Space-Time Discrete Integrable Systems with Periodic Boundary Condition

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1 Toda = KM = LV

1.1 KM = LV

KM (Kac-van Moerbeke) model:

$$\frac{dR_j}{dt} = e^{-R_{j-1}} - e^{-R_{j+1}}. \tag{1}$$

If we set

$$u_j = e^{-R_j},$$

then (1) is transformed to

LV (Lotka-Volterra) model:

$$\frac{du_j}{dt} = u_j \left(u_{j+1} - u_{j-1}\right), \tag{2}$$

since

$$\frac{du_j}{dt} = e^{-R_j} \cdot \left(-\frac{dR_j}{dt}\right) = e^{-R_j} \left(e^{-R_{j+1}} - e^{-R_{j-1}}\right) = u_j \left(u_{j+1} - u_{j-1}\right).$$

[Note] *scale invariance*

For constant α, LV eq.(2) is invariant by the scaling:

$$u_j \;\rightarrow\; \alpha u_j \quad \text{and} \quad t \;\rightarrow\; t/\alpha$$

1.2 KM = Toda

Toda lattice:

$$\frac{dQ_j}{dt} = P_j, \quad \frac{dP_j}{dt} = e^{-(Q_j - Q_{j-1})} - e^{-(Q_{j+1} - Q_j)} \tag{3}$$

Setting $r_j = Q_{j+1} - Q_j$, we have equivalent form

$$\frac{d^2 r_j}{dt^2} = 2e^{-r_j} - e^{-r_{j-1}} - e^{-r_{j+1}}. \tag{4}$$

If we suppose $r_j = R_{2j} + R_{2j+1}$, where R_j's are KM,

$$\frac{dr_j}{dt} = \left(e^{-R_{2j-1}} - e^{-R_{2j+1}}\right) + \left(e^{-R_{2j}} - e^{-R_{2j+2}}\right) = (u_{2j-1} - u_{2j+1}) + (u_{2j} - u_{2j+2}),$$

and differentiate this again, we have

$$\begin{aligned}\frac{d^2 r_j}{dt^2} =& u_{2j-1}(u_{2j} - u_{2j-2}) - u_{2j+1}(u_{2j+2} - u_{2j}) + u_{2j}(u_{2j+1} - u_{2j-1}) - u_{2j+2}(u_{2j+3} - u_{2j+1}) \\ =& 2u_{2j}u_{2j+1} - u_{2j-2}u_{2j-1} - u_{2j+2}u_{2j+3} \\ =& 2e^{-r_j} - e^{-r_{j-1}} - e^{-r_{j+1}},\end{aligned}$$

which implies KM = Toda.

1.3 Discretization

dToda:

$$I_j^{n+1} V_j^{n+1} = I_{j+1}^n V_j^n, \qquad I_j^{n+1} - I_j^n = h^2 \left(V_j^n - V_{j-1}^{n+1} \right), \tag{5}$$

where we set

$$V_j = e^{-(Q_{j+1} - Q_j)} = e^{-r_j}, \qquad I_j = 1 - hP_j.$$

Heuristic derivation:

$$\frac{V_j^{n+1}}{V_j^n} = e^{-(r_j^{n+1} - r_j^n)} \cong 1 - h\frac{dr_j}{dt} = 1 - h(P_{j+1} - P_j) \cong \frac{1 - hP_{j+1}}{1 - hP_j} \cong \frac{I_{j+1}^n}{I_j^{n+1}},$$

and

$$I_j^{n+1} - I_j^n = -h(P_j^{n+1} - P_j^n) \cong -h^2 \frac{dP_j}{dt} = -h^2 \left(e^{-(Q_j - Q_{j-1})} - e^{-(Q_{j+1} - Q_j)} \right) \cong h^2 \left(V_j^n - V_{j-1}^{n+1} \right)$$

dLV:

$$\frac{u_j^{n+1} - u_j^n}{h} = u_j^n u_{j+1}^n - u_j^{n+1} u_{j-1}^{n+1} \quad \Longleftrightarrow \quad (1 + hu_{j-1}^{n+1})u_j^{n+1} = (1 + hu_{j+1}^n)u_j^n \tag{6}$$

