Perspectives on Soliton Physics

February 17, 2007

Modulational instability in the presence of damping

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Joint work with:

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Nonlinear wave propagation in the 1960s

1960 - first successful laser

1961 - observations of optical nonlinearity

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1965 - Solitons! (Zabusky & Kruskal)

1967 - (Gardner, Greene, Kruskal, Miura)

1967 - theory of modulational instability

(Lighthill, Zakharov, Ostrovsky, Whitham, Benjamin & Feir, Benney & Newell)

Modulational instability

- Dispersive medium: waves at different frequencies travel at different speeds
- In a dispersive medium without dissipation, a uniform train of plane waves is likely to be unstable

Modulational instability

- Dispersive medium: waves at different frequencies travel at different speeds
- In a dispersive medium without dissipation, a uniform train of plane waves is likely to be unstable
- The unstable modes have nearly the same frequency as carrier wave
- Maximum growth rate:

$$\Omega \propto |A|^2$$

Experimental evidence of modulational instability in deep water - Benjamin & Feir (1967)

near the wavemaker

60 m downstream

from Benjamin (1967):

frequency = 0.85 Hz, wavelength = 2.2 m,

water depth = 7.6 m

Experimental evidence of modulational instability in an optical fiber

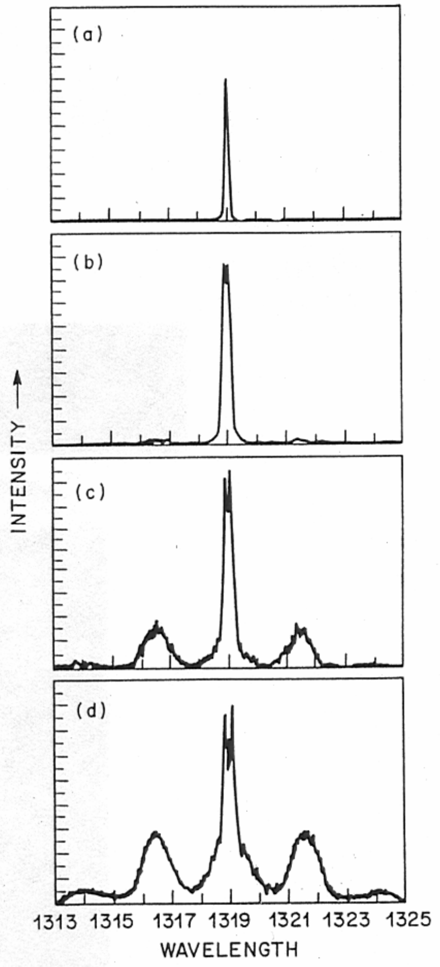


Fig.15.1 Experimental observation of modulational instability (Tai *et al.* 1986a). Input power level low (a); 5.5 *W* (b); 6.1 *W* (c); 7.1 *W* (d). For details see text.

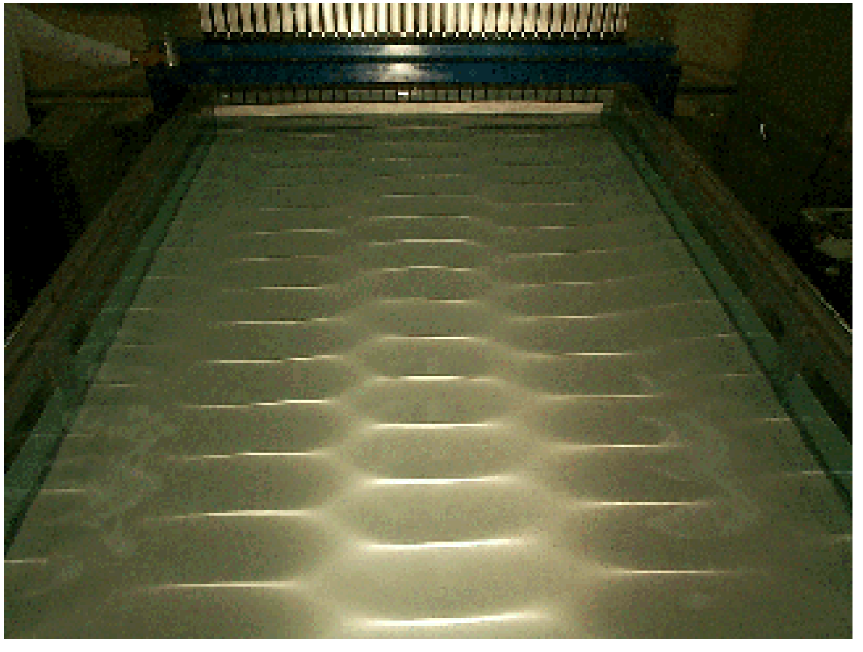
Hasegawa & Kodama

"Solitons in optical communications"

(1995)

Experimental evidence of apparently stable wave patterns in deep water

(www.math.psu.edu/dmh/FRG)



QuickTime™ and a
Motion JPEG OpenDML decompressor
are needed to see this picture.

3 Hz wave 4 Hz wave

17.3 cm wavelength 9.8 cm

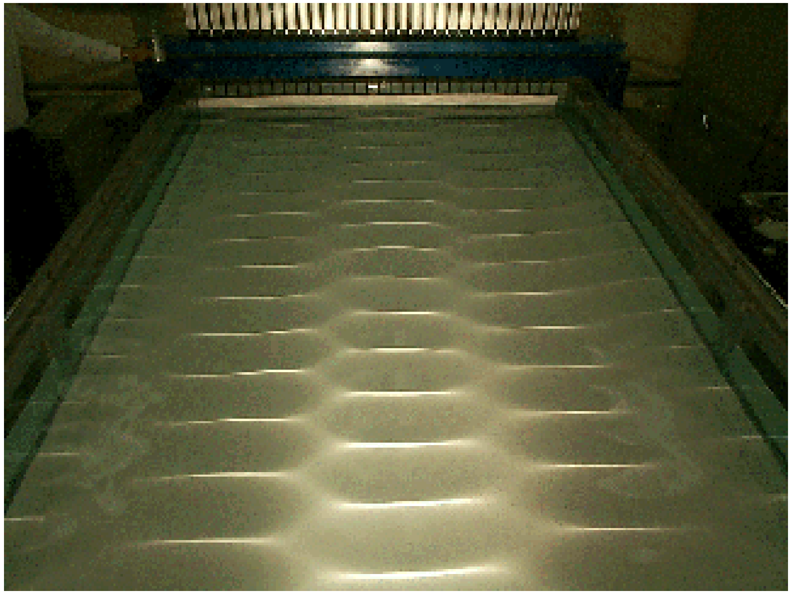
Two questions to answer:

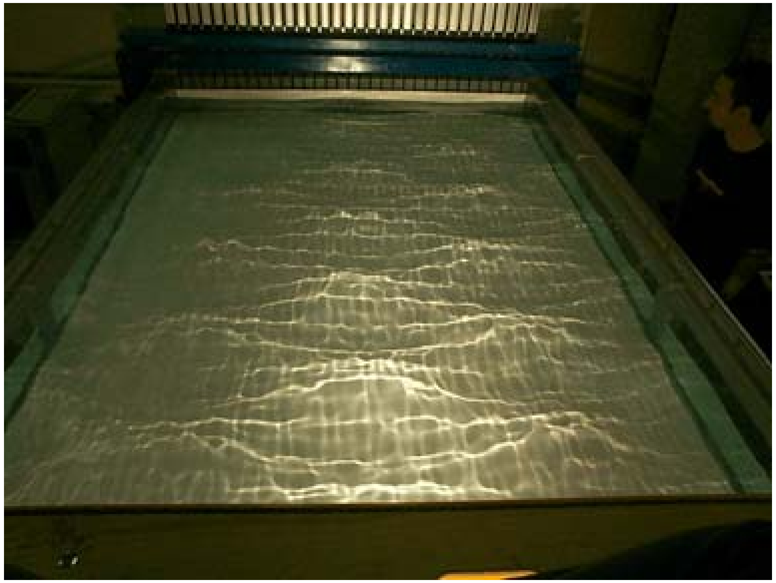
- Do we need to rethink the modulational instability? If so, why?

