Solitons in the F=1 Bose-Einstein condensates

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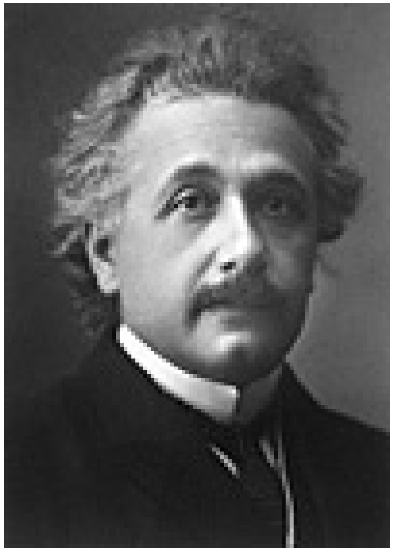
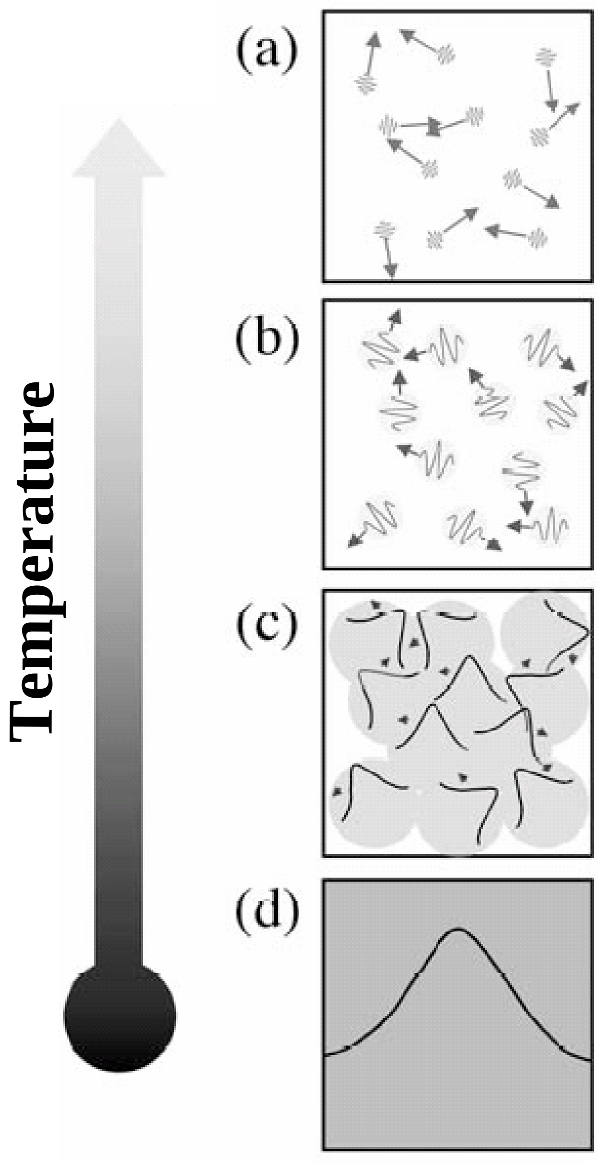
Bose-Einstein Condensation



(1925 A. Einstein)



The Nobel Prize in Physics 2001

for the achievement of BEC (1995)

Brief history and related topics of Bose-Einstein Condensation

1925 A. Einstein, the theoretical prediction

1938 F. London, rediscovery of BEC,
λ transition in ^{4}He

1961 Gross-Pitaevskii equation

1995 BEC of ultra cold atoms (JILA, MIT, Rice)

1998 hyperfine spin F=1 spinor BEC (MIT)