[Note] We can prove dToda = dLV, if we set

$$V_j^n = u_{2j}^n u_{2j+1}^n, \qquad I_j^n = (1 + hu_{2j-1}^n)(1 + hu_{2j}^n), \tag{7}$$

since

$$\begin{aligned} I_j^{n+1} V_j^{n+1} &= (1 + hu_{2j-1}^{n+1})(1 + hu_{2j}^{n+1}) \cdot u_{2j}^{n+1} u_{2j+1}^{n+1} \\ &= (1 + hu_{2j+1}^n)u_{2j}^n \cdot (1 + hu_{2j+2}^n)u_{2j+1}^n = I_{j+1}^n V_j^n, \end{aligned}$$

and

$$\begin{aligned} I_j^{n+1} - I_j^n &= (1 + hu_{2j-1}^{n+1})(1 + hu_{2j}^{n+1}) - (1 + hu_{2j-1}^n)(1 + hu_{2j}^n) \\ &= h(u_{2j-1}^{n+1} - u_{2j-1}^n + u_{2j}^{n+1} - u_{2j}^n) + h^2(u_{2j-1}^{n+1} u_{2j}^{n+1} - u_{2j-1}^n u_{2j}^n) \\ &= h^2(u_{2j-1}^n u_{2j}^n - u_{2j-1}^{n+1} u_{2j-2}^{n+1} + u_{2j}^n u_{2j+1}^n - u_{2j}^{n+1} u_{2j-1}^{n+1}) + h^2(u_{2j-1}^{n+1} u_{2j}^{n+1} - u_{2j-1}^n u_{2j}^n) \\ &= h^2(u_{2j}^n u_{2j+1}^n - u_{2j-2}^{n+1} u_{2j-1}^{n+1}) \\ &= h^2(V_j^n - V_{j-1}^{n+1}) \end{aligned}$$

1.4 Discrete Bäcklund Transformation

Bäcklund transformation for Toda: (Wadati-Toda, JPSJ 39, (1975) 1196)

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Bäcklund Transformation for the Exponential Lattice

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(Received May 14, 1975)

A Bäcklund transformation associated with the equation of motion for an exponential lattice is found. It is shown that recursive application of the transformation provides an algebraic recursion formula for the solutions. Using the recursion formula, two-soliton solution is obtained and a method for constructing N-soliton solution is presented. It is also shown that the fundamental equations of inverse method and conservation laws can be derived from the transformation.

Bäcklund transformation has the simplest form in terms of KM

$$\frac{dR_{2j}}{dt} = A\left(e^{-R_{2j+1}} - e^{-R_{2j-1}}\right), \qquad \frac{dR_{2j+1}}{dt} = \frac{1}{A}\left(e^{-R_{2j+2}} - e^{-R_{2j}}\right). \tag{8}$$

And these can be **discretized** as

$$\frac{u_{2j}^{n+1} - u_{2j}^{n}}{h} = B\left(u_{2j}^{n}u_{2j+1}^{n} - u_{2j}^{n+1}u_{2j-1}^{n+1}\right), \qquad \frac{u_{2j+1}^{n+1} - u_{2j+1}^{n}}{h} = \frac{1}{B}\left(u_{2j+1}^{n}u_{2j+2}^{n} - u_{2j+1}^{n+1}u_{2j}^{n+1}\right), \tag{9}$$

which is rewritten, by setting $h_0 = hB,\ h_1 = h/B$, as

$$(1 + h_0 u_{2j-1}^{n+1})u_{2j}^{n+1} = (1 + h_0 u_{2j+1}^{n})u_{2j}^{n}, \qquad (1 + h_1 u_{2j}^{n+1})u_{2j+1}^{n+1} = (1 + h_1 u_{2j+2}^{n})u_{2j+1}^{n}.$$

