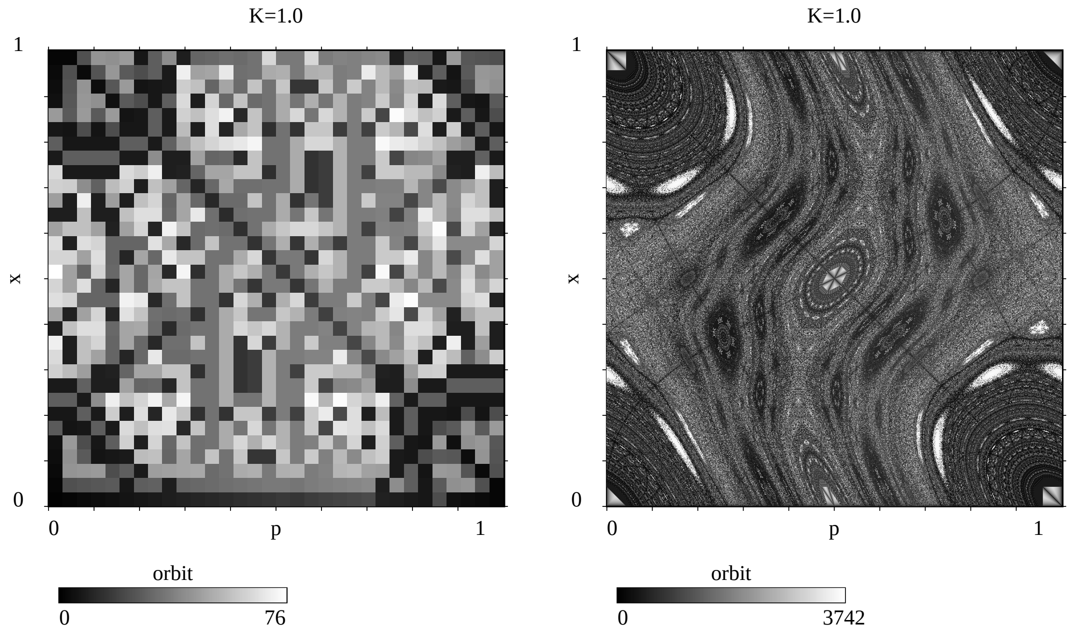
$$(\mathbf{I_x}, \mathbf{I_p}) \mapsto (\mathbf{I'_x}, \mathbf{I'_p}),\ \ \mathbf{I_x}\textbf{etc.} \in 0, 1, 2, \cdots, M-1$$

$$\mathbf{I'_p} = \left[\mathbf{I_p} + M \cdot \frac{K}{2\pi} \sin\left(\frac{2\pi \mathbf{I_x}}{M}\right) \right]$$

$$\mathbf{I'_x} = \mathbf{I_x} + \mathbf{I'_p}\ (mod\ M)\ (K > 0)$$

- **1 to 1 , and every orbit is periodic**

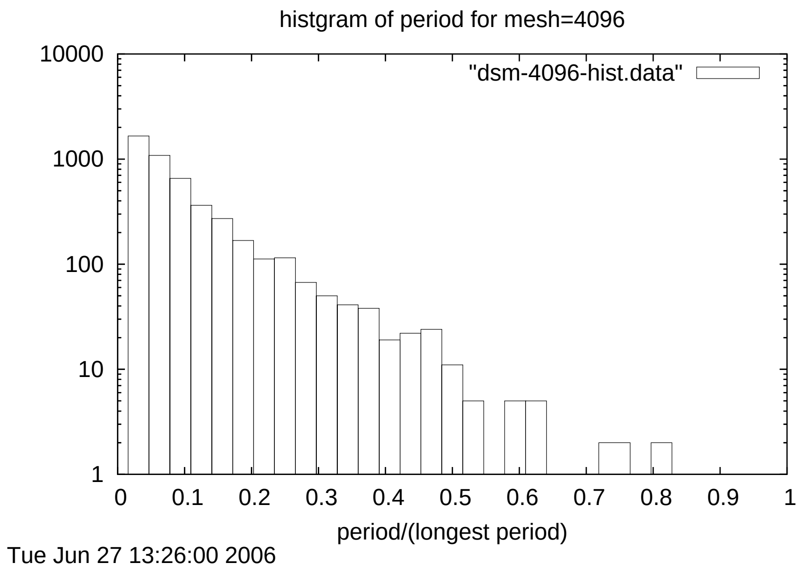
5.1 phase space of discrete standard map

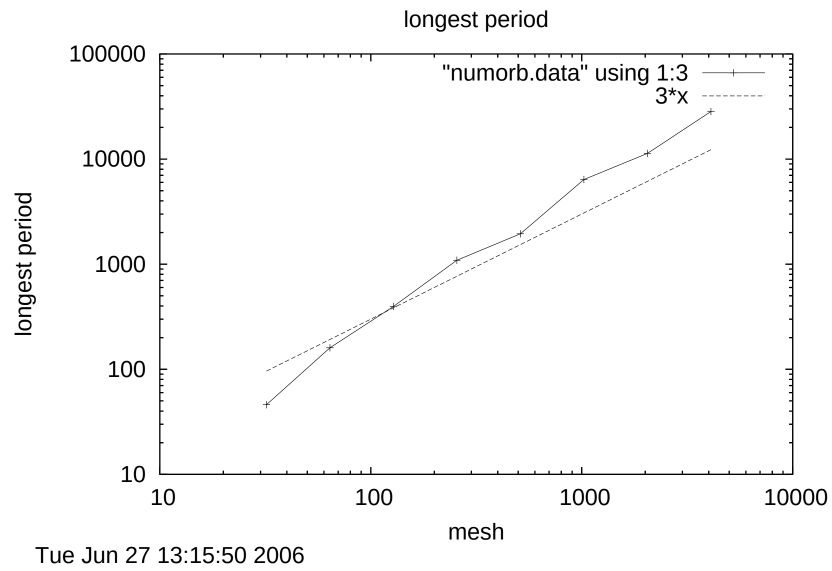


darkness represents the index of each orbit

left:low res. ($M = 32$), right: increased res.($M = 1024$)