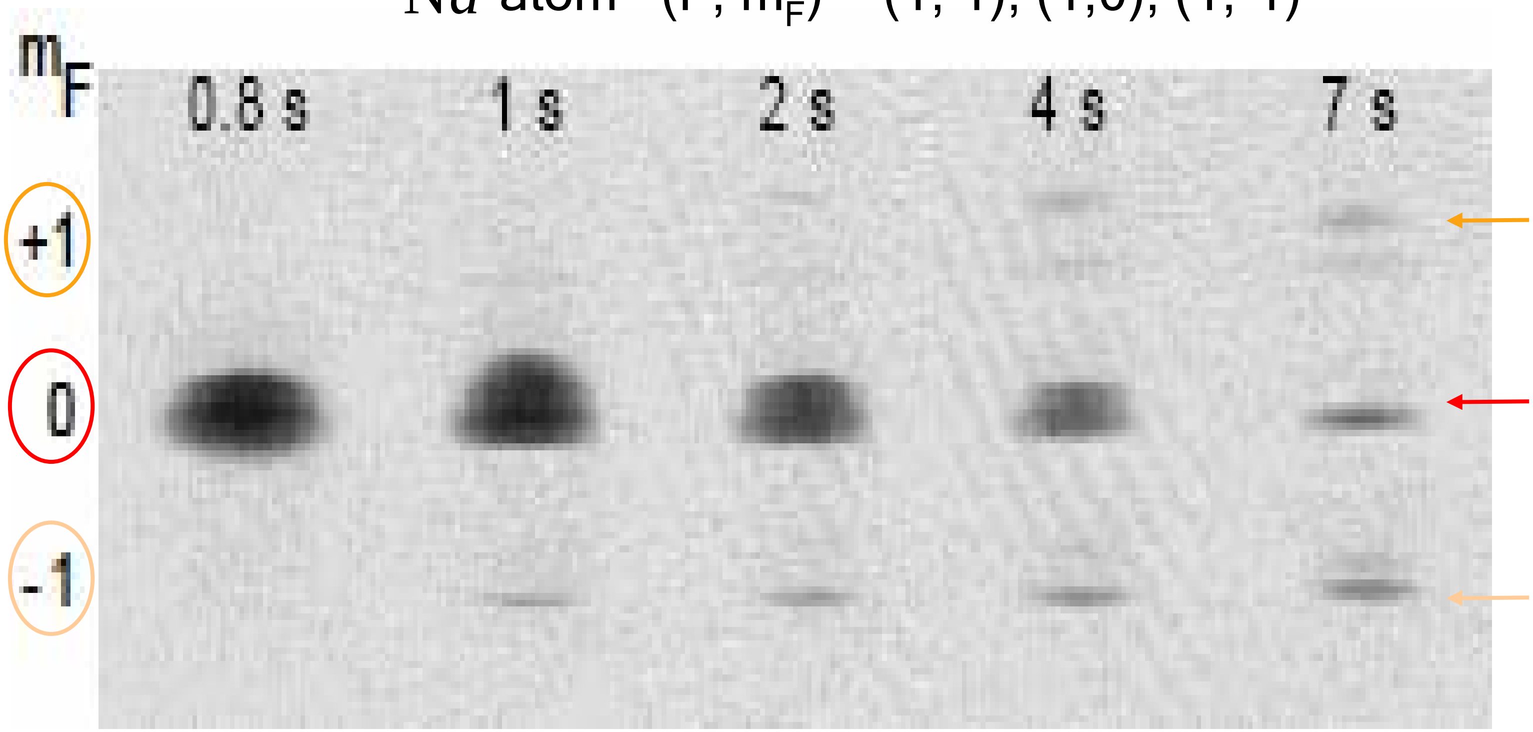
1999 dark solitons by phase imprinting method (Hannover, NIST)

2002 bright solitons in quasi-1D system (Rice, ENS)

Experiment for spinor BEC

MIT *Nature* **396**, 345 (1998)

$^{23}\mathrm{Na}$ atom (F, m_F) = (1, 1), (1,0), (1,-1)