- Are the 2-dimensional wave patterns produced in the Penn State lab stable or unstable? For either answer, why?

More experimental results

(www.math.psu.edu/dmh/FRG)





3 Hz wave (old water)

2 Hz wave (new water)

Main results

- The modulational (or Benjamin-Feir) instability is valid for waves in deep water without dissipation

Main results

- The modulational (or Benjamin-Feir) instability is valid for waves in deep water without dissipation

- But **any** amount of damping (of the right kind) stabilizes the instability
- This dichotomy (with *vs*. without damping) applies to both 1-D plane waves and to 2-D periodic surface patterns
- Segur, Henderson, Carter, Hammack, Li, Pheiff, Socha, *J. Fluid Mech*., **539**, 2005
- Controversial

To derive the nonlinear Schrödinger equation

Surface Elevation

slow modulation

fast phase

$$\zeta(x,y,t) = \varepsilon[A(\varepsilon x, \varepsilon^2 x, \varepsilon y, \varepsilon t)e^{i\theta} + A^* e^{-i\theta}] + O(\varepsilon^2)$$

Velocity Potential

$$\phi(x,y,z,t;\varepsilon) = O(\varepsilon)$$

NLS equation in 1-D

$$i(\partial_t A + c\partial_x A) + \varepsilon[\alpha\partial_t^2 A + \xi |A|^2 A \qquad] = 0$$

$$[\tau = t - \frac{x}{c}, X = \varepsilon\frac{x}{c}]$$

$$i\partial_X A + \alpha\partial_\tau^2 A + \xi |A|^2 A \qquad = 0$$

NLS equation in 1-D with damping

$$i(\partial_t A + c\partial_x A) + \varepsilon[\alpha\partial_t^2 A + \xi |A|^2 A + i\delta A] = 0$$

$$[\tau = t - \frac{x}{c}, X = \varepsilon\frac{x}{c}]$$

$$i\partial_X A + \alpha\partial_\tau^2 A + \xi |A|^2 A + i\delta A = 0$$

$$[A(\tau, X) = e^{-\delta X}\,\mathrm{A}(\tau, X)]$$

$$i\partial_X \mathrm{A} + \alpha\partial_\tau^2 \mathrm{A} + \xi \cdot e^{-2\delta X} |\mathrm{A}|^2 \mathrm{A} = 0$$

NLS in 1-D, cont'd

$$i\partial_X \mathrm{A} + \alpha\partial_\tau^2 \mathrm{A} + \xi \cdot e^{-2\delta X} \,|\, \mathrm{A} \,|^2\, \mathrm{A} = 0$$

Hamiltonian equation, but $\dfrac{dH}{dX} \neq 0$

$$H = i\int [\alpha \,|\, \partial_\tau \mathrm{A} \,|^2 - \frac{\xi}{2} e^{-2\delta X} \,|\, \mathrm{A} \,|^4] d\tau$$

Conjugate variables: A, A*

$$i\partial_X \mathrm{A} + \alpha\partial_\tau^2 \mathrm{A} + \xi \cdot e^{-2\delta X} \, | \mathrm{A} |^2 \, \mathrm{A} = 0$$, cont'd

- Uniform (in τ) wave train:

$$\mathrm{A} = \mathrm{A}_0 \exp\{i\xi \, | \mathrm{A}_0 |^2 \, (\frac{1 - e^{-2\delta X}}{2\delta})\}$$

$$i\partial_X \mathrm{A} + \alpha\partial_\tau^2 \mathrm{A} + \xi \cdot e^{-2\delta X} \ |\mathrm{A}|^2 \ \mathrm{A} = 0$$, cont'd

- Uniform (in τ) wave train:

$$\mathrm{A} = \mathrm{A}_0 \exp\{i\xi \,|\, \mathrm{A}_0 \,|^2 \,(\frac{1-e^{-2\delta X}}{2\delta})\}$$

- Perturb:

$$\mathrm{A}(\tau, X) = \exp\{i\xi \,|\, \mathrm{A}_0 \,|^2 \,(\frac{1-e^{-2\delta X}}{2\delta})\}[|\,\mathrm{A}_0\,| + \lambda(u + iv)\} + O(\lambda^2)$$

$$i\partial_X \mathrm{A} + \alpha\partial_\tau^2 \mathrm{A} + \xi \cdot e^{-2\delta X} \, |\mathrm{A}|^2 \, \mathrm{A} = 0$$, cont'd

- Uniform (in τ) wave train:

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- …algebra..

$$\frac{d^2\hat{u}}{dX^2} + \mathbf{[}\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \, |\mathrm{A}_0|^2)\mathbf{]} \cdot \hat{u} = 0$$

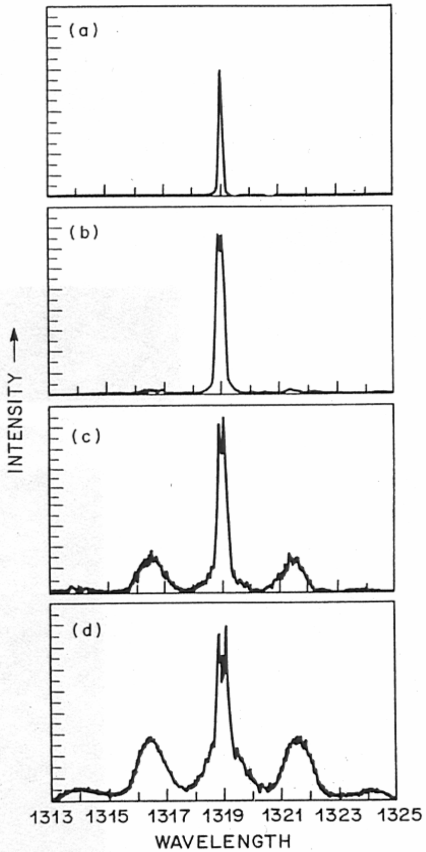


Fig.15.1 Experimental observation of modulational instability (Tai *et al.* 1986a). Input power level low (a); 5.5 W (b); 6.1 W (c); 7.1 W (d). For details see text.