5.2 statistics of orbits

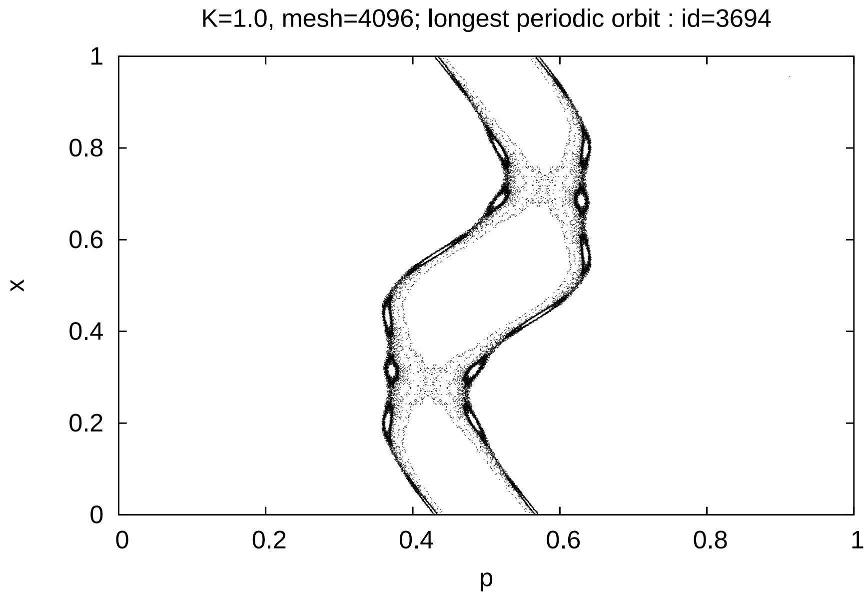


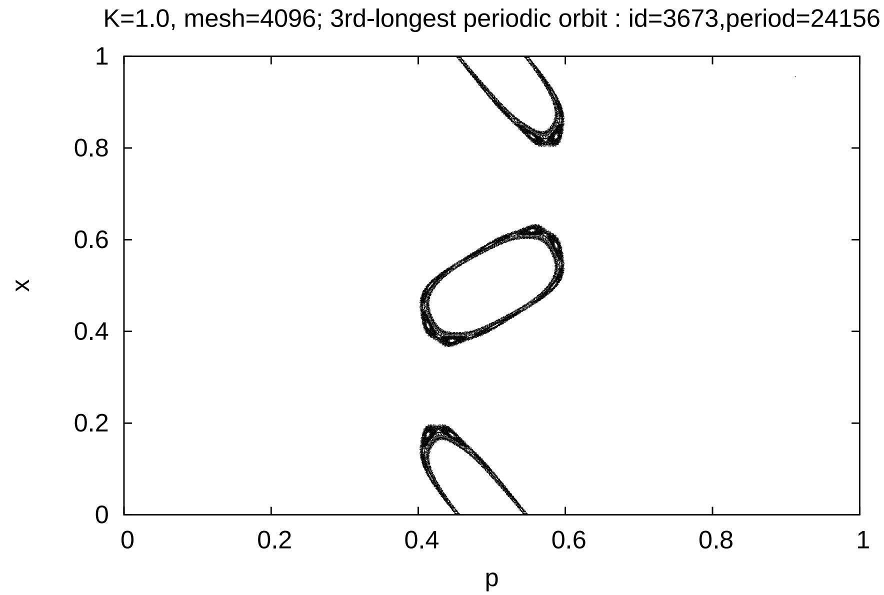


(left) distribution of periods of orbits (M=4096)
(right) longest period vs. M (resolution)

- **no dominant orbit appears**

5.3 dynamics of discrete standard map





(left) longest periodic orbit ($M = 4096$)

(right) 3rd longest periodic orbit ($M = 4096$)

Apparently the phase space is not filled uniformly.

6 Many body systems

- **the last section : failure of ergodicity in small system:**

$\longleftarrow$ **pathology caused by limitation of deg. of freedom?**

Then, many body systems help the situation?

7 failure of ergodicity in many body systems

7.1 trivial example

- **free particle system**

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} \tag{2}$$

- **harmonic chain**

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=1}^{N} \frac{k}{2}(x_{i+1} - x_i)^2 \tag{3}$$

$\Rightarrow$ **every mode energy is conserved :** $E_k(t) = E_k(0)$

7.2 less trivial example

- **1-dim. linear chain with cubic or quartic interaction (Fermi – Pasta – Ulam model)**

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i=1}^{N} \frac{k}{2}(x_{i+1} - x_i)^2 + \alpha \sum_{i=1}^{N} \frac{1}{3}(x_{i+1} - x_i)^3 \tag{4}$$

However, numerical simulation tells us that the mode energies are recurrent for the system (no relaxation).

$$E_1(t=0) = E_{total},\ \ E_k(t=0) = 0\ \ (k \geq 2)$$

$$\Rightarrow \begin{cases} \textbf{(wrong)} & E_1 = E_2 = \cdots = E_N \\ \textbf{(true)} & E_1(T) \sim E_0\ \ , E_k << E_0 (k \geq 2) \end{cases}$$

Later analysis revealed that the model (4) is quite close to the KdV equation.

$$w_\tau(\xi,\tau) + w_{\xi\xi\xi} + ww_\xi = 0\ \ , u(\xi,\tau) \in R \qquad (5)$$

8 Statistical mechanics: successful application to explain structure

Anyway, removing some difficulties, statistical mechanics succeeded in explaining order and structures:

- **ferromagnetic transition for Ising model and many other spin systems**
- **helix-coil transition for long chain of molecules**

huge number of examples..

Statistical mechanics assumes:

- **the system is in (or very close to) thermal equilibrium**
- **perturbations relaxes "quickly" :**

$$\exp\left(-\frac{t}{\tau_{relax}}\right)$$

downside : fails to describe structures which are not stationary

9 Examples which are not stationary

- **Elliptical galaxies**
- **dynamics of water molecules (Ohmine group, Sasai group)**

Purpose of this talk:

Trying to understand "order" (spatial structure) from dynamical point of view : case study

example 1 : Clustered motion in globally coupled symplectic map

Refs. TK and K. Kaneko, J. Phys. A $\underline{25}$ (1992) 6283
K. Kaneko and TK, Physica D $\underline{71}$ (1994) 146

- **model**
- **clustered motion and random motion**
- **Lyapunov spectra – "minimum" exponent?**
- **Fluctuation property of Lyapunov exponent**
- **Phase space structure – order supported by chaos**

$\boxed{\text{Model}}$

$$\left(x_i, p_i\right) \mapsto \left(x'_i, p'_i\right), \qquad i = 1, 2, \cdots, N,$$

$$p_{\mathbf{i}}' = p_{\mathbf{i}} + K \sum_{\mathbf{j}=1}^{N} \sin 2\pi(x_{\mathbf{j}} - x_{\mathbf{i}}), \tag{6}$$

