Perspectives on Soliton Physics
February 17, 2007

Modulational instability in the presence of damping

Harvey Segur
University of Colorado

Joint work with:
J. Hammack, D. Henderson,
J. Carter, W. Craig,
C-M Li, M. Oscamou, D. Pheiff, K. Socha
Nonlinear wave propagation in the 1960s

1960 - first successful laser
1961 - observations of optical nonlinearity
Nonlinear wave propagation in the 1960s

1960 - first successful laser
1961 - observations of optical nonlinearity
1961 - Gross-Pitaevski eq’n
Nonlinear wave propagation in the 1960s

1960 - first successful laser
1961 - observations of optical nonlinearity
1961 - Gross-Pitaevski eq’n
1965 - Solitons! (Zabusky & Kruskal)
1967 - (Gardner, Greene, Kruskal, Miura)
Nonlinear wave propagation in the 1960s

1960 - first successful laser
1961 - observations of optical nonlinearity
1961 - Gross-Pitaevski eq’n
1965 - Solitons! (Zabusky & Kruskal)
1967 - (Gardner, Greene, Kruskal, Miura)
1967 - theory of modulational instability
   (Lighthill, Zakharov, Ostrovsky, Whitham, Benjamin & Feir, Benney & Newell)
Modulational instability

- Dispersive medium: waves at different frequencies travel at different speeds
- In a dispersive medium without dissipation, a uniform train of plane waves is likely to be unstable
Modulational instability

- Dispersive medium: waves at different frequencies travel at different speeds
- In a dispersive medium without dissipation, a uniform train of plane waves is likely to be unstable
- The unstable modes have nearly the same frequency as carrier wave
- Maximum growth rate:

\[ \Omega \propto |A|^2 \]
Experimental evidence of modulational instability in deep water - Benjamin & Feir (1967)

near the wavemaker 60 m downstream

from Benjamin (1967):
frequency = 0.85 Hz, wavelength = 2.2 m,
water depth = 7.6 m
Experimental evidence of modulational instability in an optical fiber

Hasegawa & Kodama
“Solitons in optical communications”
(1995)

Fig. 15.1 Experimental observation of modulational instability (Tai et al. 1986a). Input power level low (a); 5.5 $W$ (b); 6.1 $W$ (c); 7.1 $W$ (d). For details see text.
Experimental evidence of apparently stable wave patterns in deep water
(www.math.psu.edu/dmh/FRG)

3 Hz wave
17.3 cm wavelength

4 Hz wave
9.8 cm
Two questions to answer:

• Do we need to rethink the modulational instability? If so, why?

• Are the 2-dimensional wave patterns produced in the Penn State lab stable or unstable? For either answer, why?
More experimental results

(www.math.psu.edu/dmh/FRG)

3 Hz wave 2 Hz wave
(old water) (new water)
Main results

• The modulational (or Benjamin-Feir) instability is valid for waves in deep water without dissipation
Main results

• The modulational (or Benjamin-Feir) instability is valid for waves in deep water without dissipation

• But any amount of damping (of the right kind) stabilizes the instability

• This dichotomy (with vs. without damping) applies to both 1-D plane waves and to 2-D periodic surface patterns

• Segur, Henderson, Carter, Hammack, Li, Pheiff, Socha, *J. Fluid Mech.*, 539, 2005

• Controversial
To derive the nonlinear Schrödinger equation

Surface slow modulation fast phase
Elevation

$$\zeta(x,y,t) = \varepsilon [A(\varepsilon x, \varepsilon^2 x, \varepsilon y, \varepsilon t)e^{i\theta} + A^* e^{-i\theta}] + O(\varepsilon^2)$$

Velocity
Potential

$$\phi(x,y,z,t;\varepsilon) = O(\varepsilon)$$
NLS equation in 1-D

\[ i(\partial_t A + c \partial_x A) + \varepsilon [\alpha \partial_t^2 A + \xi |A|^2 A] = 0 \]

\[ [\tau = t - \frac{x}{c}, X = \varepsilon \frac{x}{c}] \]

\[ i \partial_x A + \alpha \partial_{\tau} A + \xi |A|^2 A = 0 \]
NLS equation in 1-D with damping

\[ i(\partial_t A + c \partial_x A) + \varepsilon [\alpha \partial_t^2 A + \xi |A|^2 A + i\delta A] = 0 \]

\[ \tau = t - \frac{x}{c}, X = \varepsilon \frac{x}{c} \]

\[ i\partial_x A + \alpha \partial^2_{\tau} A + \xi |A|^2 A + i\delta A = 0 \]

\[ [A(\tau,X) = e^{-\delta X} A(\tau,X)] \]

\[ i\partial_x A + \alpha \partial^2_{\tau} A + \xi \cdot e^{-2\delta X} |A|^2 A = 0 \]
NLS in 1-D, cont’d

\[ i \partial_x A + \alpha \partial_{\tau}^2 A + \xi \cdot e^{-2\delta X} |A|^2 A = 0 \]