Then, if we put

$$V_j^n = u_{2j}^n u_{2j+1}^n, \qquad I_j^n = (1 + h_0 u_{2j-1}^n)(1 + h_1 u_{2j}^n), \tag{10}$$

we get dToda again

$$I_j^{n+1}V_j^{n+1} = I_{j+1}^n V_j^n, \qquad I_{j+1}^{n+1} - I_j^n = h^2(V_j^n - V_{j-1}^{n+1}),$$

where $h_0 h_1 = h^2$ is used.

2 Determinant and Simulation

2.1 Determinant Expression for dLV

dLV with Periodic Boundary Condition

We consider dLV

$$(1+hu_{j+1}^{n+1})u_j^{n+1}=(1+hu_{j-1}^{n})u_j^{n},$$

with periodic boundary condition $u_{j+N}=u_j$. Setting $x_j^n=hu_j^n$, we have

$$(1+x_{j+1}^{n+1})x_j^{n+1}=(1+x_{j-1}^{n})x_j^{n}. \tag{11}$$

Case of $N=3$:

$$(1+X_1)X_0=(1+x_2)x_0,\quad (1+X_2)X_1=(1+x_0)x_1,\quad (1+X_3)X_2=(1+x_1)x_2$$

have solution

$$X_0=x_0\frac{1+x_2+x_1x_2}{1+x_1+x_0x_1},\quad X_1=x_1\frac{1+x_0+x_2x_0}{1+x_2+x_1x_2},\quad X_2=x_2\frac{1+x_1+x_0x_1}{1+x_0+x_2x_0},$$

which are expressed as

$$X_0=x_0\frac{\Delta_1}{\Delta_0},\quad X_1=x_1\frac{\Delta_2}{\Delta_1},\quad X_2=x_2\frac{\Delta_0}{\Delta_2}.$$

Note that Δ_0 has a determinant form:

$$\Delta_0 = \begin{vmatrix} 1 & x_1 \\ -(1+x_0) & 1 \end{vmatrix}.$$

General case of N:

We have

$$X_0 = x_0 \frac{\Delta_1}{\Delta_0}, \quad \cdots, \quad X_j = x_j \frac{\Delta_{j+1}}{\Delta_j}, \quad \cdots, \quad X_{N-1} = x_{N-1} \frac{\Delta_0}{\Delta_{N-1}},$$

where

$$\Delta_0 = \begin{vmatrix} 1 & x_1 & & & \\ -(1+x_0) & 1 & x_2 & & \\ & & \cdots & & \\ & & -(1+x_{N-4}) & 1 & x_{N-2} \\ & & & -(1+x_{N-3}) & 1 \end{vmatrix}. \tag{12}$$