multicomponent condensate $\boldsymbol{\Phi} = \{\Phi_{-1}, \Phi_0, \Phi_1\}$

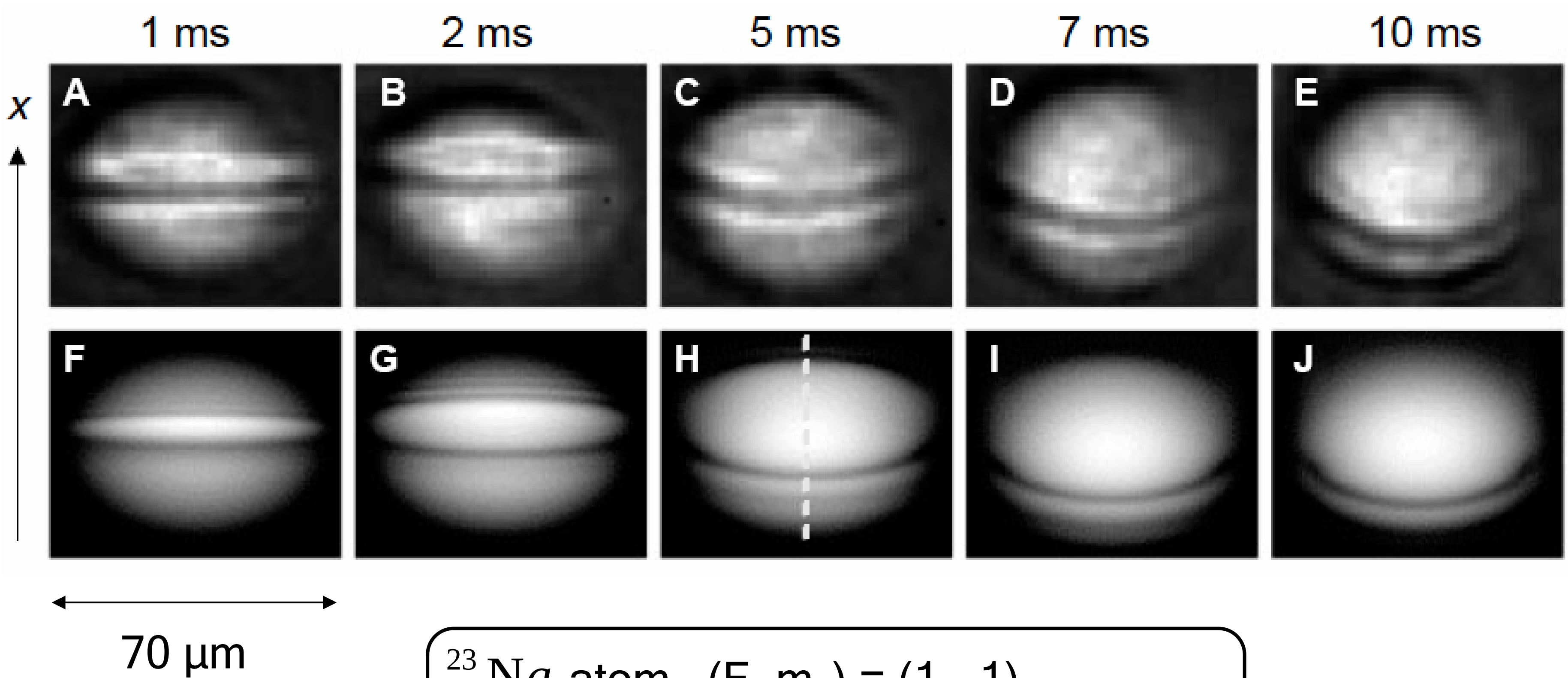
manipulations of spin dynamics:

Hamburg *PRL* **92**, 040402 (2004)

Georgia *PRL* **92**, 140403 (2004)

Generation of single-component dark soliton

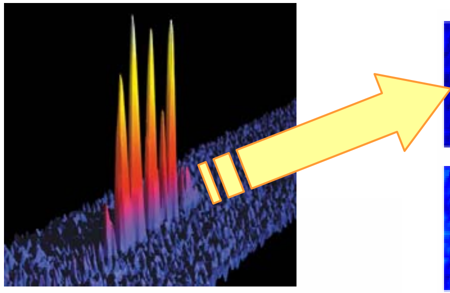
NIST *Science* **287**, 97 (2000)