If we eliminate σ_1 from (15.1.11) and (15.1.12) and construct the differential equation for the normalized side band amplitude $\overline{\rho}_1 = \rho_1/\rho_0$ (ρ_0 is given by (15.1.9)), we get

$$\frac{d^2\overline{\rho}}{dZ^2} - \Omega^2\left(\overline{\rho}_0 e^{-2\Gamma Z} - \frac{\Omega^2}{4}\right)\overline{\rho} = 0 \, . \qquad (15.2.1)$$

If we introduce a quantity R which designates the ratio of Ω^2 to ρ_0, $R = \Omega^2/\overline{\rho}_0$, R may be expressed in terms of engineering parameters as

$$R = \frac{\Omega^2}{\rho_0} = 1.1 \times 10^4 \frac{f^2 S}{P}(-\lambda^3 D) \, , \qquad (15.2.2)$$

$$\frac{d^2\hat{u}}{dX^2} + [\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \, |\mathrm{A}_0|^2)] \cdot \hat{u} = 0$$

Hasegawa &Kodama

(1995)

$$\frac{d^2\hat{u}}{dX^2} + \mathbf{[}\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \mid \mathrm{A}_0 \mid^2)\mathbf{]} \cdot \hat{u} = 0$$, cont’d

$$\frac{d^2\hat{u}}{dX^2} + \left[\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \mid \mathrm{A}_0 \mid^2)\right] \cdot \hat{u} = 0$$, cont'd

- There is a growing mode if

$$[\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \mid A_0 \mid^2)] < 0$$

$$\frac{d^2\hat{u}}{dX^2} + \mathbf{[}\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \, |\, \mathrm{A}_0\,|^2)\mathbf{]} \cdot \hat{u} = 0$$, cont'd

- There is a growing mode if

$$[\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \, |\, A_0\,|^2)] < 0$$

- For any $\delta > 0$, growth stops eventually

➔No mode grows forever

➔Total growth is bounded

What is "linearized stability"? (Lyapunov)

The uniform wave train solution is linearly stable if for every $\varepsilon > 0$ there is a $\Delta(\varepsilon) > 0$ such that if a perturbation (u,v) satisfies

$$\int [u^2(\tau,0) + v^2(\tau,0)]d\tau < \Delta(\varepsilon) \qquad \text{at } X = 0,$$

then necessarily

$$\int [u^2(\tau,X) + v^2(\tau,X)]d\tau < \varepsilon \qquad \text{for all } X > 0.$$

1-D NLS with damping, conclusion

$$\frac{d^2\hat{u}}{dX^2} + [\alpha q^2(\alpha q^2 - 2\xi \cdot e^{-2\delta X} \mid \mathrm{A}_0 \mid^2)] \cdot \hat{u} = 0$$

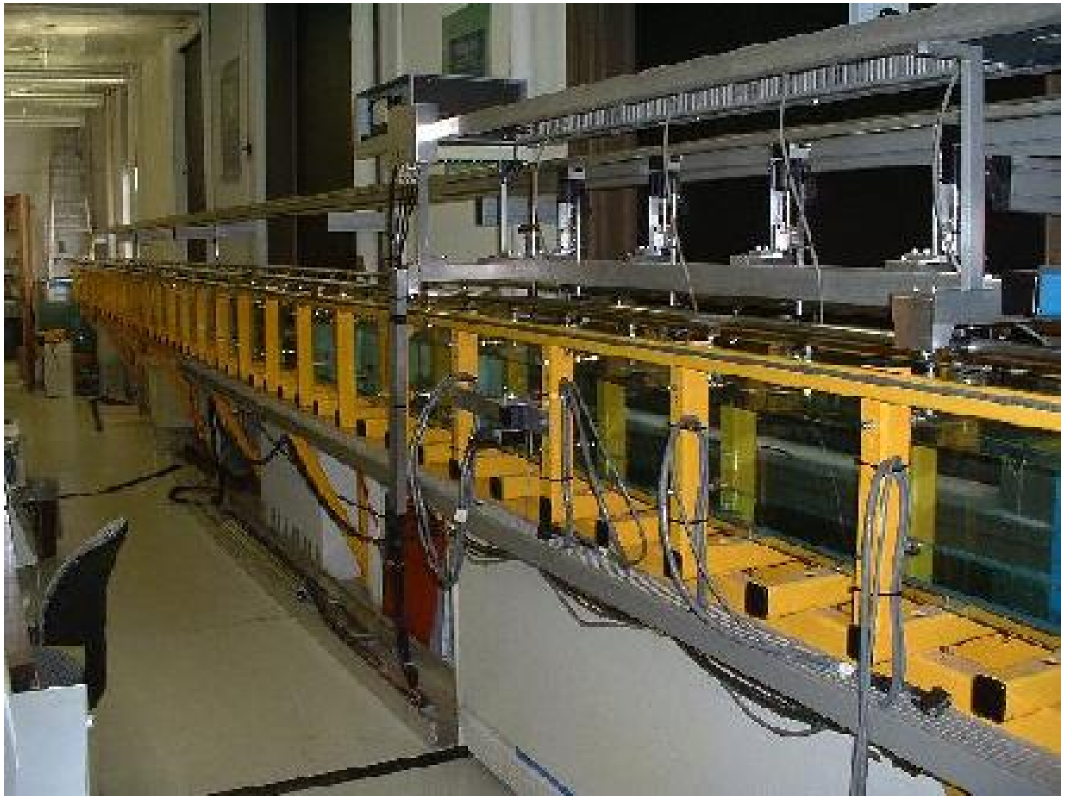
➔There is a universal bound, B: the total growth of any Fourier mode cannot exceed B

➔To demonstrate stability, choose $\Delta(\varepsilon)$ so that

$$\Delta(\varepsilon) < \frac{1}{B^2} \cdot \varepsilon$$

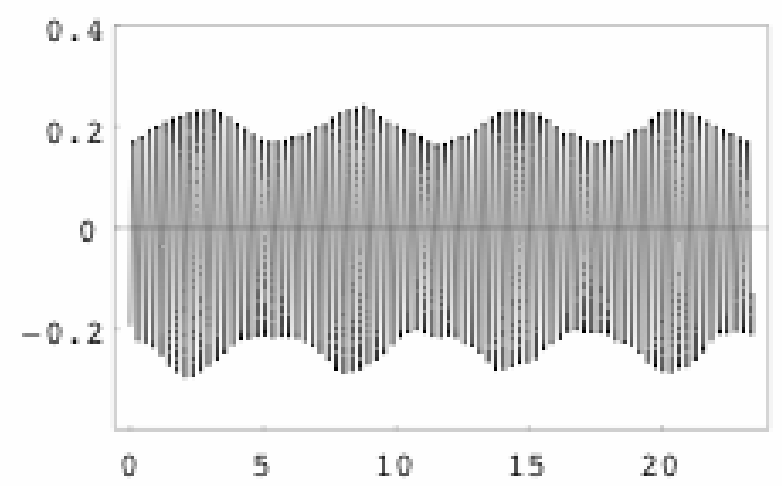
Nonlinear stability is similar, but more complicated