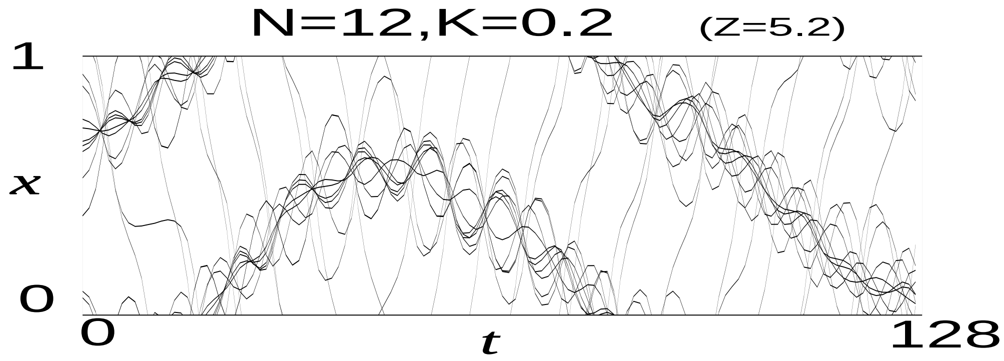
$$K > 0 : \textbf{attractive}, K < 0 : \textbf{repulsive}$$

$$x_{\mathbf{i}}' = x_{\mathbf{i}} + p_{\mathbf{i}}'.$$

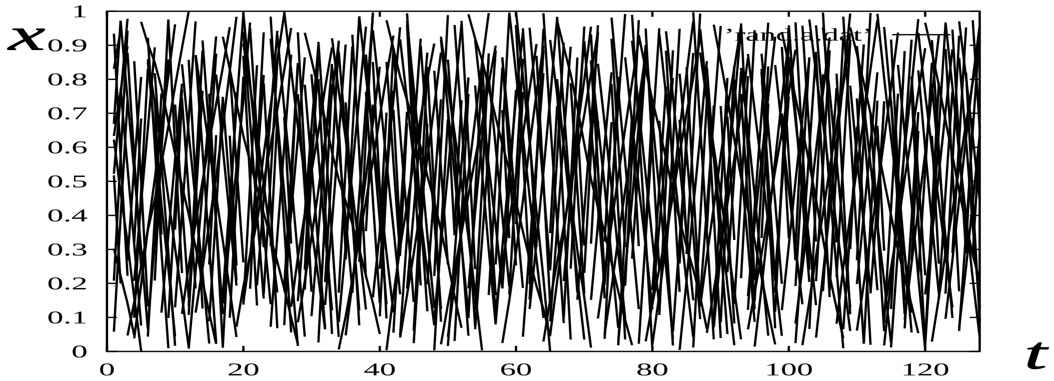
(globally coupled, long-range interaction)

$\boxed{\text{two phases of motion (with same } K\text{)}}$

clustered :



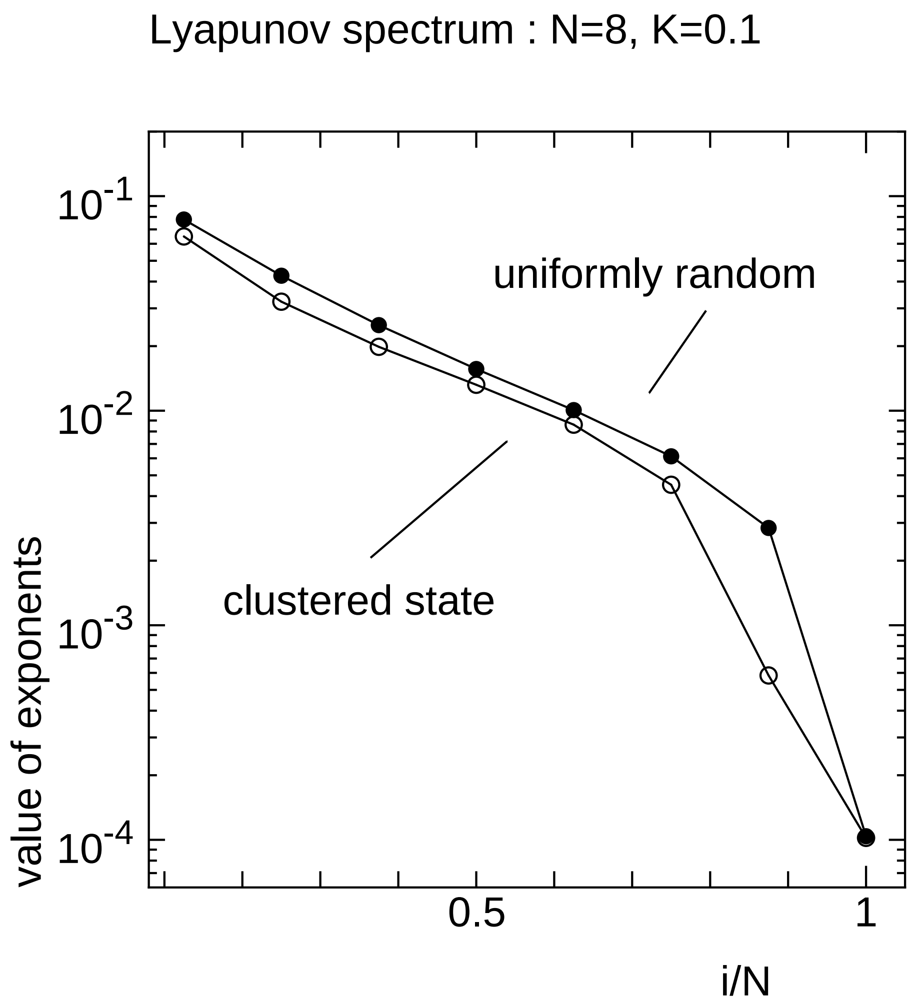
uniformly random:



$$p_i' = p_i + K \sum_{\mathbf{j}=1}^{N} \sin 2\pi(x_{\mathbf{j}} - x_i), \quad K : \mathbf{real}$$