Hamiltonian equation, but \( \frac{dH}{dX} \neq 0 \)

\[ H = i \int [\alpha |\partial_\tau A|^2 - \frac{\xi}{2} e^{-2\delta X} |A|^4]d\tau \]

Conjugate variables: \( A, A^* \)
\[ i \partial_X A + \alpha \partial^2_\tau A + \xi \cdot e^{-2\delta X} |A|^2 A = 0, \text{ cont'd} \]

- Uniform (in $\tau$) wave train:

\[ A = A_0 \exp \{i \xi |A_0|^2 \left( \frac{1-e^{-2\delta X}}{2\delta} \right) \} \]
\[i \partial_X A + \alpha \partial^2 \tau A + \xi \cdot e^{-2\delta X} |A|^2 A = 0, \text{ cont'd}\]

- Uniform (in \(\tau\)) wave train:
  \[A = A_0 \exp\{i \xi |A_0|^2 \left(\frac{1 - e^{-2\delta X}}{2\delta}\right)\}\]

- Perturb:
  \[A(\tau, X) = \exp\{i \xi |A_0|^2 \left(\frac{1 - e^{-2\delta X}}{2\delta}\right)\} |A_0| + \lambda(u + iv) + O(\lambda^2)\]
\[ i \partial_X A + \alpha \partial_\tau^2 A + \xi \cdot e^{-2\delta X} |A|^2 A = 0 \] , cont'd

- Uniform (in \( \tau \)) wave train:

\[ A = A_0 \exp\{i|\xi|A_0|^2 \left( \frac{1-e^{-2\delta X}}{2\delta} \right) \} \]

- Perturb:

\[ A(\tau,X) = \exp\{i|\xi|A_0|^2 \left( \frac{1-e^{-2\delta X}}{2\delta} \right) \}[|A_0| + \lambda(u + iv)] + O(\lambda^2) \]

- ...algebra...

\[ \frac{d^2 \hat{u}}{dX^2} + \left[ \alpha q^2 (\alpha q^2 - 2\xi \cdot e^{-2\delta X} |A_0|^2) \right] \cdot \hat{u} = 0 \]
\[
\frac{d^2 \hat{u}}{dX^2} + \left[ \alpha q^2 (\alpha q^2 - 2 \xi \cdot e^{-2\delta X} |A_0|^2) \right] \cdot \hat{u} = 0
\]

Fig. 15.1 Experimental observation of modulational instability (Tai et al. 1986a). Input power level low (a); 5.5 \( W \) (b); 6.1 \( W \) (c); 7.1 \( W \) (d). For details see text.

If we eliminate \( \sigma_1 \) from (15.1.11) and (15.1.12) and construct the differential equation for the normalized side band amplitude \( \hat{p}_1 = \rho_1 / \rho_0 \) (\( \rho_0 \) is given by (15.1.9)), we get

\[
\frac{d^2 \hat{p}}{dZ^2} - \Omega^2 \left( \tilde{\rho}_0 e^{-2\Omega Z} - \frac{\Omega^2}{4} \right) \hat{p} = 0 .
\]  

(15.2.1)

If we introduce a quantity \( R \) which designates the ratio of \( \Omega^2 \) to \( \rho_0 \), \( R = \Omega^2 / \tilde{\rho}_0 \), \( R \) may be expressed in terms of engineering parameters as

\[
R = \frac{\Omega^2}{\rho_0} = 1.1 \times 10^4 \frac{f^2 S}{P} (-\lambda^2 D) ,
\]

(15.2.2)

Hasegawa & Kodama

(1995)
\[ \frac{d^2 \hat{u}}{dX^2} + \left[ \alpha q^2 (\alpha q^2 - 2 \xi \cdot e^{-2\delta X} \mid A_0 \mid^2) \right] \cdot \hat{u} = 0 \]
\[ \frac{d^2 \hat{u}}{dX^2} + \left[ \alpha q^2 (\alpha q^2 - 2\xi \cdot e^{-2\delta X} | A_0 |^2) \right] \cdot \hat{u} = 0, \text{ cont’d} \]

