Particle position fluctuations in the asymmetric exclusion process

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(based on collaborations with A. Borodin,

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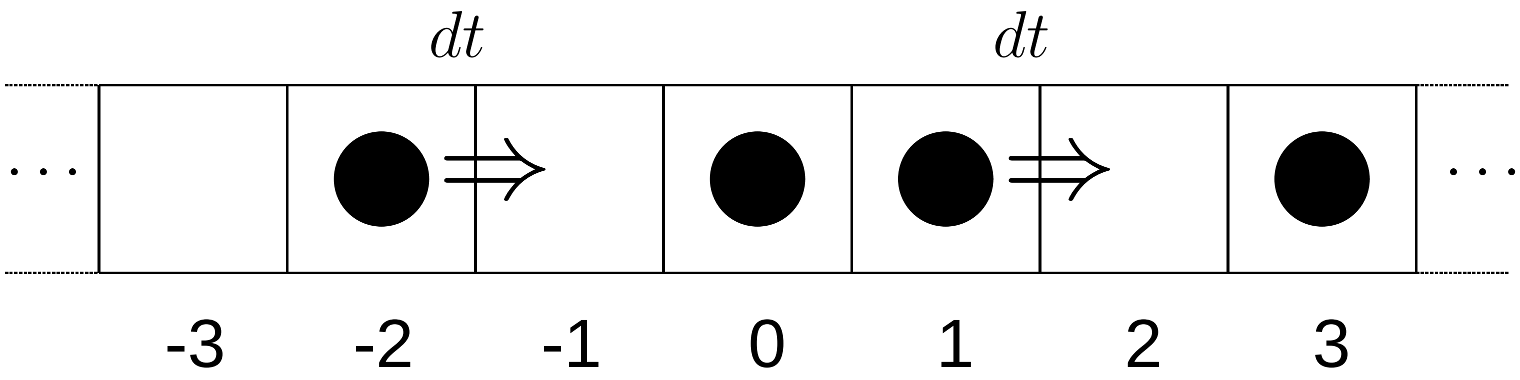
1. ASEP

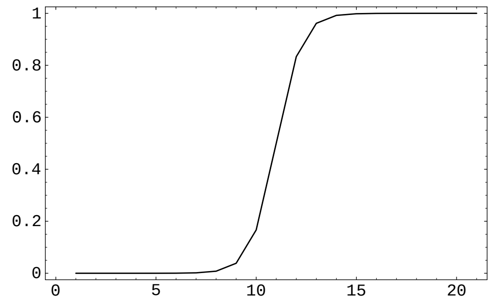
ASEP $=$ asymmetric simple exclusion process



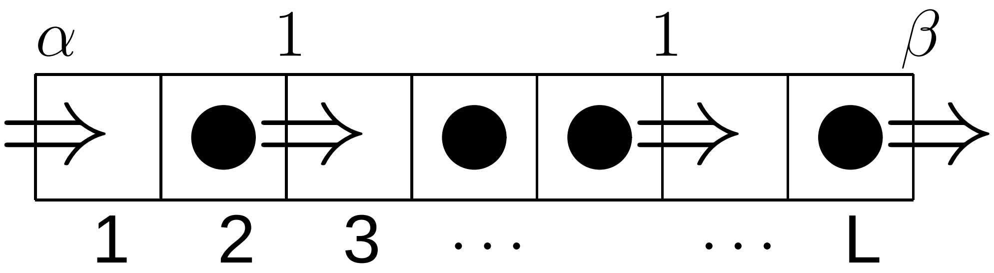
- Each site is occupied or empty.
- During short time dt each particle tries to hop to the right neighboring site w.p. dt.
- If the target site is occupied, the hopping doesn't occur (exclusion).