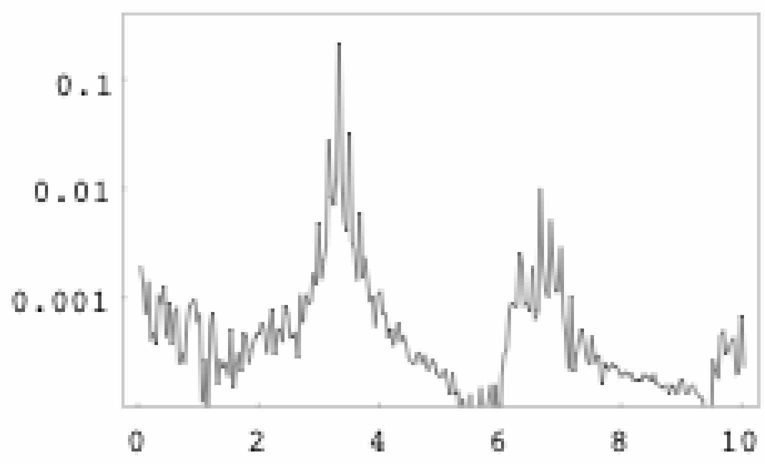
Experimental verification of theory



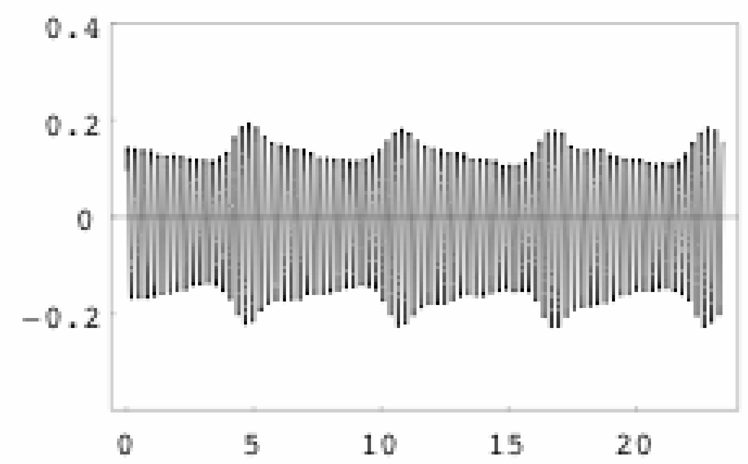
1-D tank at Penn State

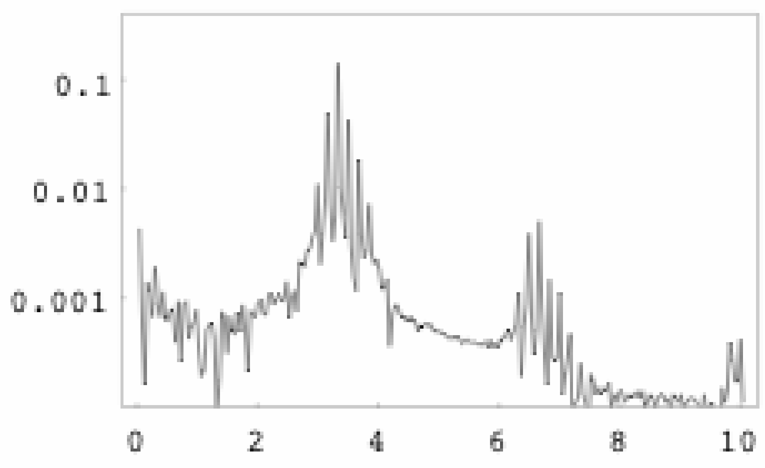
Experimental wave records





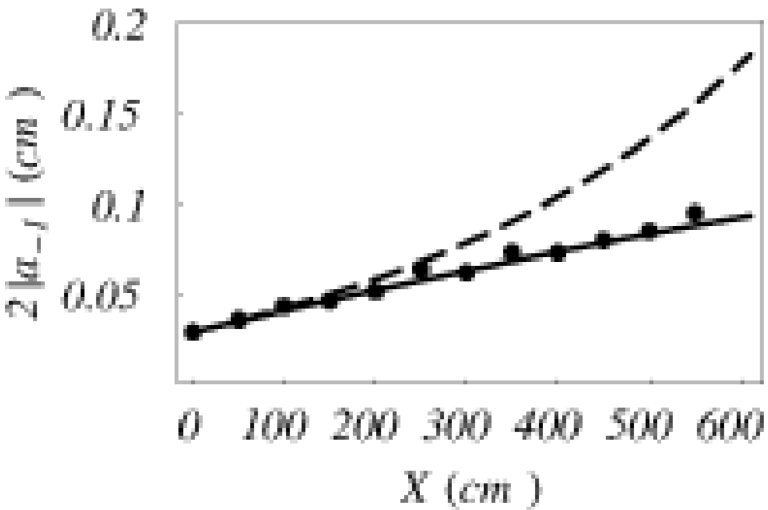
X_1

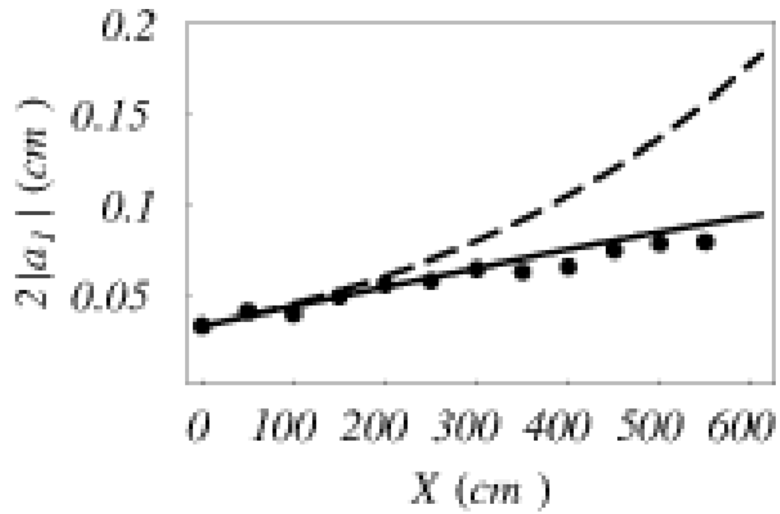




X_8

Amplitudes of seeded sidebands
(damping factored out of data)