Other Models

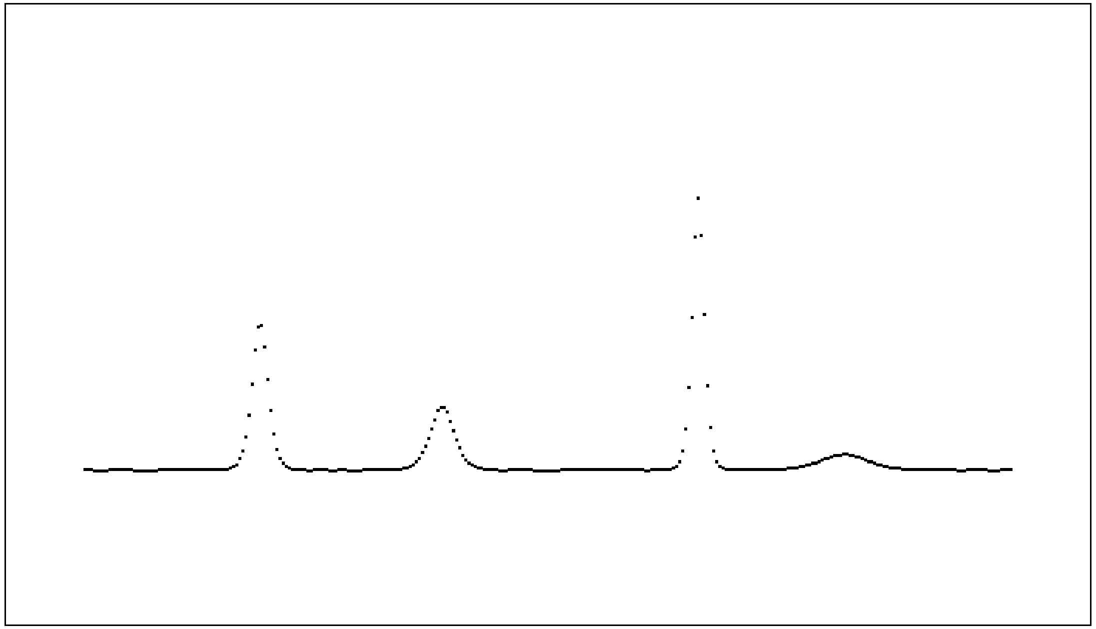
We have similar Determinant Formulas (K. Sogo, JPSJ 75, (2006) 084001) for

dKdV, dToda, dKP *etc.* .

Such explicit expressions can be used in numerical simulations quite effectively.

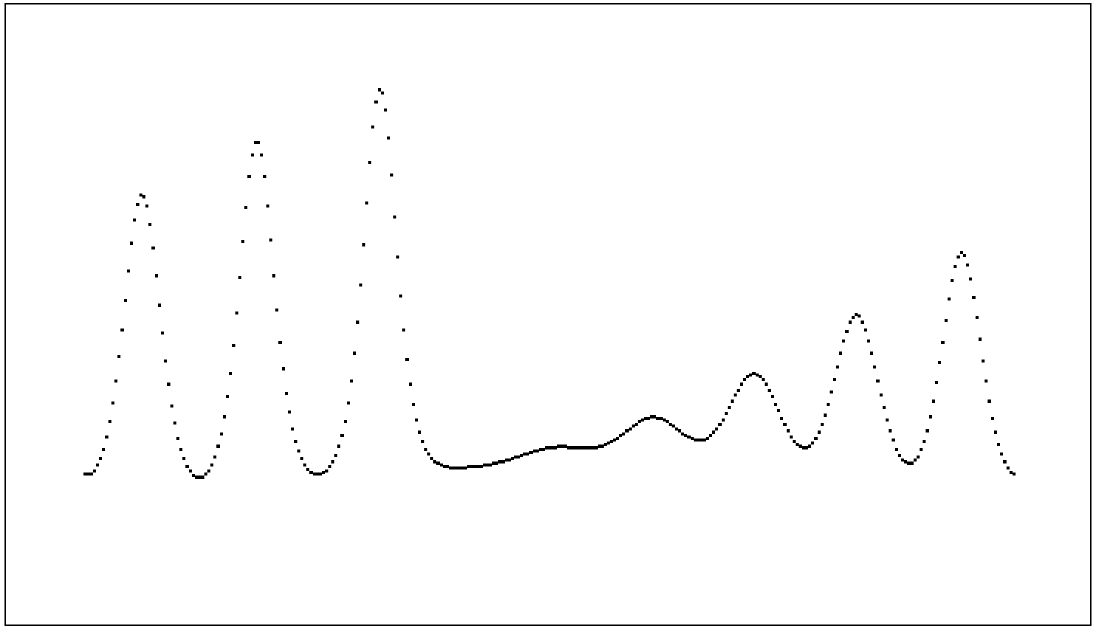
2.2 Simulations

dLV



dLV with 4 solitons.