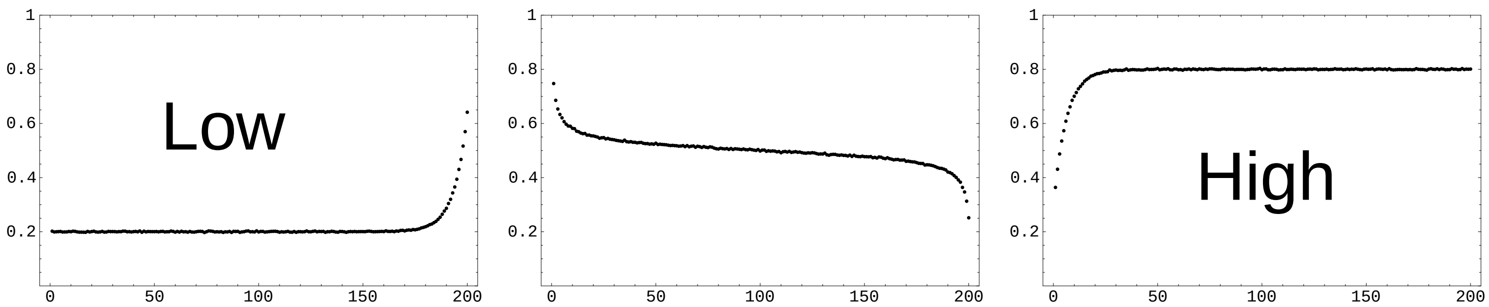
Interesting Phenomena

- Shock wave
  - Stochactic version of Burgers equation



- Boundary induced phase transition



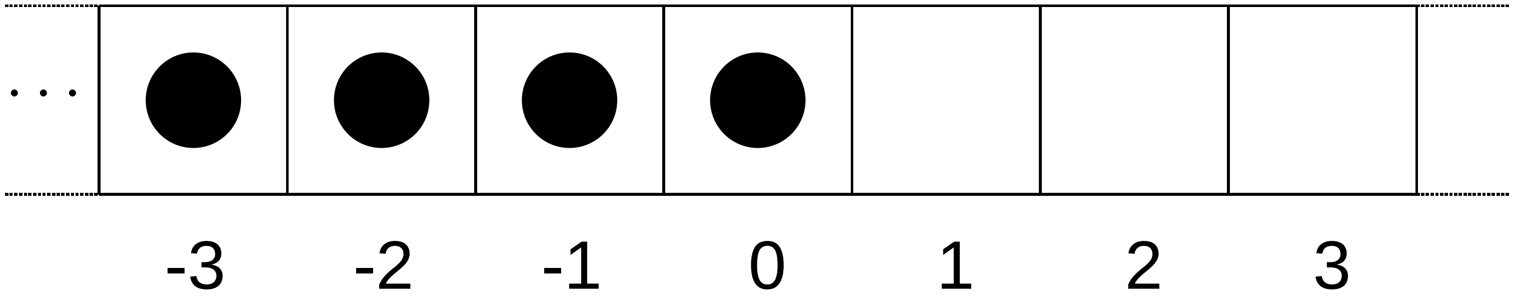


Related Fields

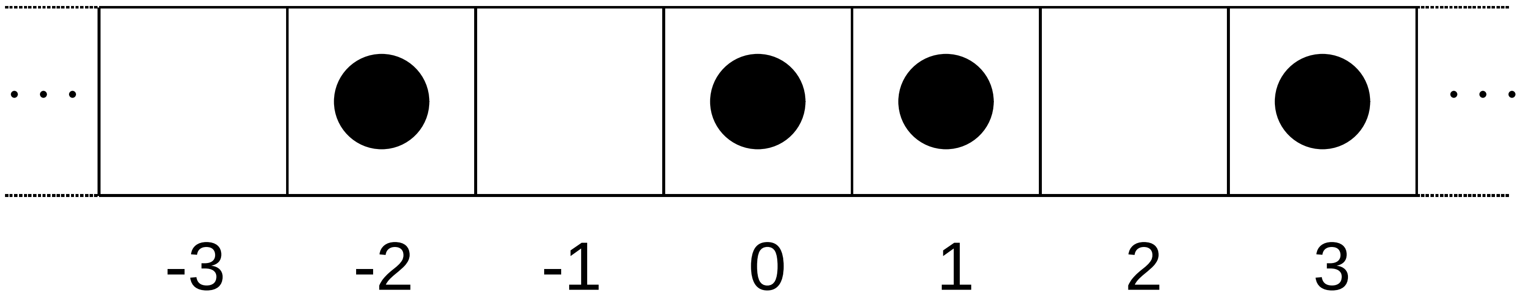
- Nonequilibrium statistical physics
- Probability...Interacting stochastic systems
- Integrable systems
  - q-orthogonal polynomials
  - Random matrix theory

2. Particle Position: Step Case

Step initial condition ($t = 0$)