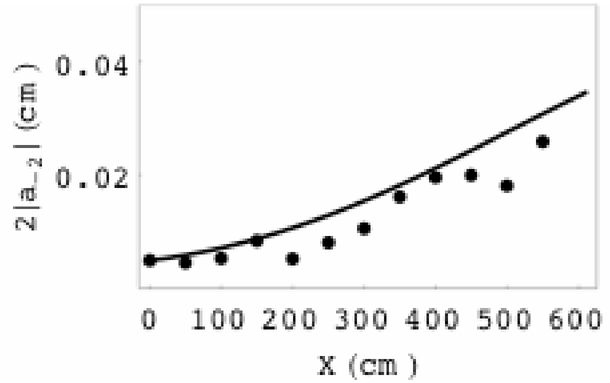


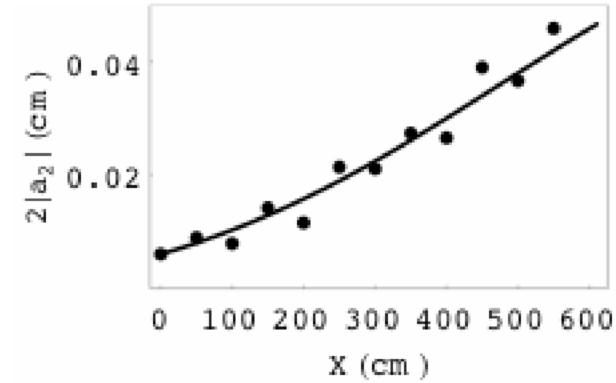
_____ damped NLS theory

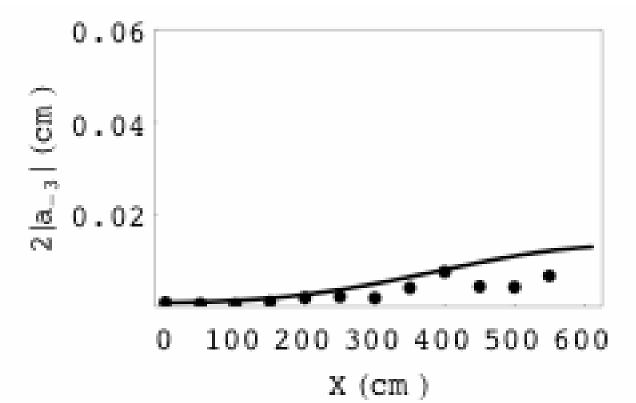
- - - Benjamin-Feir growth rate

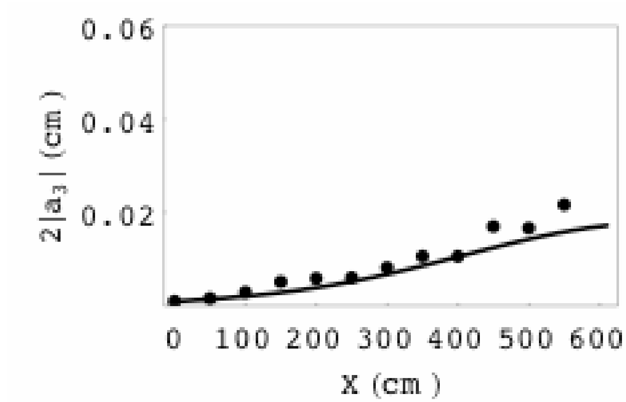
• • • experimental data

Amplitudes of unseeded sidebands
(damping factored out of data)





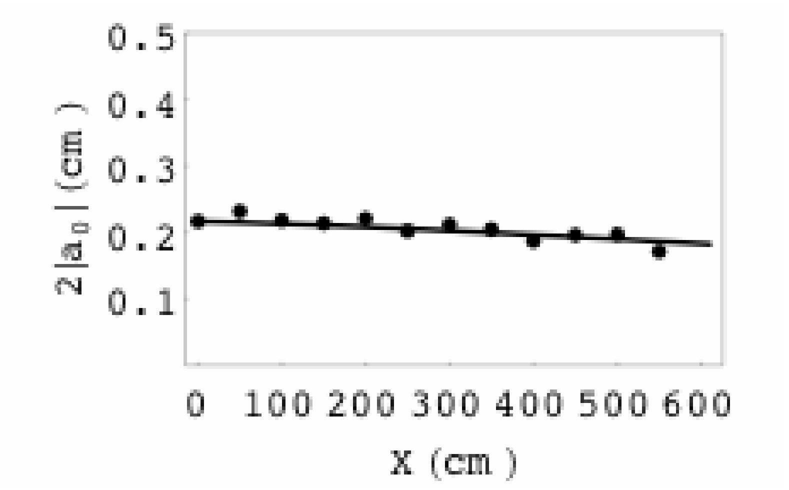


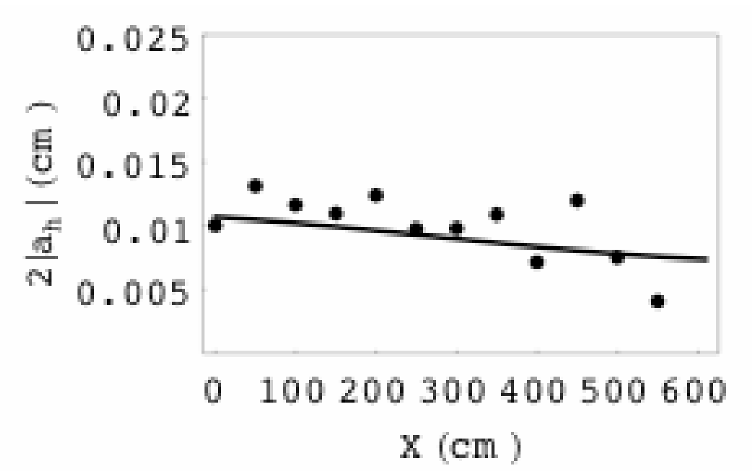


____ damped NLS theory

• • • experimental data

Amplitude of carrier wave, harmonic
(damping factored out of data)





Decay rate of 2nd harmonic is *twice* that of carrier wave. Stokes (1847):