dKdV

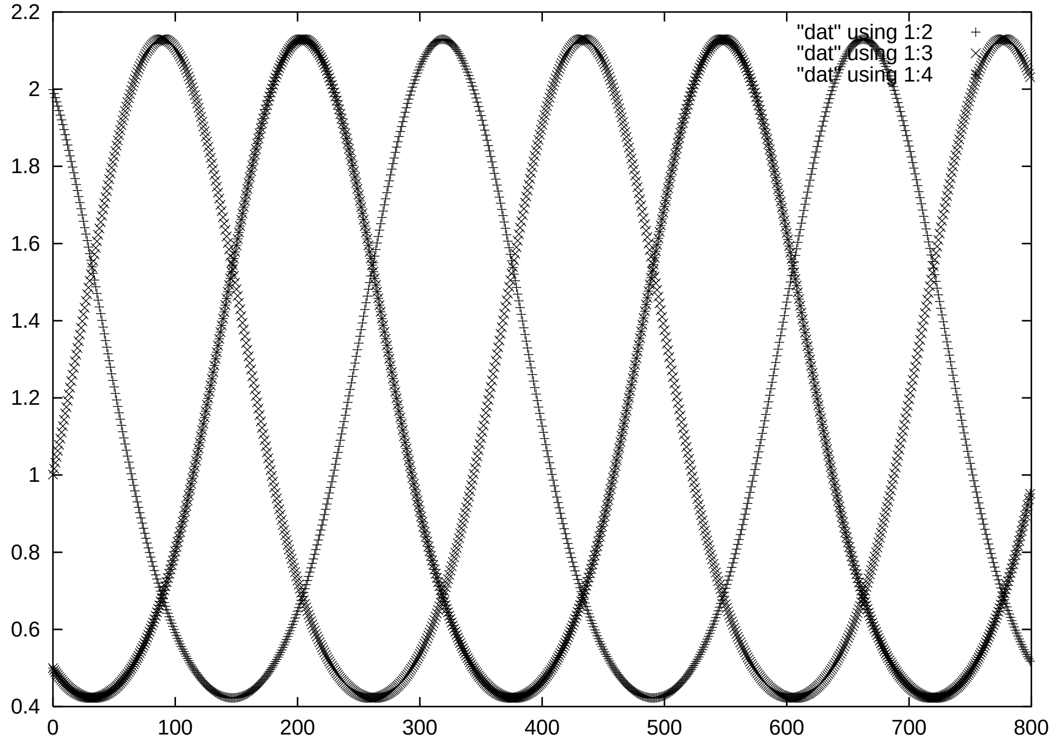


Zabusky-Kruskal case.

Real-Time Simulation of dLV of $N = 3$

```
#lv3.py
from Tkinter import *
WIDTH=800
HEIGHT=600
h,h2=0.001,h*h
x0,x1,x2=2.0,1.0,0.5
canvas=Canvas(width=WIDTH, height=HEIGHT,bg='white')
canvas.pack(expand=YES, fill=BOTH)
for n in range(0,8000):
  D0,D1,D2=1+h*x1+h2*x0*x1,1+h*x2+h2*x1*x2,1+h*x0+h2*x2*x0
  x0,x1,x2=x0*D1/D0,x1*D2/D1,x2*D0/D2
  if((n%10)==0):
    xx,y0,y1,y2=n/10,HEIGHT-x0*200,HEIGHT-x1*200,HEIGHT-x2*200
    canvas.create_rectangle(xx,y0,xx,y0,width=1,fill='black')
    canvas.create_rectangle(xx,y1,xx,y1,width=1,fill='black')
    canvas.create_rectangle(xx,y2,xx,y2,width=1,fill='black')
mainloop()
```

3 body dLV



3 body dLV

Simulate, Guess, and Verify.