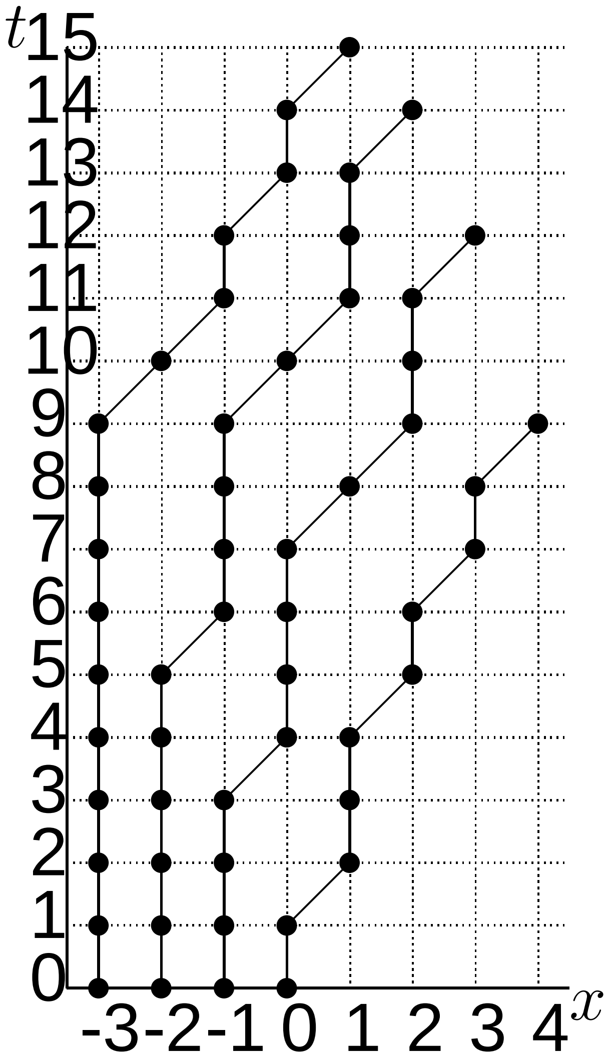


At time t



$x_N(t)$:position of the Nth particle at time t

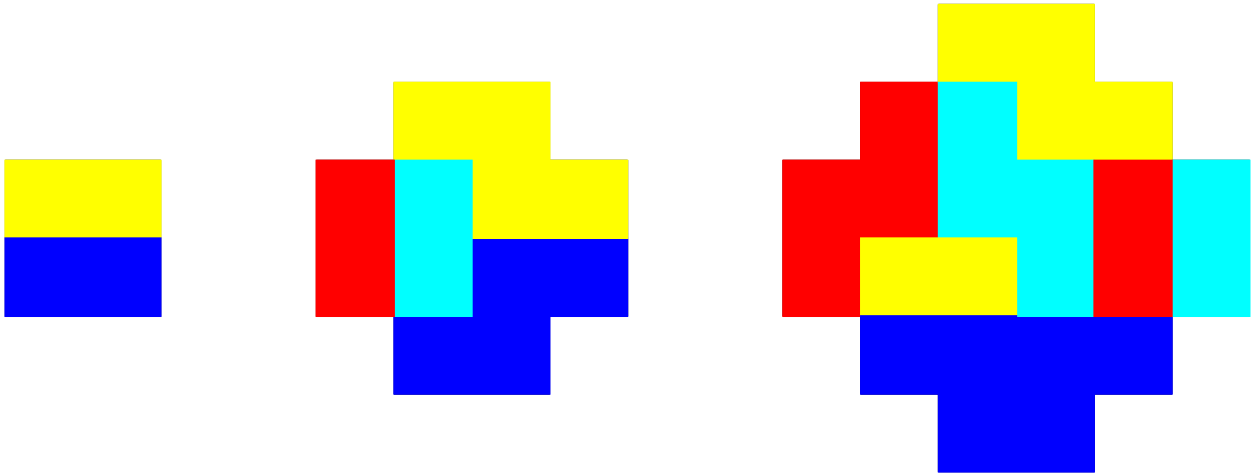
Trajectories in Discrete TASEP

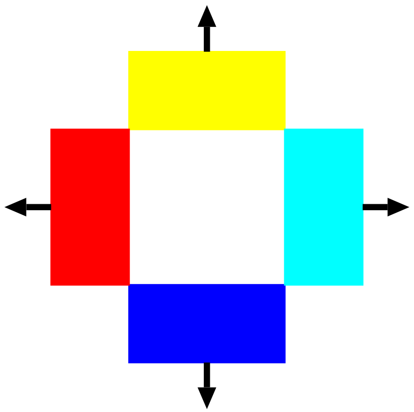


This is *not* free fermion!