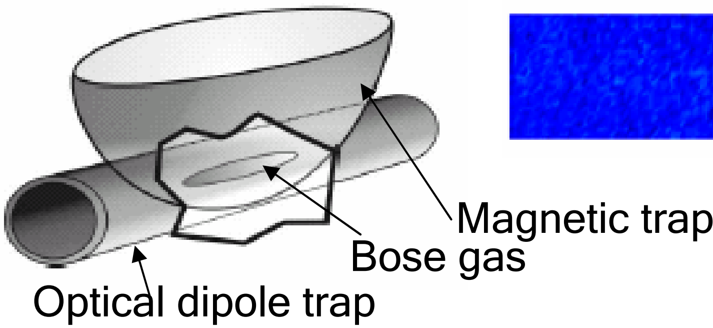


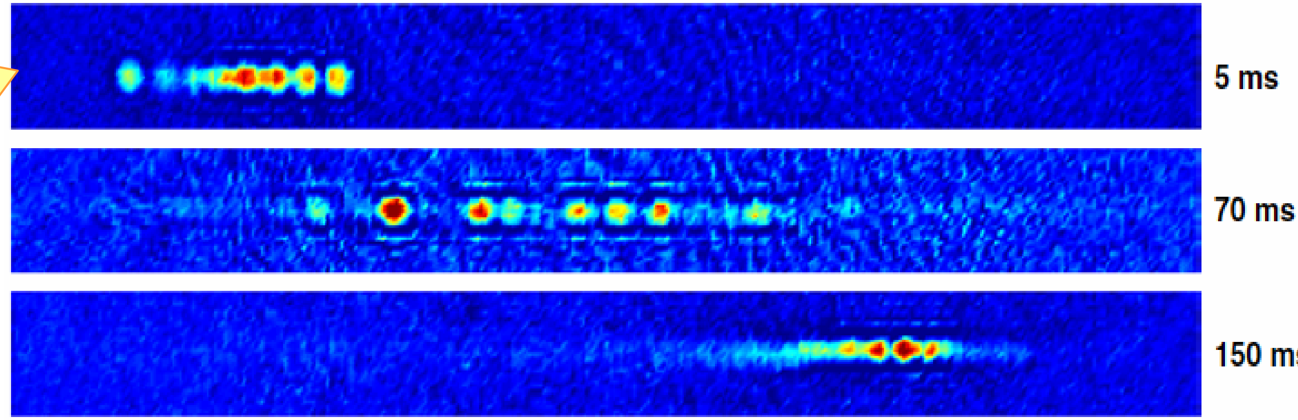
^{23}Na atom (F, m_F) = (1, -1)
A-E : experiment, F-J : theory (GPE)

Matter-wave Bright Solitons 1

Rice *Nature* 417, 150 (2002)