Lyapunov spectra

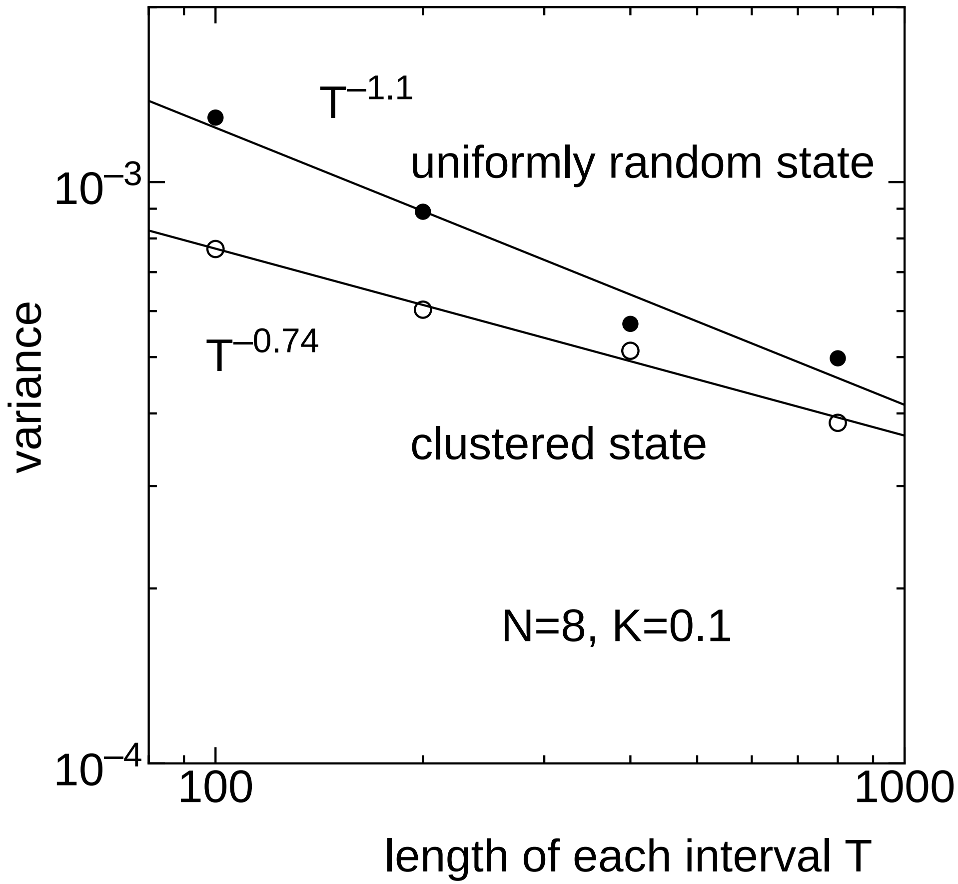
- **Both phases are chaos.**
- **difference of Lyap. exp. is observed only at the λ_{N-1}. $\Rightarrow$ This subspace represents instability of cluster?**



Fluctuation property of Lyapunov exponent

- **clustered state $\cdots$ strong temporal correlation**
- **non-clustered state $\cdots$ no temporal correlation**

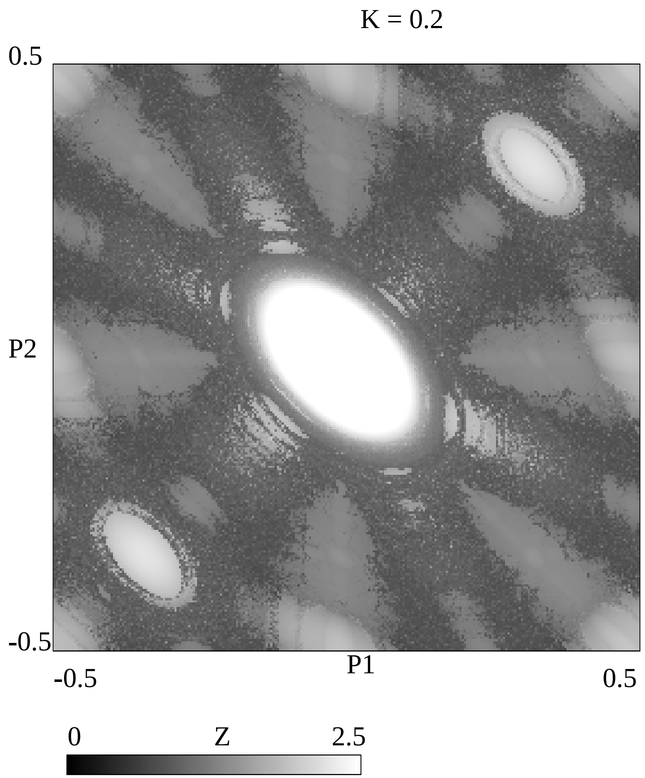
Variance of the maximum Lyap.exp.

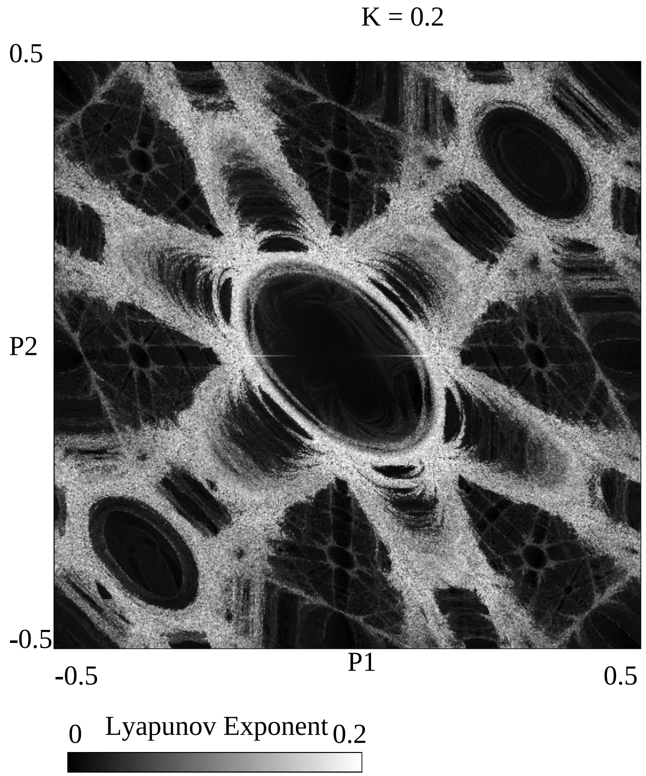


phase space structure

init. cond. for clustered motion (white)

local λ_1 (black $\cdots$ stable)