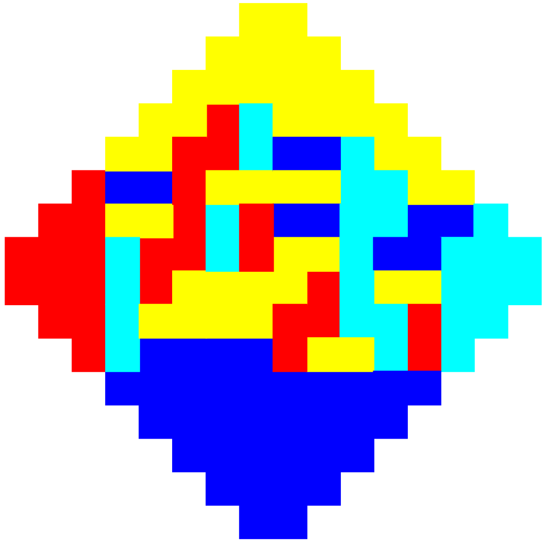
Aztec diamond

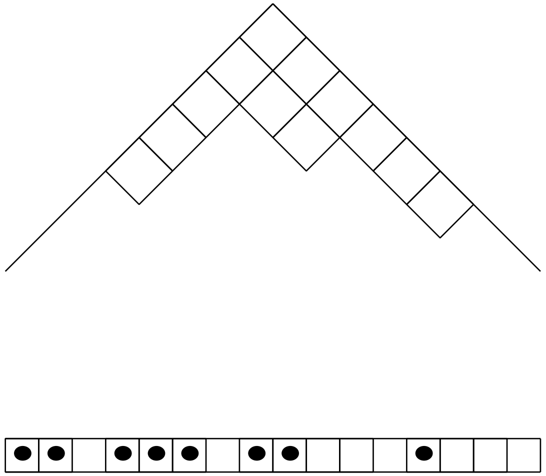
Domino tilings and Schuffling (Jockusch, Propp, Shor)

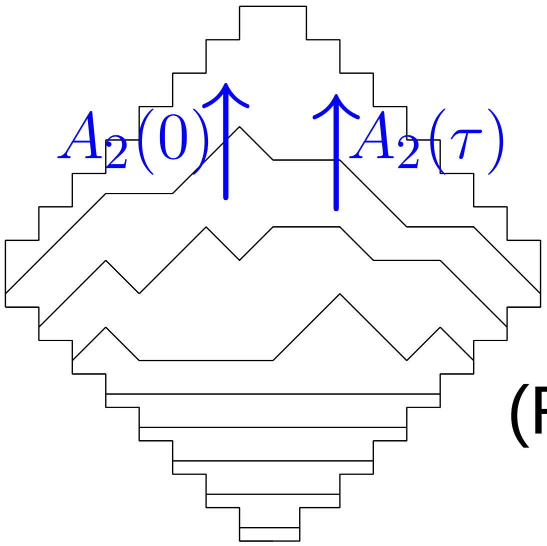




TASEP in Aztec diamond







$\Leftarrow$ Non-colliding walks $\sim$ free fermion!

(Related to Combinatorics of Young tableaux)

Airy Process

Johansson 2005

$A_2(0), A_2(\tau)$: scaled positions of N_1, N_2th particles for step

$$A_2(0) = \frac{at - x_{N_1}(t)}{dt^{1/3}}, \quad A_2(\tau) = \frac{at - x_{N_1+ct^{2/3}}(t)}{dt^{1/3}}$$

$$\mathbb{P}[A_2(0) < s_1, A_2(\tau) < s_2 \ (j = 1,2)] = \det(1 - K_2 \chi_{s_1,s_2})$$

where the kernel is

$$\mathcal{K}_2 = \begin{cases} \int_0^\infty d\lambda e^{-\lambda(\tau_1-\tau_2)} \mathsf{Ai}(\xi_1 + \lambda)\mathsf{Ai}(\xi_2 + \lambda) & \tau_1 < \tau_2 \\ -\int_{-\infty}^0 d\lambda e^{-\lambda(\tau_1-\tau_2)} \mathsf{Ai}(\xi_1 + \lambda)\mathsf{Ai}(\xi_2 + \lambda) & \tau_1 \geq \tau_2 \end{cases}$$

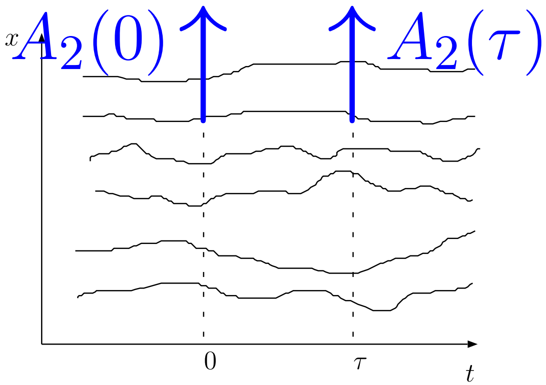
GUE Dyson's BM (tGUE)