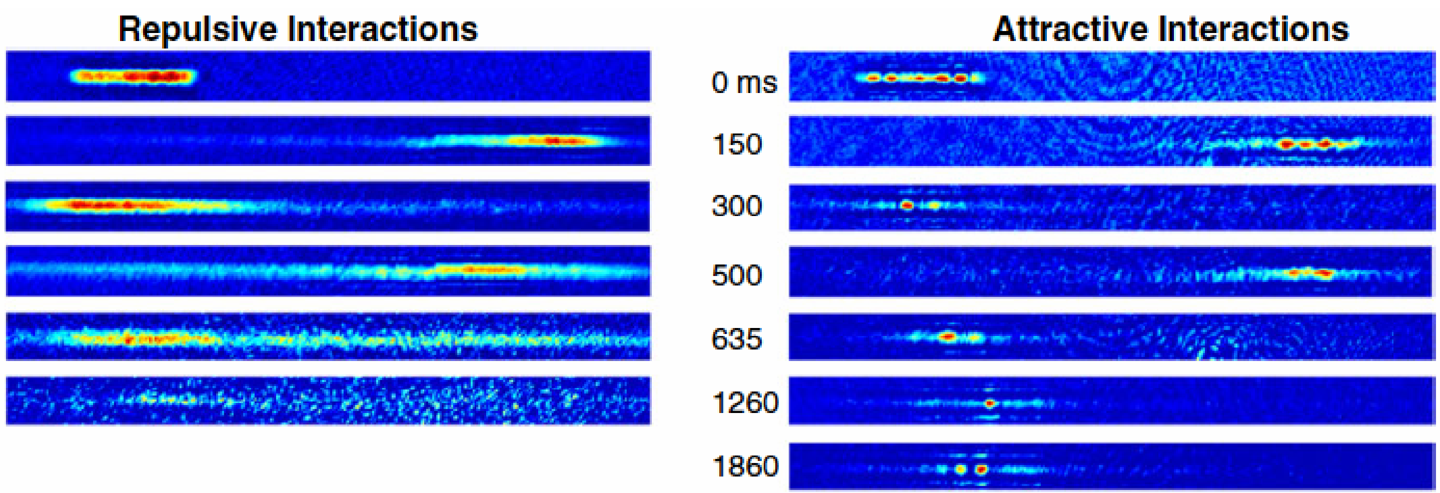






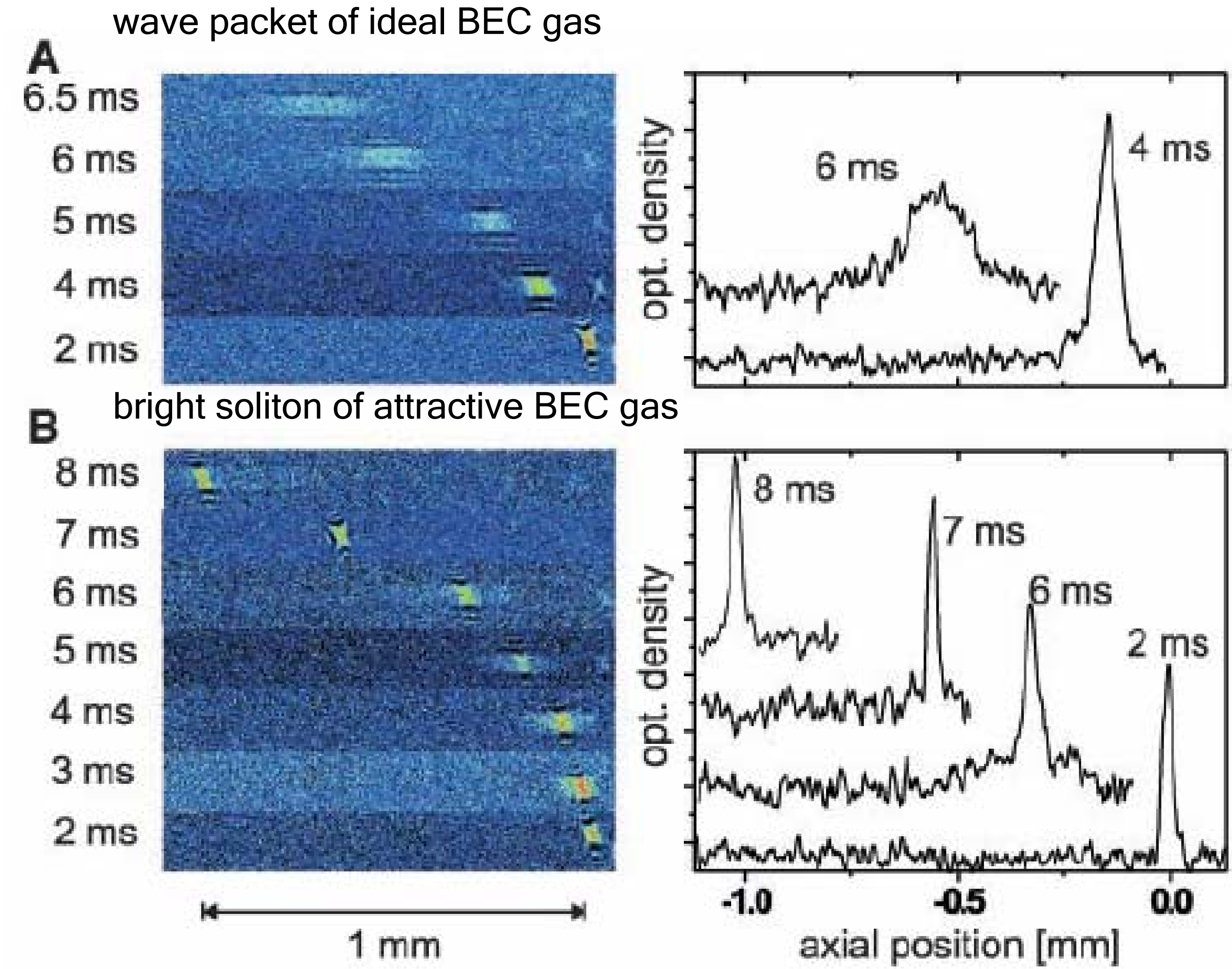
wave guide : 47 µm × 230 µm

quasi 1 dimensional regime



Matter-wave Bright Solitons 2

ENS *Science* **296**, 1290 (2002)



Gross-Pitaevskii (GP) equation for F=1 BEC (nonlinear Schrödinger equation) in 1D system