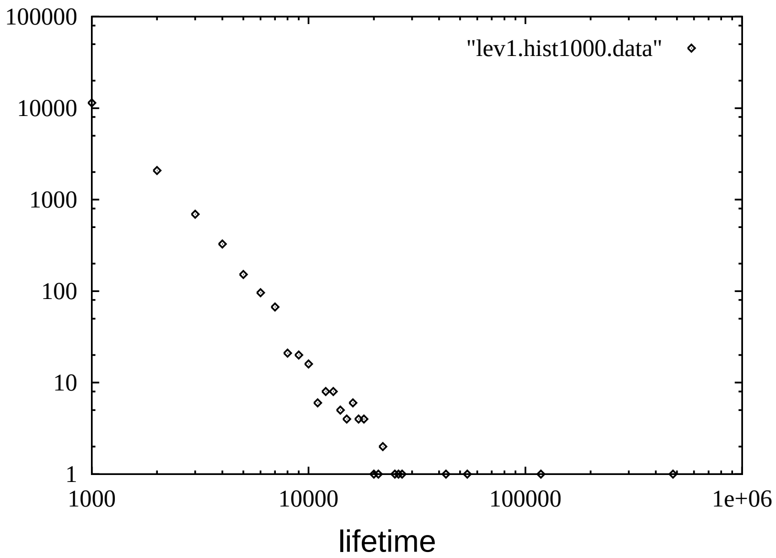




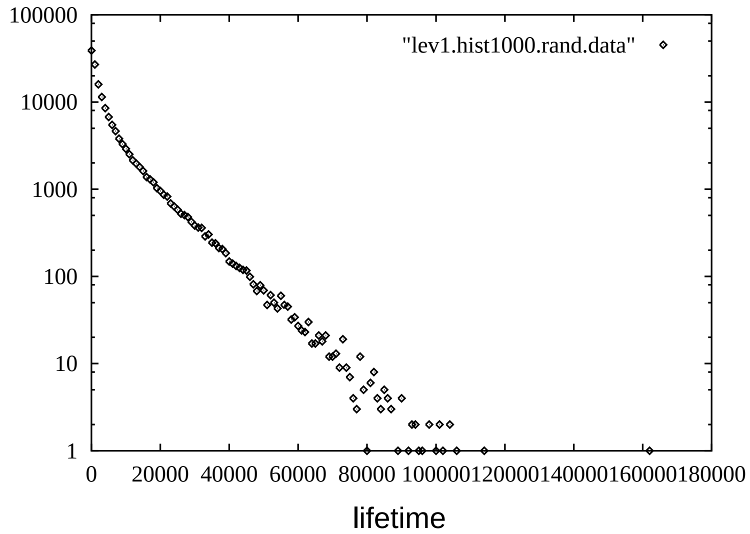
strong similarity $\Rightarrow$ their origins ?

Lifetime distribution of each state

distribution of lifetime of clustered state : Z>1.05 (N=8,K=0.2)



distribution of lifetime of non-clusterd state : Z<1.05 (N=8,K=0.2)



cluster ... $\tau^{-\alpha}$

non-cluster .. $\exp(-\gamma t)$

- **Self-similar structures in the phase space – common to most Hamiltonian systems**
- **chaos supports the order**

example 2 : Itinerant behavior in the mass-sheet model

Refs.

T. Tsuchiya, N. Gouda and TK, Astrophysics and Space Science, 257,pp. 319-341, 1998

T. Tsuchiya, N. Gouda, and TK, Phys. Rev. E 53(1996)2210.

T. Tsuchiya, TK, and N. Gouda, Phys.Rev. E vol.50 (1994) 2607.

- **model**
- **equilibrium behavior**
- **itinierancy**

introduction

important point in the previous part:

- **dynamical coexistence of macroscopically distinguishable states (spatial structures) in conservative system**
- **non-uniform (heterogeneous) phase space**

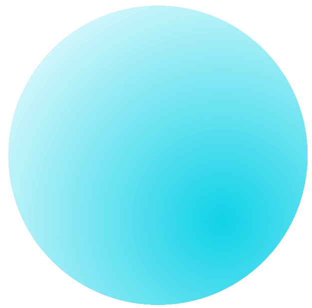
("chaotic itinerancy" Kaneko, Ikeda, Tsuda, ...)

So what about the stellar systems? (e.g., galaxies)

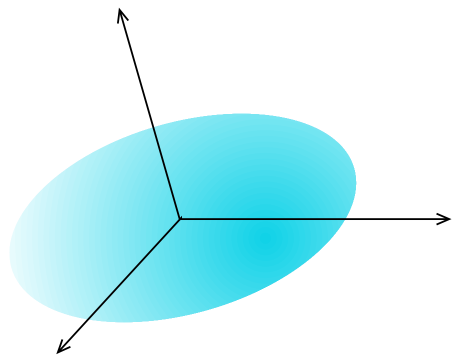
motivation and original idea (Gouda):
elliptic galaxies

[photo of the elliptic galaxy M87]

- equilibrium shape : spherically symmetric



- observed shape : triaxial



- Previous studies : investigating special solutions

with several shapes without spherical symmetry.

- **However, it is quite unlikely that many-body system interacting with non-linear potential with (almost surely) irregular initial condition run on some special exact solution. They should behave rather chaotically.**
- **idea (Gouda): we may understand the spatial structure as "dynamically supported", not in equilibrium.**

- **But 3-dim. Newtonian N-body problem $\dfrac{1}{r}$ is hard**
- **Then, can we observe similar phenomena in more simple models?**

$\boxed{\text{model}}$

"mass-sheet model" (1-dimensional self-gravitating model)

$$H = \sum_{\mathbf{i}=1}^{N} \frac{{p_{\mathbf{i}}}^2}{2m} + 2\pi G m^2 \sum_{\mathbf{i}>\mathbf{j}} |x_{\mathbf{i}} - x_{\mathbf{j}}| \ , \quad -\infty < x_i < \infty \ , \tag{7}$$

(globally coupled system, long-range interaction)

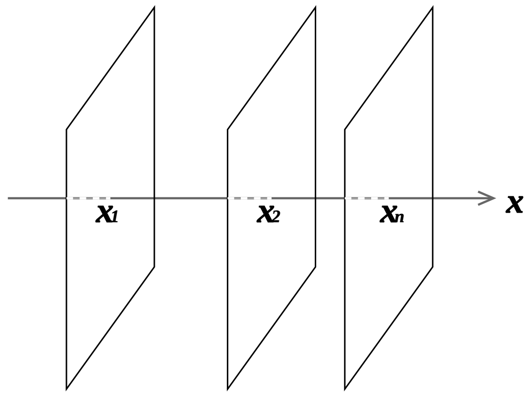


Fig. 1 A schematic picture of the model (7)