Time-dependent random matrix

$$H = \begin{bmatrix} u_{11}(t) & u_{12}(t) + iv_{12}(t) & \cdots & u_{1N}(t) + iv_{1N}(t) \\ u_{12}(t) - iv_{12}(t) & u_{22}(t) & \cdots & u_{2N}(t) + iv_{2N}(t) \\ \vdots & \vdots & \ddots & \vdots \\ u_{1N}(t) - iv_{1N}(t) & u_{2N}(t) - iv_{2N}(t) & \cdots & u_{NN}(t) \end{bmatrix}$$

$u_{jk}(t), v_{jk}(t)$: OU process (Brownian motion in potential)

Dynamics of eigenvalues

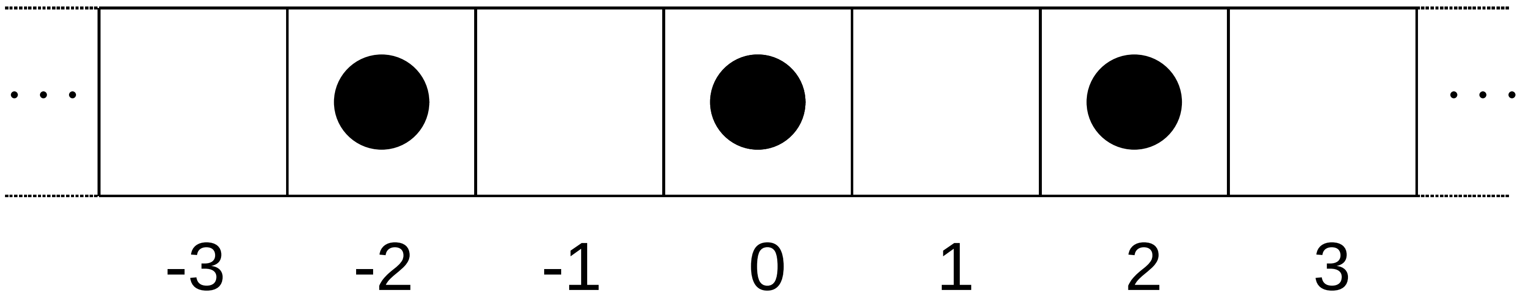


$A_2(0), A_2(\tau) \cdots$ Airy process

Step $\Leftrightarrow$ tGUE

3. Particle Position: Alternating Case

Alternating initial condition ($t = 0$)



One(N_1th) particle (Baik, Rains, Prähofer, Spohn) $\cdots$ GOE

Conjecture

	step	alternating
1p	GUE	GOE
2p	tGUE	tGOE?

tGOE$=$ GOE Dyson's BM ($=$ a special case of Calogero model)

Determinantal Green's function

Schütz 1997

Probability $G(x_1, \cdots, x_N; t)(= G(x_1, \cdots, x_N; t|y_1, \cdots, y_N; 0))$ that N particles starting from $y_1, y_2, \cdots, y_N$ ($y_N < \cdots < y_1$) are on $x_1, x_2, \cdots, x_N$ ($x_N < \cdots < x_1$) at time t

$$G(x_1, x_2, \cdots, x_N; t|y_1, y_2, \cdots, y_N; 0) \quad (= G(x_1, x_2, \cdots, x_N; t))$$

$$= \det[F_{k-j}(x_{N-k+1} - y_{N-j+1}; t)]_{j,k=1,\cdots,N}$$

$$= \begin{vmatrix} F_0(x_N - y_N; t) & F_{-1}(x_N - y_{N-1}; t) & \cdots & F_{-N+1}(x_N - y_1; t) \\ F_1(x_{N-1} - y_N; t) & F_0(x_{N-1} - y_{N-1}; t) & \cdots & F_{-N+2}(x_{N-1} - y_1; t) \\ \vdots & \vdots & & \vdots \\ F_{N-1}(x_1 - y_N; t) & F_{N-2}(x_1 - y_{N-1}; t) & \cdots & F_0(x_1 - y_1; t) \end{vmatrix}$$

where $F_n(x; t) = e^{-t} \frac{t^x}{x!} \sum_{k=0}^{\infty} (-1)^k \frac{(n)_k}{(x+1)_k} \frac{t^k}{k!}$ $\Rightarrow$ NS2004