3 Toda = Baxter ?

Coffee Break

高木貞治の『近世数学史談』

高木貞治(1875–1960)は，日本の生んだ世界的数学者のひとりである．いわゆる「類体論」を完成させ，「クロネッカーの青春の夢」を実現したことで有名である．また，『解析概論』や『初等整数論講義』など，数多くの著書がある．なかでも『解析概論』は，著者の世代の必読書のひとつであった．

表題の『近世数学史談』(岩波文庫および共立出版)は，19 世紀の数学の展開を，ガウスを中心にアーベルやヤコービなどの業績を通して論じたものである．抜群におもしろい本なので，読者にもぜひ一読を薦めたい．現役の数学者で，この本を読んで数学を志したという人も多いと聞く．ちなみに著者は，この本から「楕円関数」のおもしろさを学んだ(この本を楕円関数入門として読むのは，少し邪道であるが)．

楕円関数は，近年の理論物理学でもよく用いられている．例えば，ソリトン方程式の周期解は，楕円関数を用いて書かれるし，バクスター(R. J. Baxter, 1940–)の「8 頂点模型」という可積分格子模型にも楕円関数が登場する．1971 年にバクスターの論文が出たとき，戸田盛和(1917–)先生は「ここにも楕円関数が出てくるんだね」と愉快そうに話された(戸田格子の周期解は 1967 年である)．のちに両者が大いに関係するとは，予想されていたのであろうか．

『近世数学史談』によると，算術幾何平均(例題 1.7)，楕円積分(例題 4.14)，超幾何関数(問題 7-2[5])の 3 つを題材にした著書を書く計画を，ガウスが持っていたという．その本は実現しなかったが，「多変数超幾何関数論」として現代数学にその花を咲かせているとも言える．

3.1 Solution of 3 body LV

Differential equations

$$\frac{dx}{dt} = x(y - z), \quad \frac{dy}{dt} = y(z - x), \quad \frac{dz}{dt} = z(x - y) \tag{13}$$

can be solved by using the conservation law

$$x + y + z = a, \quad xyz = 1.$$

Eliminating y and z, we have

$$\left(\frac{dx}{dt}\right)^2 = x(x^3 - 2ax^2 + a^2x - 4),$$

where the RHS is factorized by real numbers $\alpha > \beta > \gamma > 0$ as

$$\left(\frac{dx}{dt}\right)^2 = x(x - \alpha)(x - \beta)(x - \gamma), \tag{14}$$

which satisfy

$$\sqrt{\alpha} = \sqrt{\beta} + \sqrt{\gamma}.$$

Equation (14) is solved as

$$\frac{\gamma}{x} = 1 - \frac{\beta - \gamma}{\beta}\mathrm{sn}^2\xi, \qquad \left(\xi = \omega t,\ \omega = \frac{\sqrt{\beta(\alpha - \gamma)}}{2}\right). \tag{15}$$

The modulus of the elliptic functions is defined by

$$\kappa^2 = \frac{\alpha}{\beta} \cdot \frac{\beta - \gamma}{\alpha - \gamma} \tag{16}$$

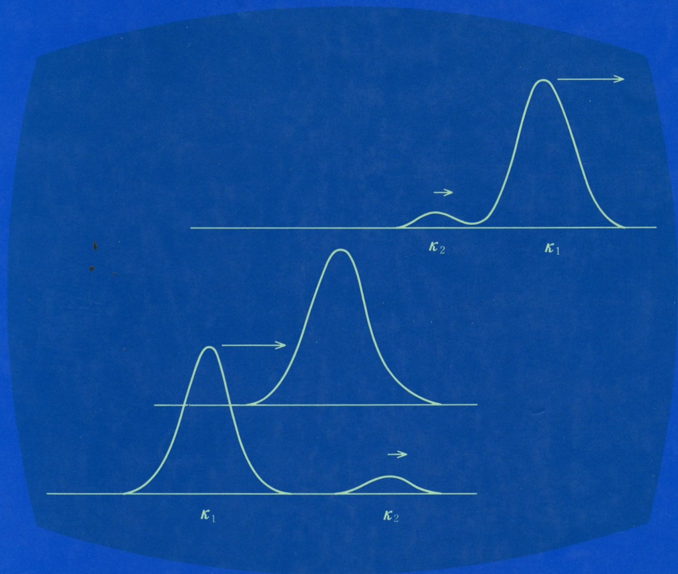
Others are

$$\frac{\gamma}{y} = 1 - \frac{\beta - \gamma}{\beta}\mathrm{sn}^2(\xi + \zeta), \qquad \frac{\gamma}{z} = 1 - \frac{\beta - \gamma}{\beta}\mathrm{sn}^2(\xi + 2\zeta),$$