- There is a growing mode if

\[ [\alpha q^2 (\alpha q^2 - 2\xi \cdot e^{-2\delta X} | A_0 |^2)] < 0 \]
\[
\frac{d^2 \hat{u}}{dX^2} + \left[ \alpha q^2 (\alpha q^2 - 2 \xi \cdot e^{-2\delta X} \left| A_0 \right|^2) \right] \cdot \hat{u} = 0, \text{ cont'd}
\]

- There is a growing mode if

\[
\left[ \alpha q^2 (\alpha q^2 - 2 \xi \cdot e^{-2\delta X} \left| A_0 \right|^2) \right] < 0
\]

- For any $\delta > 0$, growth stops eventually
  - No mode grows forever
  - Total growth is bounded
What is “linearized stability”? (Lyapunov)

The uniform wave train solution is linearly stable if for every $\varepsilon > 0$ there is a $\Delta(\varepsilon) > 0$ such that if a perturbation $(u,v)$ satisfies

$$\int [u^2(\tau,0) + v^2(\tau,0)]d\tau < \Delta(\varepsilon)$$

at $X = 0$, then necessarily

$$\int [u^2(\tau,X) + v^2(\tau,X)]d\tau < \varepsilon$$

for all $X > 0$. 
1-D NLS with damping, conclusion

\[ \frac{d^2 \hat{u}}{dX^2} + \left[ \alpha q^2 (\alpha q^2 - 2 \xi \cdot e^{-2\delta X} | A_0 |^2) \right] \cdot \hat{u} = 0 \]

- There is a universal bound, B: the total growth of any Fourier mode cannot exceed B

- To demonstrate stability, choose \( \Delta(\varepsilon) \) so that

\[ \Delta(\varepsilon) < \frac{1}{B^2 \cdot \varepsilon} \]

Nonlinear stability is similar, but more complicated
Experimental verification of theory

1-D tank at Penn State
Experimental wave records

\[ \mathbf{X}_1 \]

\[ \mathbf{X}_8 \]
Amplitudes of seeded sidebands (damping factored out of data)

- - - - Benjamin-Feir growth rate
• • • experimental data
Amplitudes of unseeded sidebands (damping factored out of data)

---

**damped NLS theory**

• • • experimental data
Amplitude of carrier wave, harmonic
(damping factored out of data)

Decay rate of 2nd harmonic is \textit{twice} that of carrier wave. Stokes (1847):

\[ A_2(\tau, X) = k_o \{ A(\tau, X) \}^2 \]
Numerical simulations of full water wave equations, plus damping

Wu, Liu & Yue,
*J Fluid Mech*, 556, 2006
Are the 2-D patterns also stable?
Q: How to make 2-D periodic surface wave patterns experimentally?

One method:

\[ \eta = a \cdot \cos(kx + ly - \omega t) + a \cdot \cos(kx - ly - \omega t) + O(a^2) \]

\[ = 2a \cdot \cos(kx - \omega t) \cdot \cos(ly) + O(a^2) \]
2 coupled NLS equations, with damping:

\[
0 = i(\partial_t A + u \partial_x A + v \partial_y A) + \\
\varepsilon[\alpha \partial_t^2 A + \beta \partial_y^2 A + \gamma \partial_t \partial_y A + \xi |A|^2 A + \zeta |B|^2 A + i \delta A],
\]

\[
0 = i(\partial_t B + u \partial_x B - v \partial_y B) + \\
\varepsilon[\alpha \partial_t^2 B + \beta \partial_y^2 B - \gamma \partial_t \partial_y B + \xi |B|^2 B + \zeta |A|^2 B + i \delta B].
\]
Linear stability, with damping:
(preliminary result)
Thank you for your attention
How to measure $\delta$?

$$i \partial_X A + \alpha \partial^2_\tau A + \xi |A|^2 A + i\delta A = 0$$

Integral quantities of interest:

$$M(X) = \int |A(\tau,X)|^2 d\tau, \quad M(X) = M(0) \cdot e^{-2\delta X}$$

$$P(X) = i \int [A \partial_\tau A^* - A^* \partial_\tau A] d\tau, \quad P(X) = P(0) \cdot e^{-2\delta X}$$
Theory for 2-D periodic surface patterns

- 2 coupled NLS equations with damping
- No preferred coordinate system
- Change variables

\[ A(x,y,t) = e^{-\varepsilon \delta x} A(x,y,t), \]
\[ B(x,y,t) = e^{-\varepsilon \delta x} B(x,y,t). \]

- The new equations are Hamiltonian, \( \frac{dH}{dx} \neq 0 \)
- \( \Rightarrow \) linearized stability (in Lyapunov sense)
Stabilizing the Benjamin-Feir (or modulational) instability
(no relation to tsunamis)