$$A_2(\tau, X) = k_o\{A(\tau, X)\}^2$$

Numerical simulations of full water wave equations, plus damping

A note on stabilizing the Benjamin-Feir instability 5

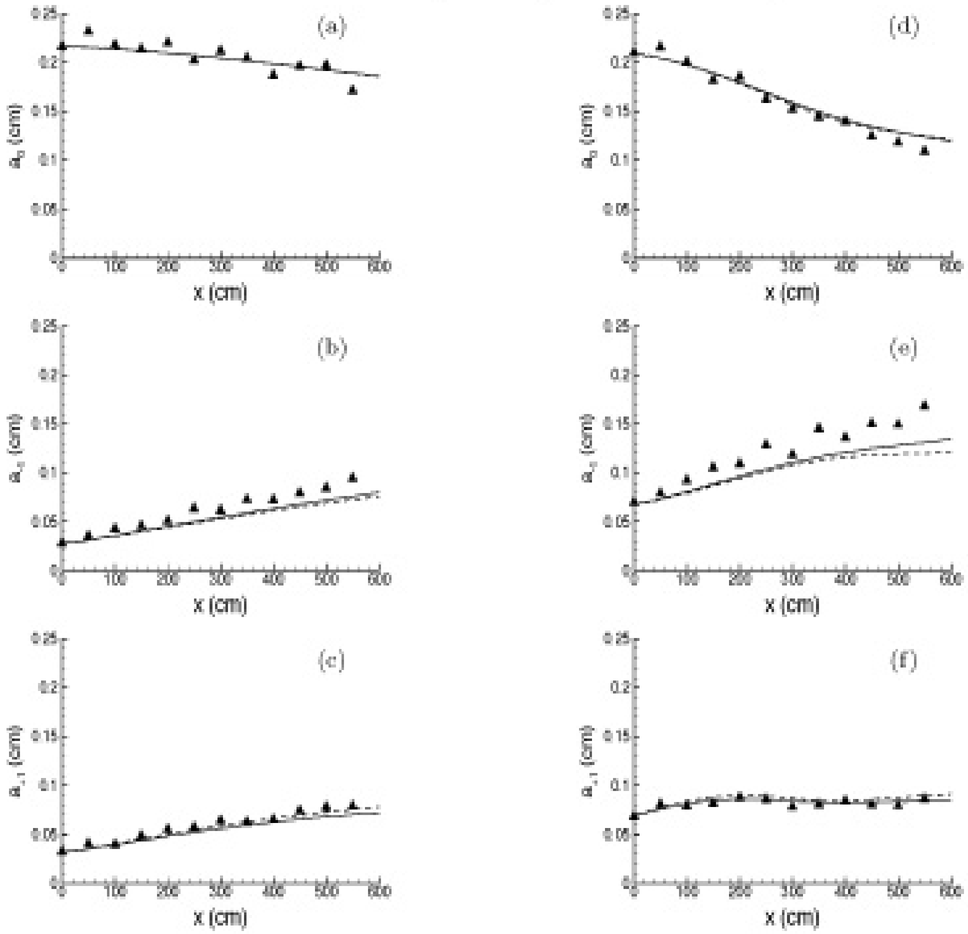
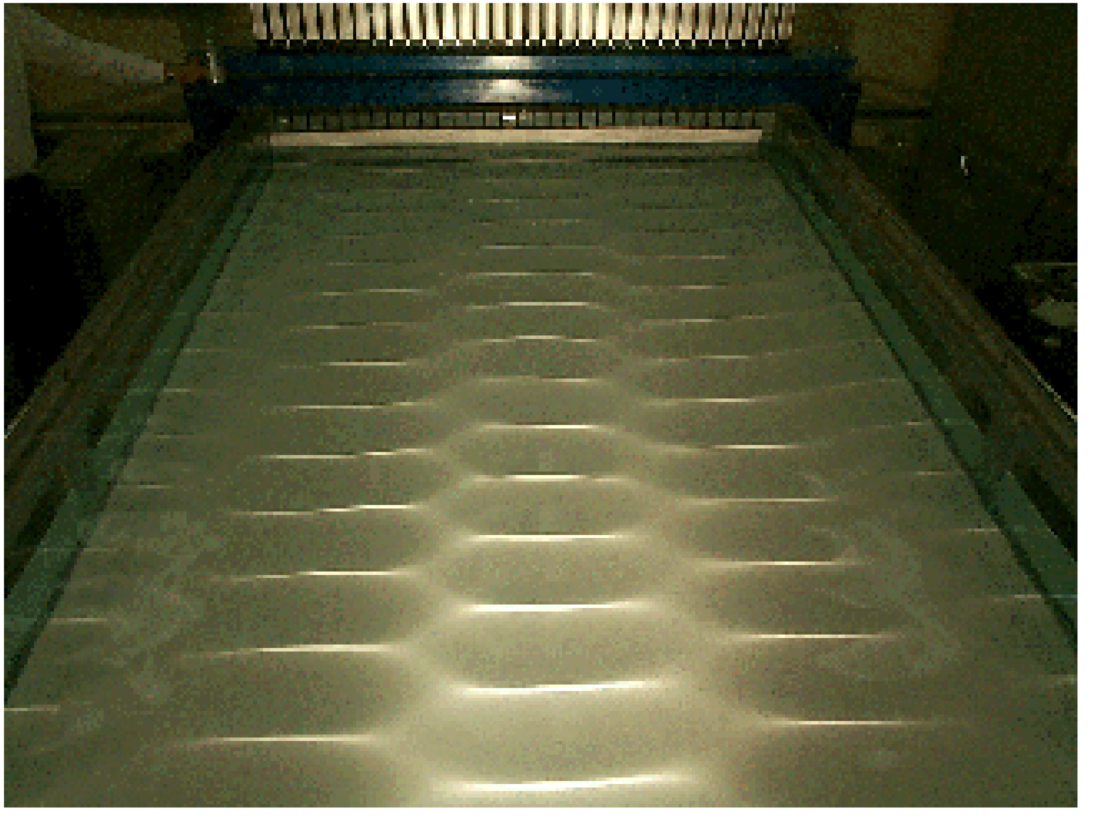


FIGURE 1. Comparisons of the HOS simulations (Model I: - - -; Model II: ——) with the experiments of **BF** (▲) for wave amplitudes in the decaying frame. (a), (d): carrier wave a_0; (b), (e): lower sideband a_{-1}; and (c), (f): upper sideband a_{+1}; as functions of distance from the wavemaker for the evolution of small-amplitude ((a), (b), (c)) and large-amplitude ((d), (e), (f)) wave trains. (x=0 is 128 cm from the wavemaker.)

Wu, Liu & Yue,
J Fluid Mech, **556**,
2006

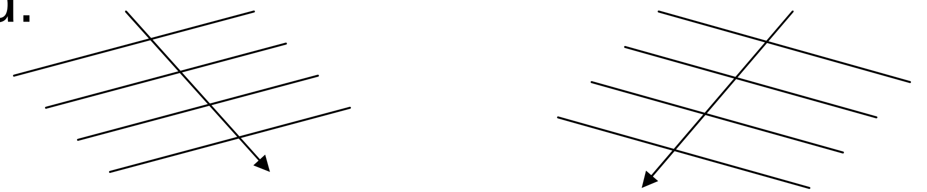
Are the 2-D patterns also stable?



2-D periodic surface patterns

Q: How to make 2-D periodic surface wave patterns experimentally?

One method:

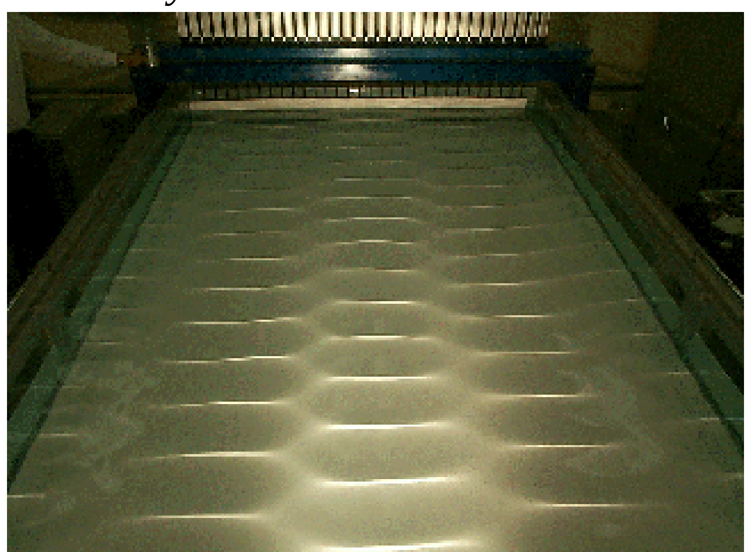


$$\eta = a \cdot \cos(kx + ly - \omega t) + a \cdot \cos(kx - ly - \omega t) + O(a^2)$$
$$= 2a \cdot \cos(kx - \omega t) \cdot \cos(ly) + O(a^2)$$