- Order parameter in the mean-field theory

$$\mathbf{\Phi} = (\Phi_1, \Phi_0, \Phi_{-1})$$

{1,0,-1}: z-component of spin

- GP eq. is obtained as $i\hbar\partial_t \mathbf{\Phi} = \dfrac{\delta E_{\mathrm{GP}}}{\delta \mathbf{\Phi}^*}$

with energy functional $E_{\mathrm{GP}} = \displaystyle\int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m} |\partial_x \mathbf{\Phi}|^2 + \frac{\bar{c}_0}{2} n^2 + \frac{\bar{c}_2}{2} \boldsymbol{f}^2 \right)$

where $n = \displaystyle\sum_{\alpha=1,0,-1} \Phi_\alpha^* \Phi_\alpha$ is the number density

and $f^i = \displaystyle\sum_{\alpha,\beta=1,0,-1} \Phi_\alpha^* \mathsf{f}^i_{\alpha\beta} \Phi_\beta$ (i=x,y,z) is the spin density

$$\mathsf{f}^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{f}^y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{f}^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

: su(2) matrices for spin-1

$$i\hbar\partial_t\Phi_1 = -\frac{\hbar^2}{2m}\partial_x^2\Phi_1 + (\bar{c}_0+\bar{c}_2)(|\Phi_1|^2+|\Phi_0|^2)\Phi_1 + (\bar{c}_0-\bar{c}_2)|\Phi_{-1}|^2\Phi_1 + \bar{c}_2\Phi_{-1}^*\Phi_0^2,$$

$$i\hbar\partial_t\Phi_0 = -\frac{\hbar^2}{2m}\partial_x^2\Phi_0 + (\bar{c}_0+\bar{c}_2)(|\Phi_1|^2+|\Phi_{-1}|^2)\Phi_0 + \bar{c}_0|\Phi_0|^2\Phi_0 + 2\bar{c}_2\Phi_0^*\Phi_1\Phi_{-1},$$

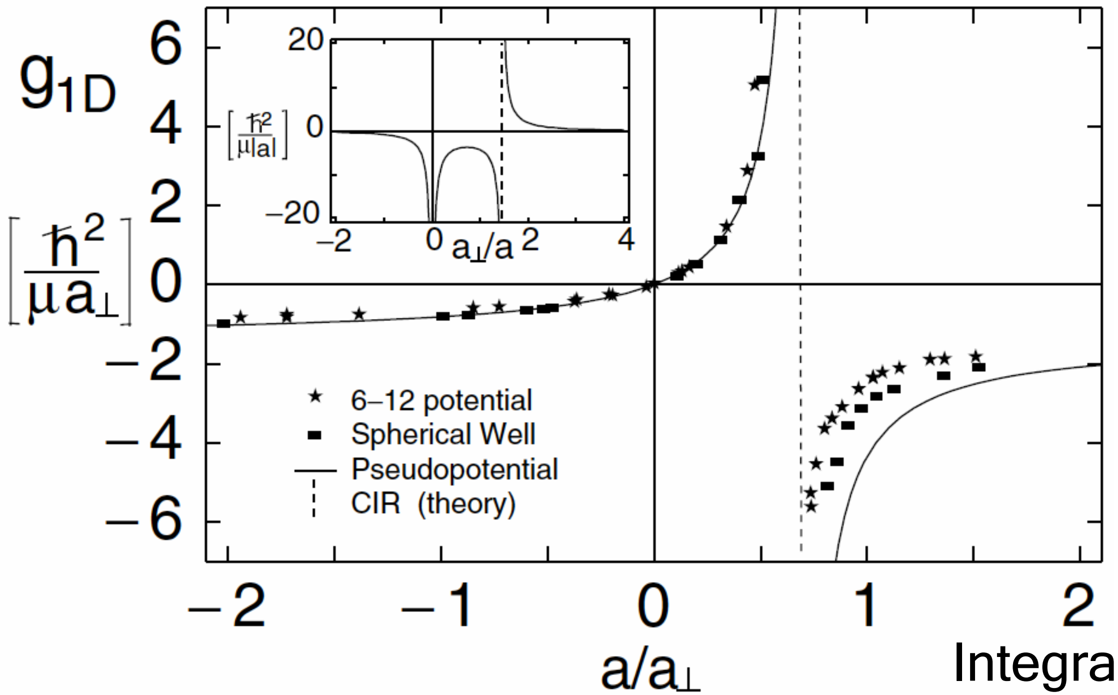
$$i\hbar\partial_t\Phi_{-1} = -\frac{\hbar^2}{2m}\partial_x^2\Phi_{-1} + (\bar{c}_0+\bar{c}_2)(|\Phi_{-1}|^2+|\Phi_0|^2)\Phi_{-1} + (\bar{c}_0-\bar{c}_2)|\Phi_1|^2\Phi_{-1} + \bar{c}_2\Phi_1^*\Phi_0^2.$$