Harvey Segur
University of Colorado, USA
Joint work with:
D. Henderson, J. Carter, W. Craig, J. Hammack, C-M Li,
M. Oscamou, D. Pheiff, K. Socha
Stable patterns of surface waves in deep water

by

Joe Hammack† (1944-2004)
Diane Henderson (Penn State)
Harvey Segur (Colorado)
Maribeth Bleymaier, John Carter,
Cong-Ming Li, Dana Pheiff, Katherine Socha

NCAR workshop on
Coherent Structures in Atmosphere and Ocean
Boulder, CO
July 13, 2005
If stable patterns of surface waves exist in deep water, then they are *Coherent Structures*.

Do stable wave patterns exist in deep water?
Stabilizing the Benjamin-Feir (or modulational) instability

Harvey Segur
University of Colorado, USA
Joint work with:
Equations of (inviscid) water waves

(i) On bottom, $z = -h(x,y)$

$$\vec{u} \cdot \nabla (z + h(x,y)) = 0$$

(ii) In fluid, $-h < z < \zeta(x,y,t)$

$$\vec{u} = \nabla \phi, \quad \nabla^2 \phi = 0$$

(iii) At free surface, $z = \zeta(x,y,t)$

$$\partial_t \zeta + \nabla \phi \cdot \nabla \zeta = \partial_z \phi,$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g \zeta = 0.$$  

(iv) Ignore viscosity, surface tension, variable density, fish, …
What about a higher order NLS model (like Dysthe)?

___, damped NLS    ----, NLS    - - -, Dysthe

• • •, experimental data
Stable, periodic wave patterns in deep water

Harvey Segur
University of Colorado

Joint work with:
J. Hammack, D. Henderson,
J. Carter, W. Craig,
C-M Li, M. Oscamou, D. Pheiff, K. Socha
Equations of (inviscid) water waves, in deep water

(i) In fluid, $z < \zeta(x,y,t)$

$$\vec{u} = \nabla \phi, \quad \nabla^2 \phi = 0$$

(ii) As $z \to -\infty$, $|u| \to 0$

(iii) At free surface, $z = \zeta(x,y,t)$

$$\partial_t \zeta + \nabla \phi \cdot \nabla \zeta = \partial_z \phi,$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g \zeta = 0.$$

(iv) Ignore viscosity, surface tension, variable density, fish, ...
Basic facts about wave propagation
(according to linear theory)

Sound waves
• All travel at the same speed (speed of sound)

Water waves
• Longer waves travel faster than shorter waves
  (for gravity-induced surface water waves)
Basic question:
Is a uniform train of 1-D surface waves of finite amplitude on deep water stable?
Basic question:
Is a uniform train of 1-D surface waves of finite amplitude on deep water stable?

Slight variation:
Is a uniform train of 1-D electromagnetic waves of finite amplitude in a dispersive optical fiber stable?
Basic question:
Is a uniform train of 1-D surface waves of finite amplitude on deep water stable?

Variations:
Is a uniform train of 1-D electromagnetic waves of finite amplitude in an optical fiber stable?

What about Langmuir waves in a plasma? Spin waves in a thin magnetic film?
How to reconcile the experimental observations with Benjamin-Feir instability?

Recall: In deep water without dissipation, a uniform train of monochromatic plane waves (with 1-D surface patterns) with finite amplitude (|A|) is unstable to small perturbations with nearly the same frequency.

The maximum growth rate of the instability is

$$\Omega_{\text{max}} = C |A|^2$$
How to reconcile the experimental observations with Benjamin-Feir instability?

Options

• Modulational instability afflicts 1-D plane waves, but not 2-D periodic patterns

• The Penn State tank is too short to observe the (relatively slow) growth of the instability

• Other (please specify)
Skip the detailed analysis

How does damping affect the modulational instability?

• Common effect
  calculate theoretical growth rate, without damping

Observed decay rate = predicted growth rate - decay rate
Skip the detailed analysis

How does damping affect the modulational instability?

- 2nd (important) effect
  predicted growth rate without damping

\[ \Omega \propto |A|^2 \]
How does damping affect the modulational instability?

- 2nd (important) effect

Predicted growth rate without damping

\[ \Omega \propto |A|^2 \]

Predicted "growth rate" with damping

\[ \Omega \propto |A|^2 e^{-2t\delta} \]
Skip the detailed analysis

How does damping affect the modulational instability?

\[ \Omega \propto |A|^2 e^{-2t\delta} \]

- For any \( \delta > 0 \), growth stops eventually
- No mode grows forever
- Total growth is bounded