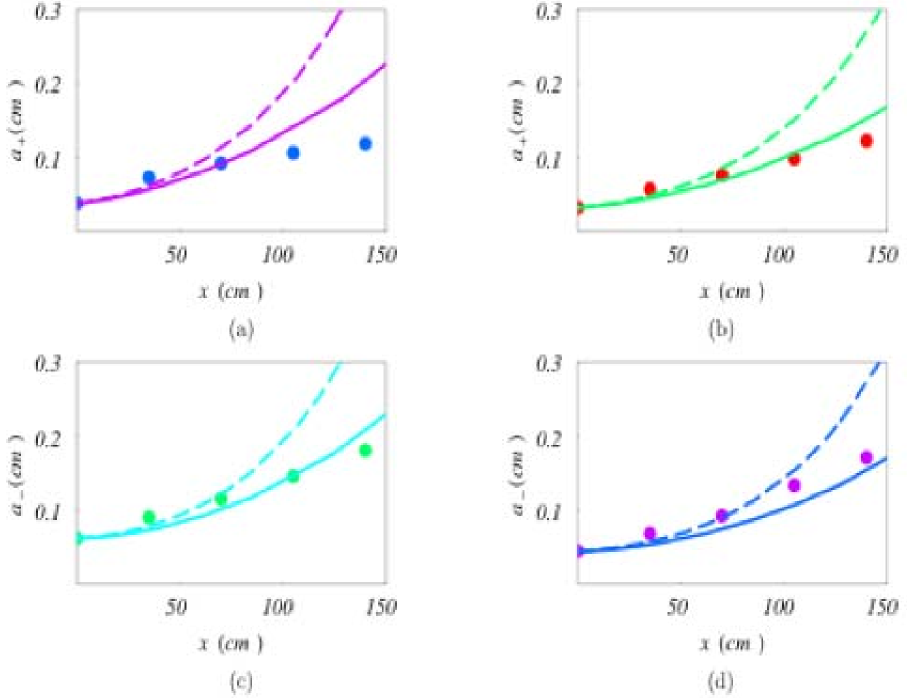
2 coupled NLS equations, with damping:

$$0 = i(\partial_t A + u\partial_X A + v\partial_y A) +$$
$$\varepsilon[\alpha\partial_t^2 A + \beta\partial_y^2 A + \gamma\partial_t\partial_y A + \xi\,|A|^2\,A + \zeta\,|B|^2\,A + i\delta A],$$

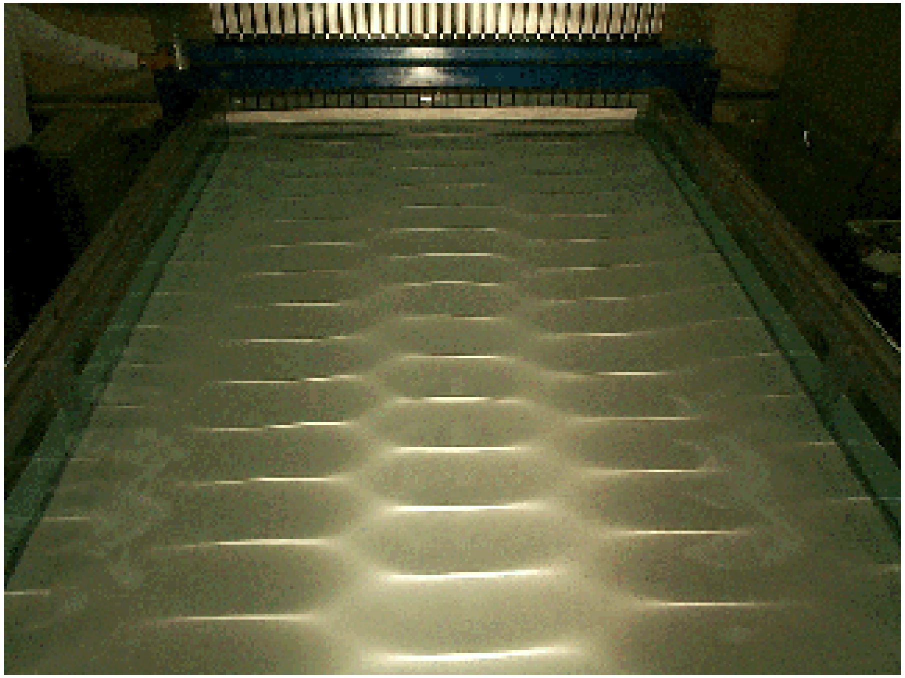
$$0 = i(\partial_t B + u\partial_x B - v\partial_y B) +$$
$$\varepsilon[\alpha\partial_t^2 B + \beta\partial_y^2 B - \gamma\partial_t\partial_y B + \xi\,|B|^2\,B + \zeta\,|A|^2\,B + i\delta B].$$



Linear stability, with damping:
(preliminary result)



Thank you for your attention



How to measure δ?

$$i\partial_X A + \alpha\partial_\tau^2 A + \xi \,|\, A \,|^2\, A + i\delta A = 0$$

Integral quantities of interest:

$M(X) = \int |\, A(\tau, X)\,|^2 d\tau$, $\qquad M(X) = M(0)\cdot e^{-2\delta X}$

$P(X) = i\int [A\partial_\tau A^* - A^*\partial_\tau A] d\tau$, $\qquad P(X) = P(0)\cdot e^{-2\delta X}$

Theory for 2-D periodic surface patterns

- 2 coupled NLS equations with damping
- No preferred coordinate system
- Change variables

$$A(x,y,t) = e^{-\varepsilon\delta x}\mathrm{A}(x,y,t),$$
$$B(x,y,t) = e^{-\varepsilon\delta x}\mathrm{B}(x,y,t).$$

- The new equations are Hamiltonian, $\frac{dH}{dx} \neq 0$

- => linearized stability (in Lyapunov sense)

Conference on tsunami and nonlinear waves

Stabilizing the Benjamin-Feir (or modulational) instability

(no relation to tsunamis)

Harvey Segur

University of Colorado, USA

Joint work with:

D. Henderson, J. Carter, W. Craig, J. Hammack, C-M Li, M. Oscamou, D. Pheiff, K. Socha

Stable patterns of surface waves in deep water

by

Joe Hammack† (1944-2004)

Diane Henderson (Penn State)

Harvey Segur (Colorado)

Maribeth Bleymaier, John Carter,

Cong-Ming Li, Dana Pheiff, Katherine Socha

NCAR workshop on

Coherent Structures in Atmosphere and Ocean

Boulder, CO

July 13, 2005

If stable patterns of surface waves exist in deep water, then they are *Coherent Structures*.

Do stable wave patterns exist in deep water?

Universiteit van Stellenbosch
April 11, 2006

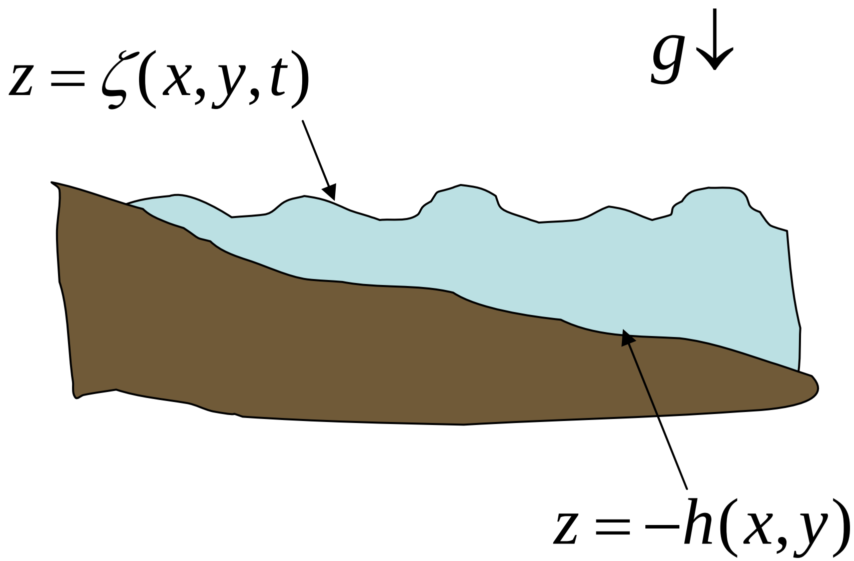
Stabilizing the Benjamin-Feir (or modulational) instability