where $\zeta = 2K(\kappa)/3$. These recover the results of simulation.

非線形波動とソリトン

戸田盛和＝著



日本評論社

3.2 Calculation by Toda

Toda gave a calculation of N=3 Toda lattice with a special *symmetric* initial condition, which contains several hard calculations such as

$$\mathrm{sn}\left(\frac{K}{3}\right) = \sqrt{\frac{\beta}{\alpha}}.$$

And his results have one to one correspondence with ours of $N = 3$ LV.

Why ?

Although the solutions of $N = 3$ Toda are, in general, expressed by hyperelliptic functions of $g = 2$, Toda's solution is expressed by elliptic functions of $g = 1$. The situation may be interpreted as follows.

In general, we have an equivalence:

$$\textbf{(3 body Toda)} = \textbf{(6 body LV)}.$$

There is the case, however, that the period of LV is reduced from 6 to 3, that is,

$$R_3 = R_0, \quad R_4 = R_1, \quad R_5 = R_2.$$

Using the Bäcklund relations, these are satisfied when

$$Q_0' = Q_2, \quad Q_1' = Q_0, \quad Q_2' = Q_1,$$

and simulations of this case give a behavior of $g = 1$.

Further we have another similarity of 3 body LV with **Baxter's eight vertex model**. For example, the RHS of (15) can be factorized as

$$1 - \frac{\mathrm{sn}^2\xi}{\mathrm{sn}^2\eta} = \frac{\Theta^2(0)H(\eta-\xi)H(\eta+\xi)}{H^2(\eta)\Theta^2(\xi)}, \qquad \left(\mathrm{sn}^2\eta = \frac{\beta}{\beta-\gamma}\right) \tag{17}$$

which is a familiar formula in the eight vertex model.

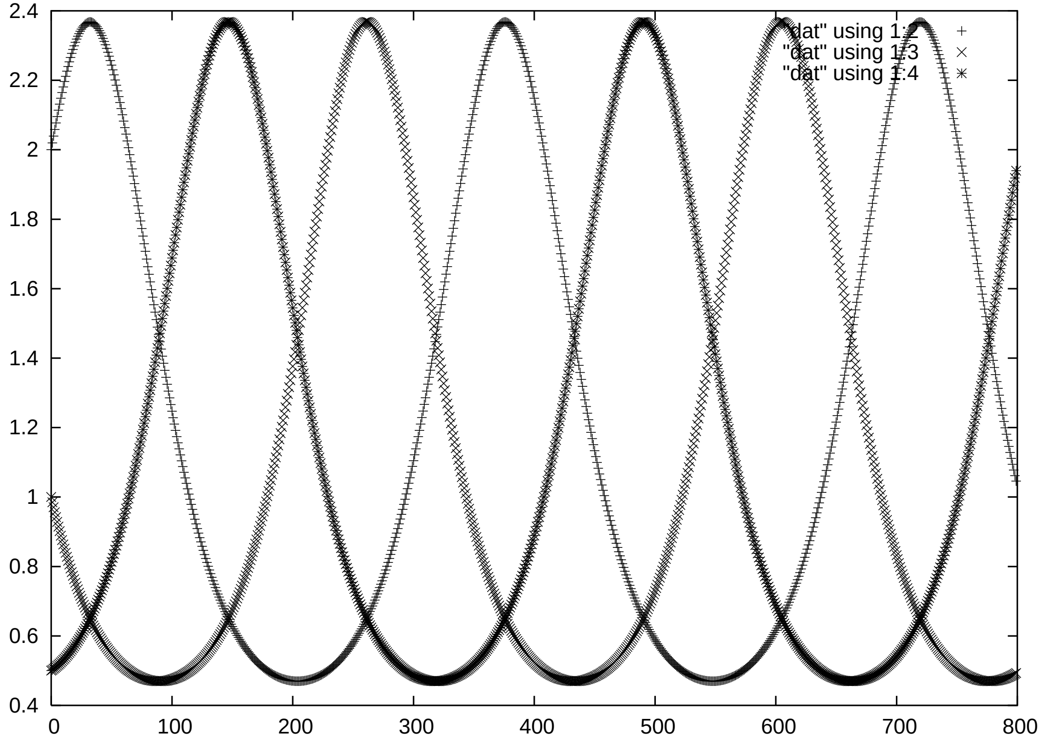
This coincidence indicates also that the *spectral parameter* of the eight vertex model corresponds to *time* variable ξ of LV = Toda with PBC.