initial condition and equilibrium

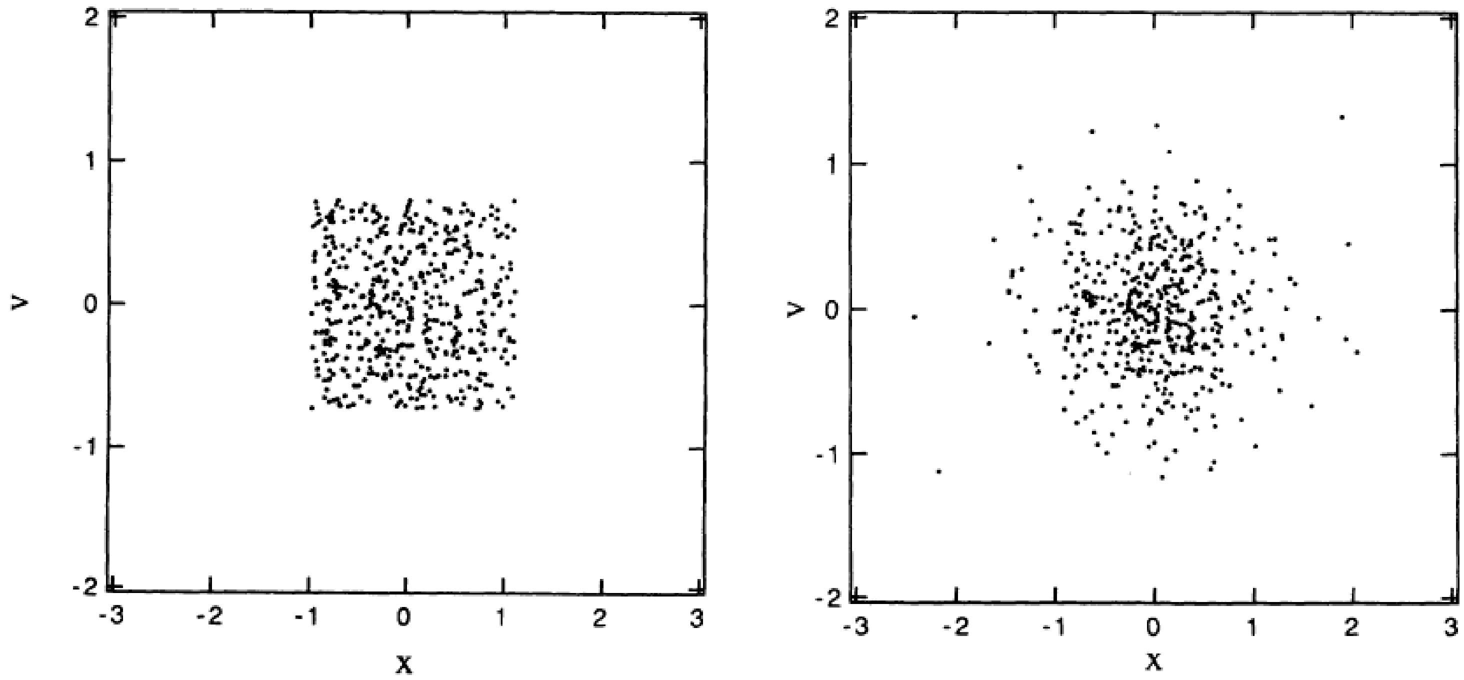


Fig. 2 initial condition ("waterbag")(left) and equilibrium("isothermal")(right) : μ-space back from equilibrium?

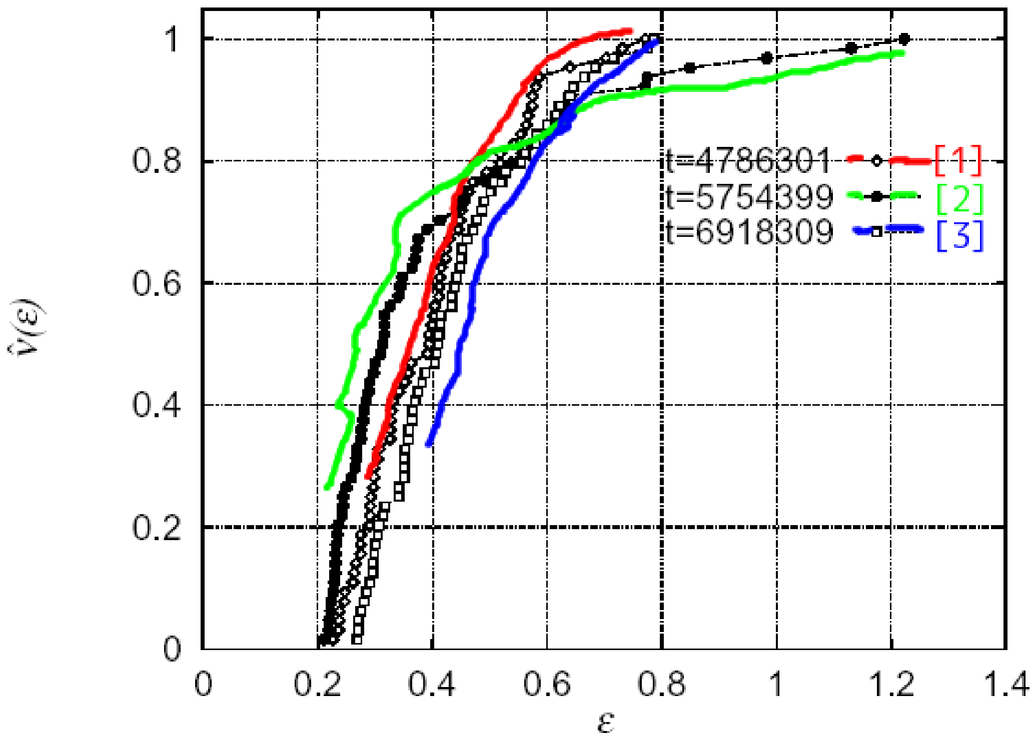


Fig. 3 time evolution of cumulative energy distribution

Spilt milk back to the cup??