Reinterpretation of G

Two particles

$$G(x_1,x_2;t) = \begin{vmatrix} F_0(x_2-y_2;t) & F_1(x_1-y_2;t) \\ F_{-1}(x_2-y_1;t) & F_0(x_1-y_1;t) \end{vmatrix}$$

$$= \sum_{x_2^2(>x_1^2)} \begin{vmatrix} \phi(x_1^1,x_1^2) & \phi(x_1^1,x_2^2) \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \psi_0^{(2)}(x_1^2) & \psi_0^{(2)}(x_2^2) \\ \psi_1^{(2)}(x_1^2) & \psi_1^{(2)}(x_2^2) \end{vmatrix}$$

where $x_1^1 = x_1, x_1^2 = x_2$ and

$$\phi(x_1,x_2) = 0(x_1 > x_2), \quad -1(x_1 \le x_2)$$

$$\psi_0^{(2)}(x) = -F_{-1}(x-y_1;t)$$

$$\psi_1^{(2)}(x) = F_0(x-y_2;t)$$

Auxiliary weight

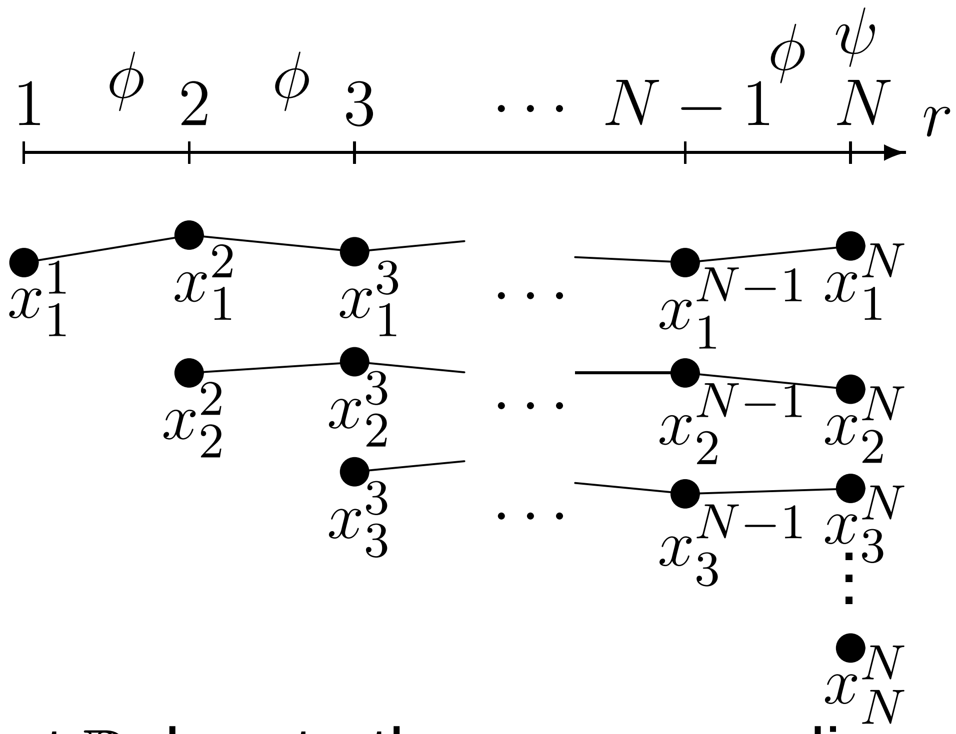
$$\prod_{r=1}^{N-1} \det[\phi(x_j^r, x_k^{r+1})]_{j,k=1}^{r+1} \cdot \det[\psi_j^{(N)}(x_{k+1}^N)]_{j,k=0}^{N-1}$$

$$= \begin{vmatrix} \phi(x_1^1, x_1^2) & \phi(x_1^1, x_2^2) \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \phi(x_1^2, x_1^3) & \phi(x_1^2, x_2^3) & \phi(x_1^2, x_3^3) \\ \phi(x_2^2, x_1^3) & \phi(x_2^2, x_2^3) & \phi(x_2^2, x_3^3) \\ 1 & 1 & 1 \end{vmatrix} \cdots$$

$$\times \begin{vmatrix} \psi_0^{(N)}(x_1^N) & \psi_0^{(N)}(x_2^N) & \cdots & \psi_0^{(N)}(x_N^N) \\ \psi_1^{(N)}(x_1^N) & \psi_1^{(N)}(x_2^N) & \cdots & \psi_1^{(N)}(x_N^N) \\ \vdots & \vdots & & \vdots \\ \psi_{N-1}^{(N)}(x_1^N) & \psi_1^{(N)}(x_{N-1}^N) & \cdots & \psi_{N-1}^{(N)}(x_N^N) \end{vmatrix}$$