Harvey Segur
University of Colorado, USA
Joint work with:
D. Henderson, J. Carter, W. Craig, J. Hammack,
C-M Li, M. Oscamou, D. Pheiff, K. Socha

Equations of (inviscid) water waves

(i) On bottom, $z = -h(x,y)$

$$\vec{u} \cdot \nabla(z + h(x,y)) = 0$$



(ii) In fluid, $-h < z < \zeta(x,y,t)$

$$\vec{u} = \nabla \phi, \qquad \nabla^2 \phi = 0$$

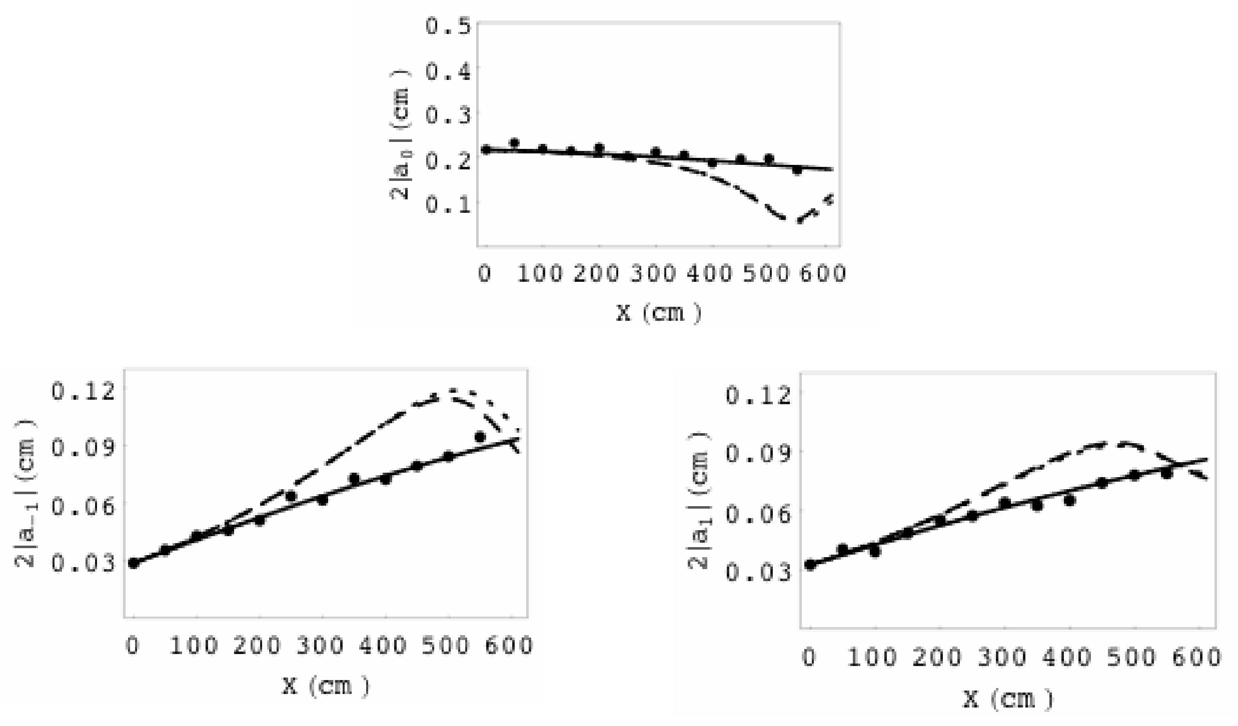
(iii) At free surface, $z = \zeta(x,y,t)$

$$\partial_t \zeta + \nabla \phi \cdot \nabla \zeta = \partial_z \phi,$$

$$\partial_t \phi + \tfrac{1}{2} |\nabla \phi|^2 + g\zeta = 0.$$

(iv) Ignore viscosity, surface tension, variable density, fish, …

What about a higher order NLS model (like Dysthe) ?



____, damped NLS ----, NLS - - -, Dysthe

• • •, experimental data

Colorado State University
November 13, 2006

Stable, periodic wave patterns in deep water

Harvey Segur
University of Colorado

Joint work with:
J. Hammack, D. Henderson,
J. Carter, W. Craig,
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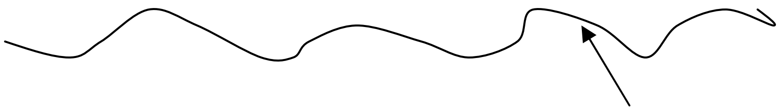
Equations of (inviscid) water waves, in deep water

(i) In fluid, $z < \zeta(x,y,t)$

$$\vec{u} = \nabla\phi, \qquad \nabla^2\phi = 0$$

$g\downarrow$



$z = \zeta(x,y,t)$

(ii) As $z \to -\infty, \qquad |u| \to 0$

(iii) At free surface, $z = \zeta(x,y,t)$

water

$$\partial_t\zeta + \nabla\phi\cdot\nabla\zeta = \partial_z\phi,$$

$$\partial_t\phi + \tfrac{1}{2}|\nabla\phi|^2 + g\zeta = 0.$$

(iv) Ignore viscosity, surface tension, variable density, fish, …

Basic facts about wave propagation

(according to linear theory)

Sound waves

- All travel at the same speed (speed of sound)

Water waves

- Longer waves travel faster than shorter waves

(for gravity-induced surface water waves)

Basic question:

Is a uniform train of 1-D surface waves of finite amplitude on deep water *stable*?

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Is a uniform train of 1-D surface waves of finite amplitude on deep water *stable*?

Slight variation:

Is a uniform train of 1-D electromagnetic waves of finite amplitude in a dispersive optical fiber *stable*?

Basic question:

Is a uniform train of 1-D surface waves of finite amplitude on deep water *stable*?

Variations:

Is a uniform train of 1-D electromagnetic waves of finite amplitude in an optical fiber *stable*?

What about Langmuir waves in a plasma?

Spin waves in a thin magnetic film?

How to reconcile the experimental observations with Benjamin-Feir instability?

Recall: In deep water without dissipation, a uniform train of monochromatic plane waves (with 1-D surface patterns) with finite amplitude (|A|) is unstable to small perturbations with nearly the same frequency.

The maximum growth rate of the instability is

$$\Omega_{\max} = C\,|A|^2$$

How to reconcile the experimental observations with Benjamin-Feir instability?

Options

- Modulational instability afflicts 1-D plane waves, but not 2-D periodic patterns

- The Penn State tank is too short to observe the (relatively slow) growth of the instability

- Other (please specify)

Skip the detailed analyis

How does damping affect the modulational instability?

- Common effect

  calculate theoretical growth rate,
  without damping

Observed decay rate =

predicted growth rate - decay rate

Skip the detailed analyis

How does damping affect the modulational instability?

- 2nd (important) effect

  predicted growth rate without damping

$$\Omega \propto |A|^2$$

Skip the detailed analyis

How does damping affect the modulational instability?

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predicted "growth rate" with damping

$$\Omega \propto |A|^2 e^{-2t\delta}$$

Skip the detailed analyis

How does damping affect the modulational instability?

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- For any $\delta > 0$, growth stops eventually

➔No mode grows forever

➔Total growth is bounded