energetic particles and quasi-equilibrium

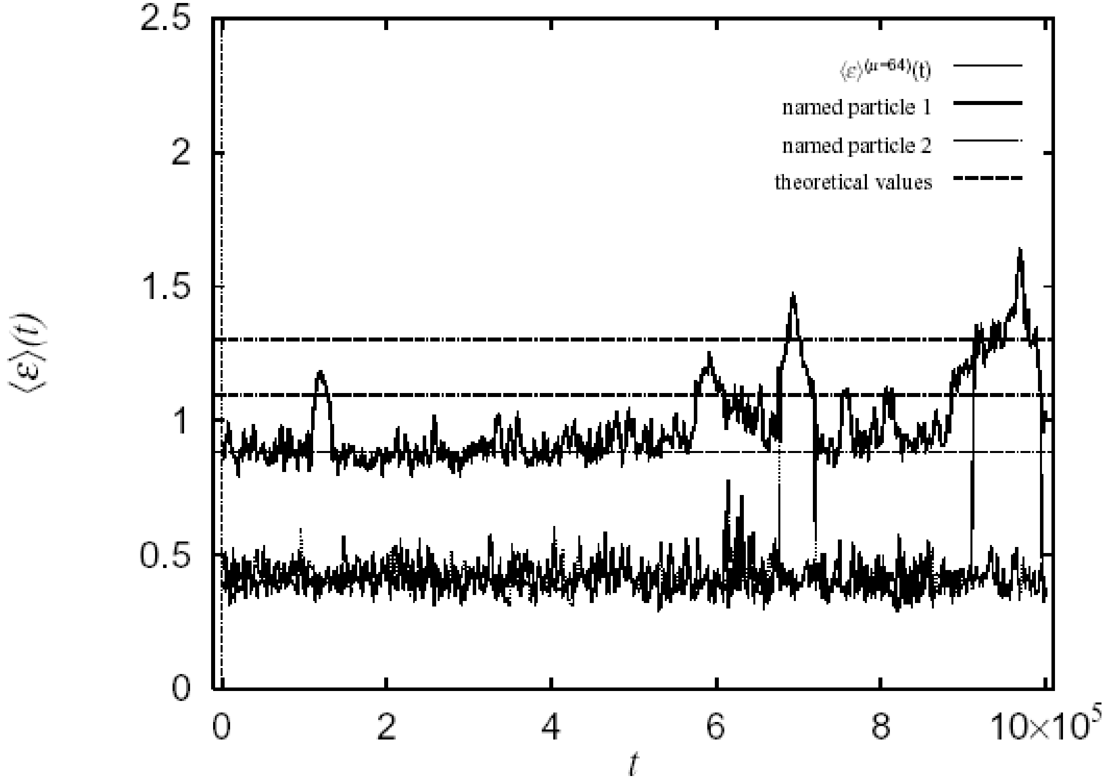


Fig. 4 emergence of energetic particles

- **This system have several (many?) quasi equilibria**
- **the quasi equilibria are not isolated : they are dynamically connected via creation and annihiration of energetic particles**

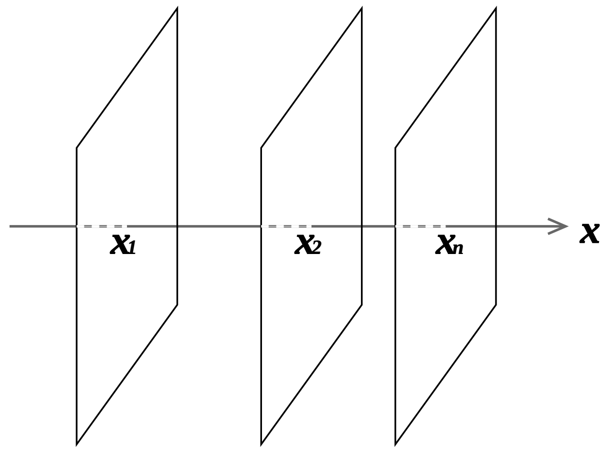
example 3 : Emergence of power-law correlation in the mass-sheet model

Refs. H. Koyama and TK, Phys. Lett. A vol. <u>295</u> (2002) 109

H. Koyama and TK, Europhys. Lett., vol. <u>58</u>(2002) 356.

H. Koyama and TK, Phys. Lett. A vol.<u>279</u> (2001) 226.

$\boxed{\text{model}}$



A schematic picture of the model (8) : 1-dimensional self-gravitating system, or "sheet model"

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + 2\pi G m^2 \sum_{i>j} |x_i - x_j| \ , \quad -\infty < x_i < \infty \ , \ (8)$$

formation of fractal structure

dynamics seen in μ-space

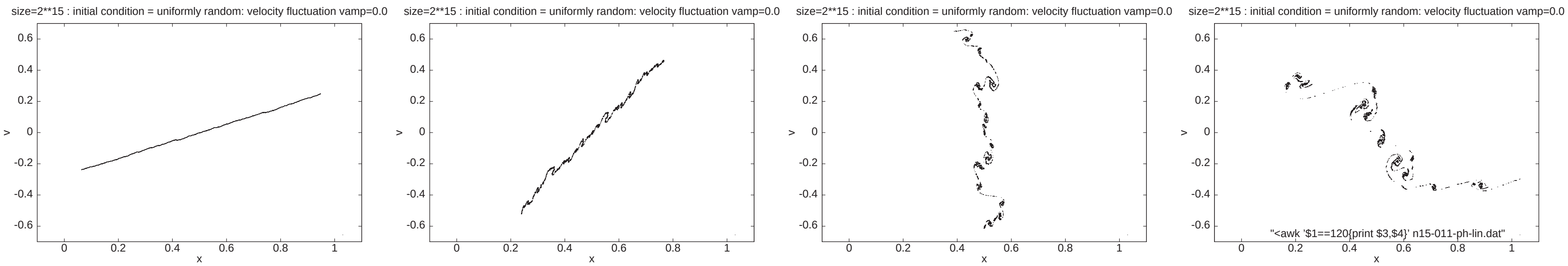


Fig. 5 Formation of fractal structure. Initial condition : x_i : uniformly random in [0,1], $u_i = 0$, $N = 2^{15}$. Time are 2.34375,4.6875,7.03125, and 9.375.

box-counting dimension in the μ-space

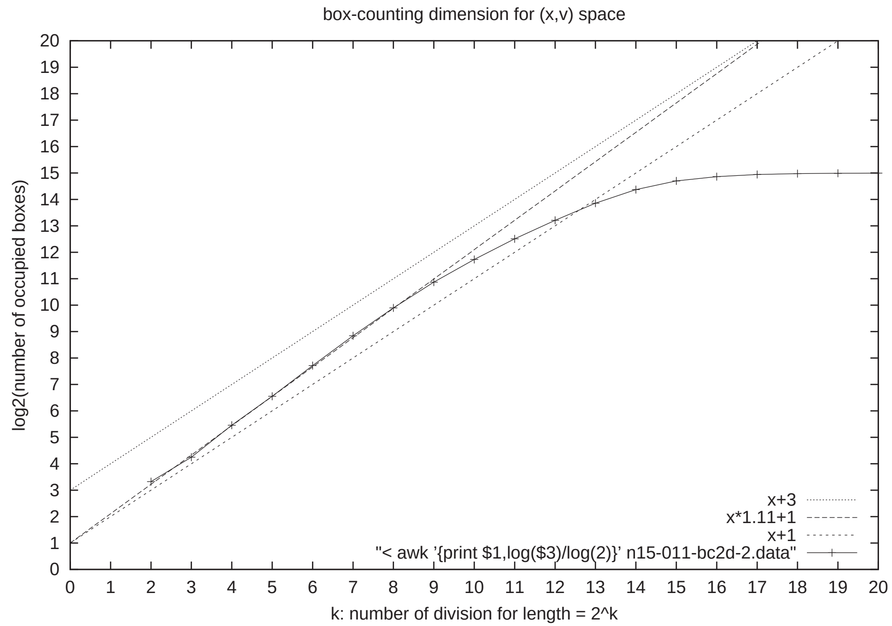
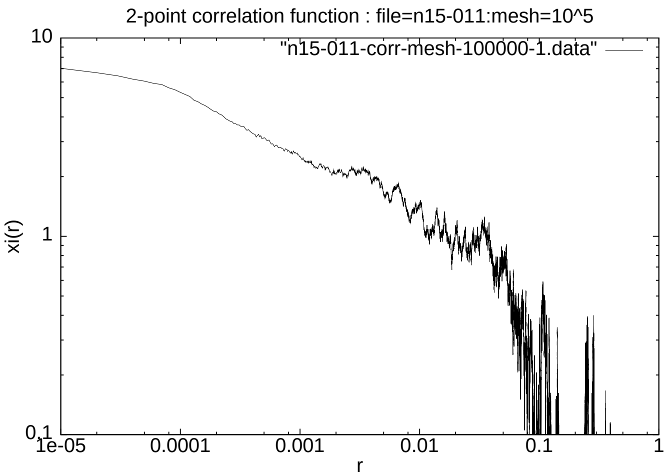