$\bar{c}_0 = \bar{c}_2 = c$ an integrable point that gives the matrix nonlinear Schrödinger equation

Confinement Induced Resonance

M. Olshanii: *PRL* **81**, 938 (1998),

T. Bergeman, *et. al*: *PRL* **91**, 163201 (2003).



$$g_{1\mathrm{D}} = \frac{2\hbar^2 a}{\mu a_\perp^2}\frac{1}{(1 - Ca/a_\perp)}$$

$$a_\perp = \sqrt{\frac{\hbar}{\mu\omega_\perp}}$$

$$C = -\zeta(1/2) = 1.4603\ldots$$

Monte Carlo studies:

PRL **92**, 030402 (2004)

Integrable condition: $c_0 = c_1$

$$\bar{g}_0 = -2\bar{g}_2$$

$$a_\perp = \frac{3Ca_0a_2}{2a_0+a_2}$$

f-channel coupling

(*f*=0,2 for F=1 bosons)

$$\bar{g}_f = \frac{2\hbar^2 a_f}{\mu a_\perp^2}\frac{1}{(1 - Ca_f/a_\perp)}$$

matrix nonlinear Schrödinger equation (MNLSE)

- At $\bar{c}_0 = \bar{c}_2 = c$, GP eq. is equivalent to MNLSE:

$$i\partial_t Q + \partial_x^2 Q - 2cQQ^\dagger Q = O$$

$$(\hbar = 2m = 1)$$

with an identification

$$Q = \begin{pmatrix} \Phi_1 & \Phi_0/\sqrt{2} \\ \Phi_0/\sqrt{2} & \Phi_{-1} \end{pmatrix}$$

In terms of the matrix Q, physical densities are expressed as

number density: $n = \mathrm{tr}(Q^\dagger Q)$

spin density: $\boldsymbol{f} = \mathrm{tr}(Q^\dagger \boldsymbol{\sigma} Q)$

momentum density: $p = -i\ \mathrm{tr}(Q^\dagger Q_x)$

energy density: $e = \mathrm{tr}(Q_x^\dagger Q_x + c\ Q^\dagger Q Q^\dagger Q)$

Nonvanishing boundary conditions

- We consider the boundary conditions:

$$Q(x) \to Q_\pm \quad x \to \pm\infty \quad \text{with} \quad Q_\pm^\dagger Q_\pm = Q_\pm Q_\pm^\dagger = \lambda_0^2 I$$

- We can fix

$$|Q| \to \lambda_0 I \qquad x \to \infty$$

by an SU(2) transformation, in terms of matrix, given by

$$Q' = \mathcal{U} Q \mathcal{U}^T \qquad \mathcal{U} = \exp[i\boldsymbol{\theta} \cdot \boldsymbol{\sigma}]$$

- This generalizes the analysis for the vanishing condition

$$Q(x) \to O \qquad x \to \pm\infty$$

Inverse Scattering Method for MNLSE

initial value for MNLSE

$$Q(x,0)$$

$$\partial_x \Psi = U\Psi$$

direct problem

scattering coefficients at t=0

$$\{\zeta_j, R(\zeta), \Pi_j\}$$

$$\partial_t \Psi = V\Psi$$

linearized evolution

scattering coefficients at time t

$$\{\zeta_j, R(\zeta;t), \Pi_j(t)\}$$

$$\partial_x \Psi = U\Psi$$

inverse problem

Gelfand-Levitan eq.

time-evolution by MNLSE

$$Q(x,t)$$

$$i\partial_t Q + \partial_x^2 Q - 2cQQ^\dagger Q = O$$

MNLSE $\longleftrightarrow$ $[\partial_x - U, \partial_t - V] = O$

The analysis is quite complicated.

- Initial value problem

  The solution is given by the Gelfand-Levitan equation.

- Soliton solutions

  are obtained by imposing the reflectionless condition in the Gelfand-Levitan equation.