Such is the reason of the statement

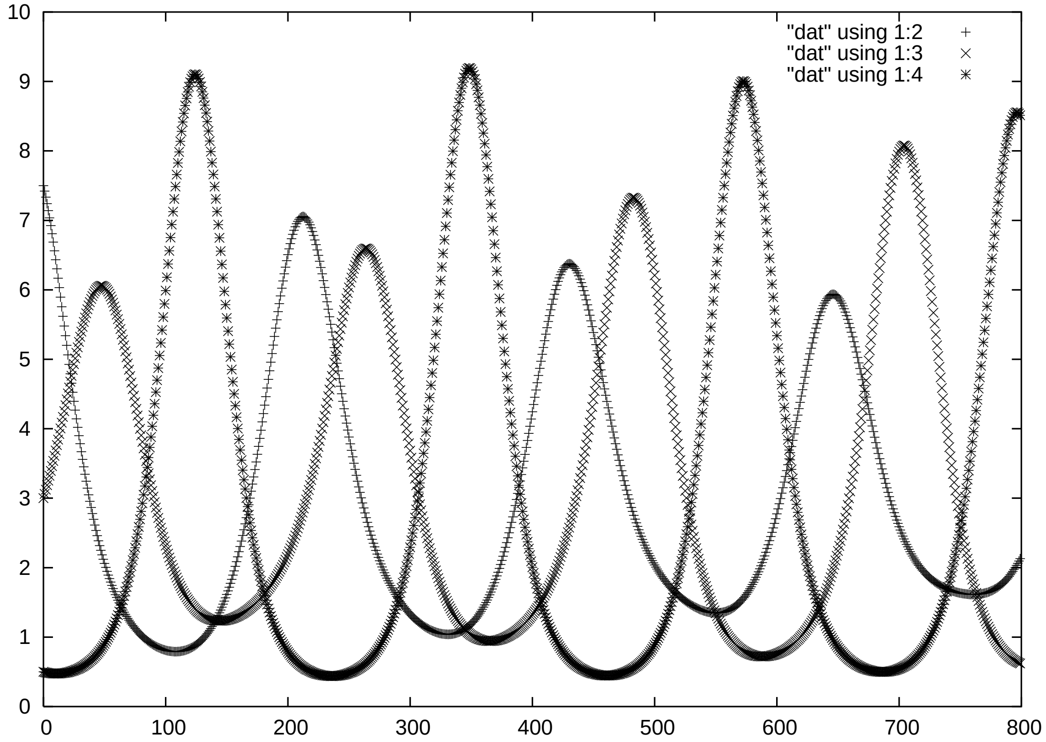
$$\mathbf{Toda} = \mathbf{Baxter}.$$

3.3 Simulations of 3 body Toda by using 6 body LV

Symmetric Case:



Asymmetric Case:



Thanks to Audiences,

Especially to Prof. Wadati !