Fig. 6 Box counting dimension of the μ-space distribution shown in the bottom of Fig.5. Sample orbits with the same class of initial condition with different random number gives dimension $D = 1.1 \pm 0.04$ for 24 samples. Lines with $D = 1$ are also shown for comparison.

2-body correlation function $\xi(r)$

$$dP = ndV(1+\xi(r)) \tag{9}$$

$$\xi(r) \propto r^{-\alpha} \tag{10}$$



two-point correlation function $\xi(r)$ for $t = 9.375$ in Fig. 5. Exponent α of $\xi \propto r^{-\alpha}$ is $\alpha = 0.20 \pm 0.03$ for 24 samples.

$\boxed{\text{development of power-law structure}}$

spatial scale : small $\rightarrow$ large ("hierarchical clustering")

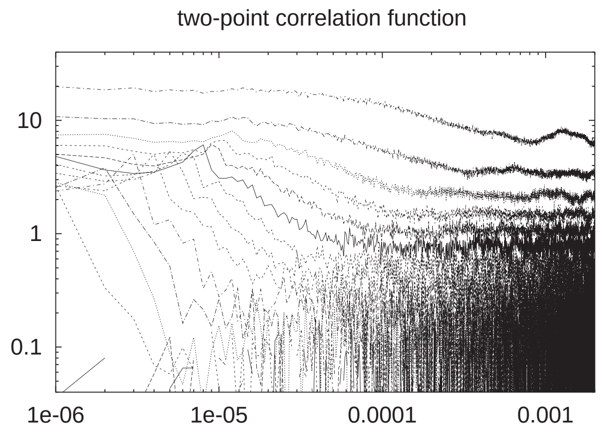


Fig. 7 Evolution of correlation function $\xi(r)$ at $t = \frac{5}{64}\ell$, $\ell = 3, 4, \cdots, 17$ (from bottom to top). System size $N = 2^{14}$, Initial condition: x_i =random, $v_i = 0$.

relaxation of the structure

Actually, the power-law structure $\xi(r) \propto r^{-\alpha}$ decays, but long after virialization.

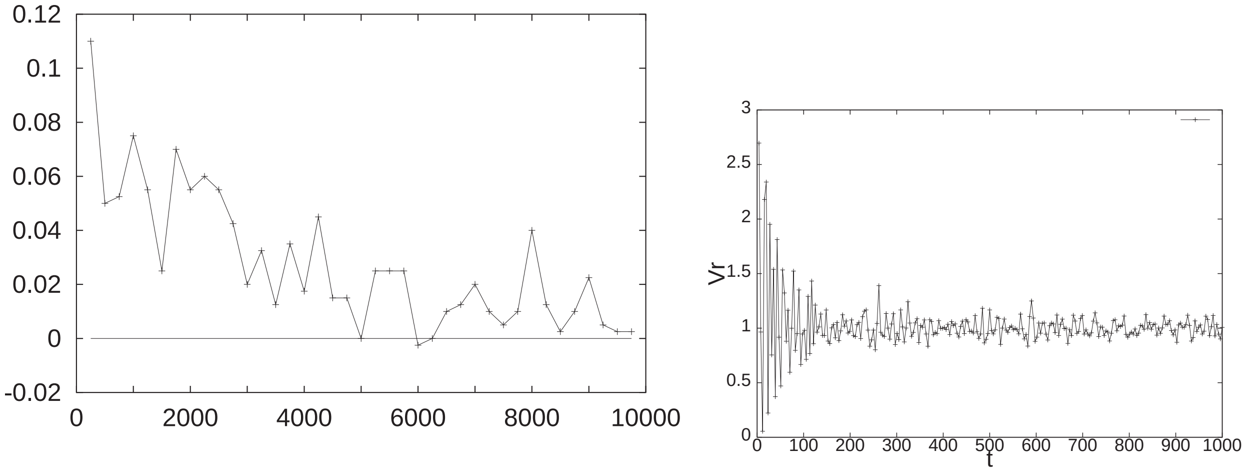
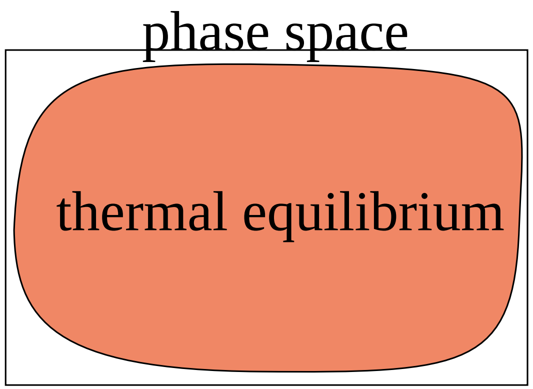
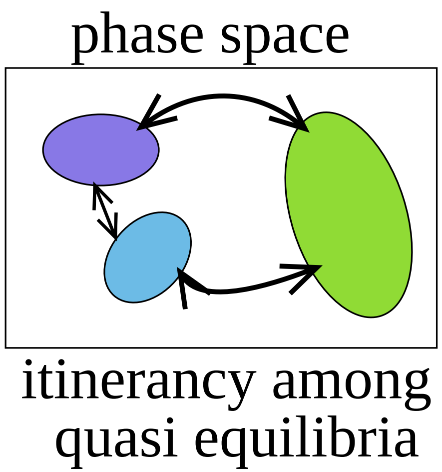


Fig. 8 time dependence of the exponent $\alpha(t)$ (left) and the virial ratio$2E_{kin}/E_{pot}$ (right)(10)

Summary

Spatial structures can emerge in non-equilibrium situations. Their origins are non-uniformity of the phace space, where dynamical properties of the system is reflected:





equilibrium structure (left) and dynamical order(right)

- **chaos does not necessarily imply “mess” nor “darkness”:**
  they can be rich sources of dynamic order.
- **long-range interaction may be important in creating structure/order which are dynamically changing.**