where $\psi_j^{(r)}(x) = (-1)^{r-1-j} F_{-r+1+j}(x - y_{j+1}; t)$

Non-colliding walk interpretation



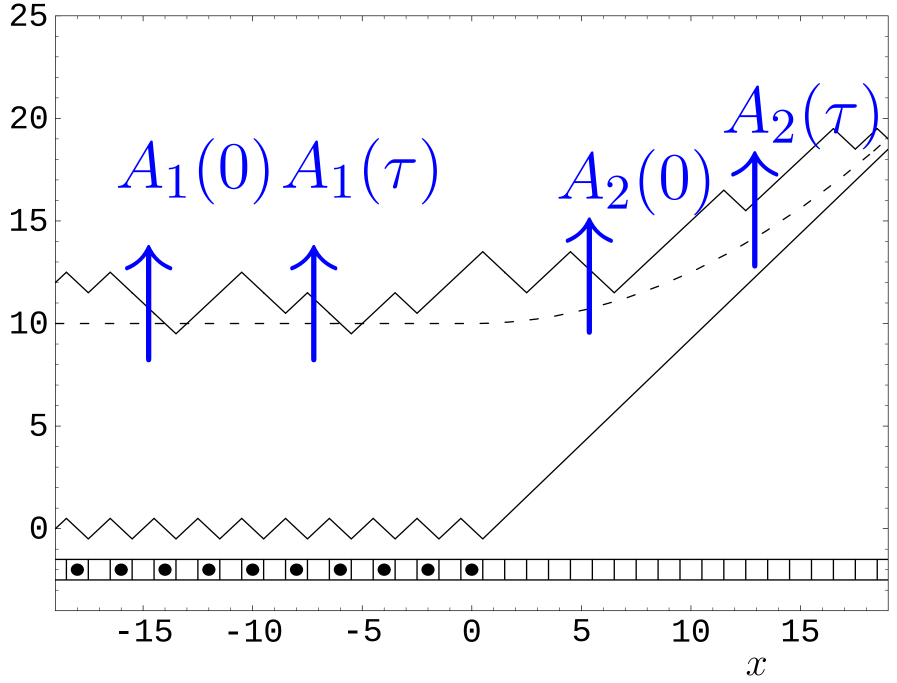
$\Leftarrow$ Non-colliding walk
$\sim$ free fermion

Let $\mathbb{P}$ denote the corresponding measure. We have

$$\boxed{G(x_1, \cdots, x_N; t) = \mathbb{P}[x_1^r = x_r \ (r = 1, \cdots, N)]}$$

TASEP particle configuration $=$ dynamics of the 1st walker

”Half alternating” initial condition



- Each particle cannot affect the particles on its right
- Same as alternating case deep inside the negative ($x < 0$) region

Joint distribution for Alternating

S2005, BFPS 2006

$A_1(0), A_1(\tau)$: scaled positions of 2 particles for alternating

$$\text{Prob}[A_1(0) < s_1, A_1(\tau) < s_2] = \det(1 - K_1 \chi_{s_1,s_2})$$

$$K_1(\tau_1, \xi_1; \tau_2, \xi_2) = \tilde{K}_1(\tau_1, \xi_1; \tau_2, \xi_2) - \Phi_1(\tau_1, \xi_1; \tau_2, \xi_2)$$

where

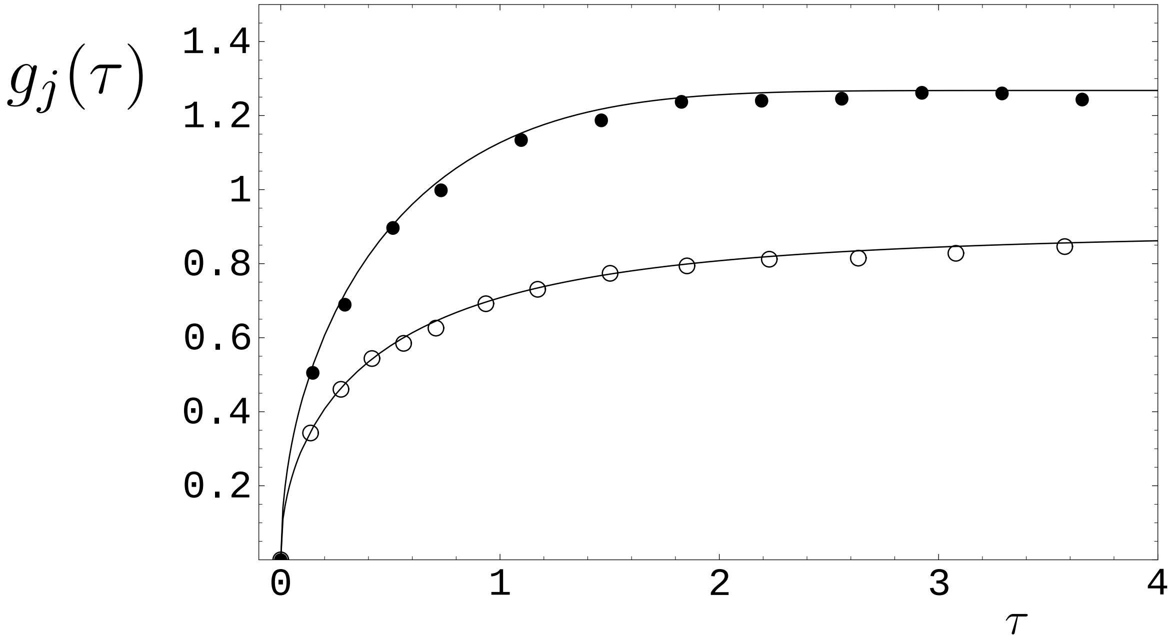
$$\tilde{K}_1(\tau_1, \xi_1; \tau_2, \xi_2) = \frac{1}{2} e^{\frac{(\tau_2-\tau_1)(\xi_1+\xi_2)}{4} + \frac{(\tau_2-\tau_1)^3}{12}} \text{Ai}\left(\frac{\xi_1 + \xi_2}{2} + \frac{(\tau_2 - \tau_1)^2}{4} \right)$$

$$\Phi_1(\tau_1, \xi_1; \tau_2, \xi_2) = \begin{cases} \frac{1}{\sqrt{8\pi(\tau_2-\tau_1)}} \exp\left[-\frac{(\xi_2-\xi_1)^2}{8(\tau_2-\tau_1)}\right] & \tau_1 < \tau_2 \\ 0 & \tau_1 \geq \tau_2 \end{cases}$$

Alternating $\overset{?}{\Leftrightarrow}$ tGOE (GOE Dyson's BM)

2pt correlation

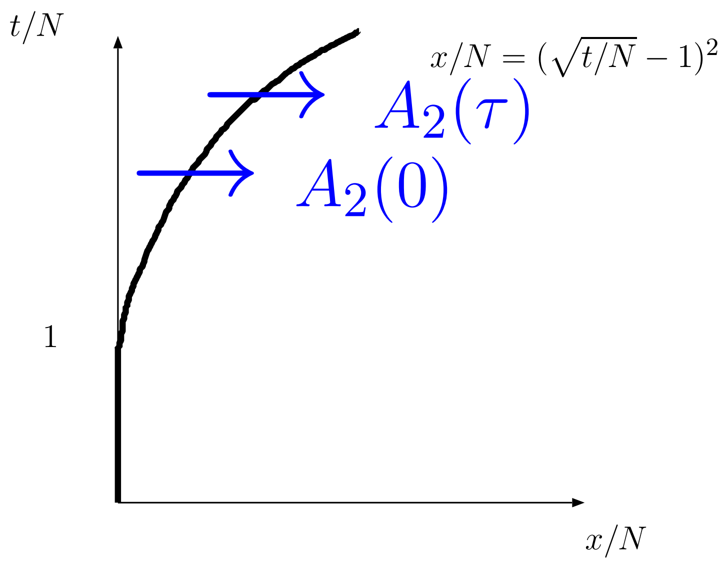
$$g_j(\tau) = \sqrt{\frac{\langle (A_j(\tau) - A_j(0))^2 \rangle}{2}} \quad (j = 1, 2)$$



4. Tagged particle problem

IS 2007

Position of Nth particle for the step initial condition.



Airy process $\Rightarrow$ Poster by Imamura

5. Summary

- Fluctuations of ASEP is often described by RMT
- Aztec diamond, Green's function $\rightarrow$ free fermion
- Spatial correlation for the alternating case

Future Problems

- Two-time correlation
- Finite geometry
- Partially ASEP

Conclusion

Wadati group is a good group

- Good people (students, postdocs, colleagues, etc)
- Good atmosphere (academic, international, free)
- Good Professor

When asked ”What is your best paper ?”,
Wadati-sensei replied ”The next one”.