1-soliton (c>0: repulsive & anti-ferromagnetic)

- Explicit expression

$$Q(x,t) = \lambda_0 e^{i\phi(x,t)} \frac{(e^{2(\chi+\rho+i\varphi)} \det \Pi + e^{\chi+\rho} \mathrm{tr}\Pi + 1) I + 2ie^{\chi+i\varphi}\Pi}{e^{2(\chi+\rho)} \det \Pi + e^{\chi+\rho} \mathrm{tr}\Pi + 1}$$

where $e^{-\rho} = \sin\varphi$ and

the phase function for the carrier wave: $\phi(x,t) = kx - (k^2 + 2\lambda_0^2)t + \delta$

the coordinate function for the enveloping soliton: $\chi(x,t) = -2\lambda_0 \sin\varphi (x - 2(\lambda_0 \cos\varphi + k)t)$

There are 3 parameters $\lambda_0,\ k,\ \varphi$ and a matrix Π (polarization matrix)

- Physical densities are calculated as:

number density:

$$n(x,t) = 2\lambda_0^2 - \frac{4\lambda_0^2 e^{\chi-\rho}}{(\det S)^2}(e^{2\chi+2\rho}\det\Pi\ \mathrm{tr}\Pi + 4e^{\chi+\rho}\det\Pi + \mathrm{tr}\Pi)$$

spin density:

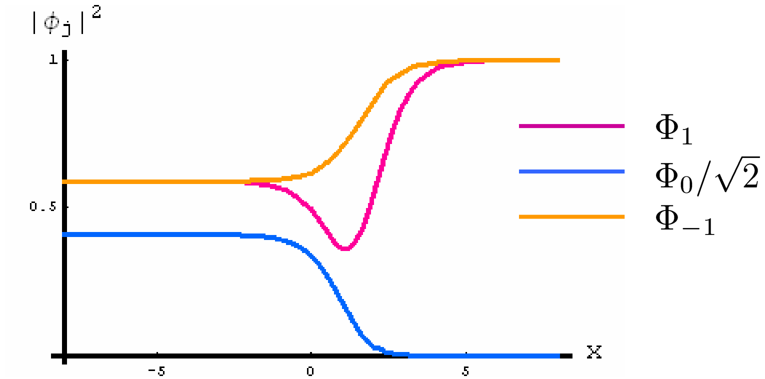
$$\boldsymbol{f}(x,t) = \frac{4\lambda_0^2 e^{\chi-\rho}}{(\det S)^2}(e^{2\chi+2\rho}\det\Pi - 1)\,\mathrm{tr}(\Pi\boldsymbol{\sigma})$$

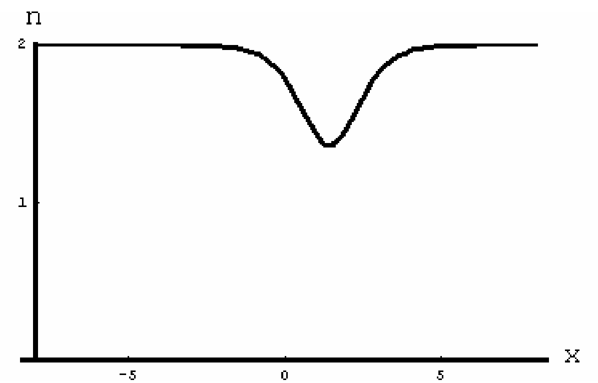
where

Polarization matrix Π plays to distribute the spin components.

$$\det S = e^{2\chi+2\rho}\det\Pi + e^{\chi+\rho}\mathrm{tr}\Pi + 1$$

- Ferromagnetic state $\det \Pi = 0$

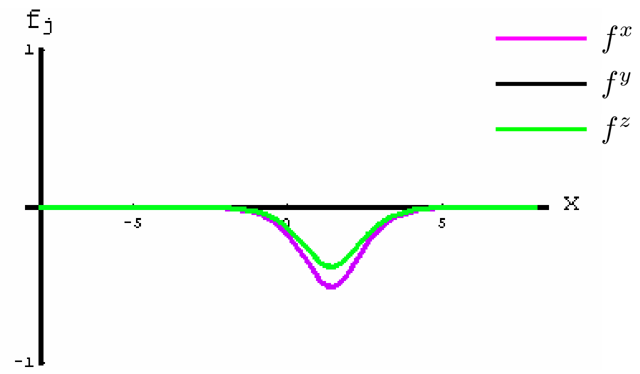




- domain-wall wavefunctions
- a single-valley dark soliton

  for number density
- total spin is nonzero

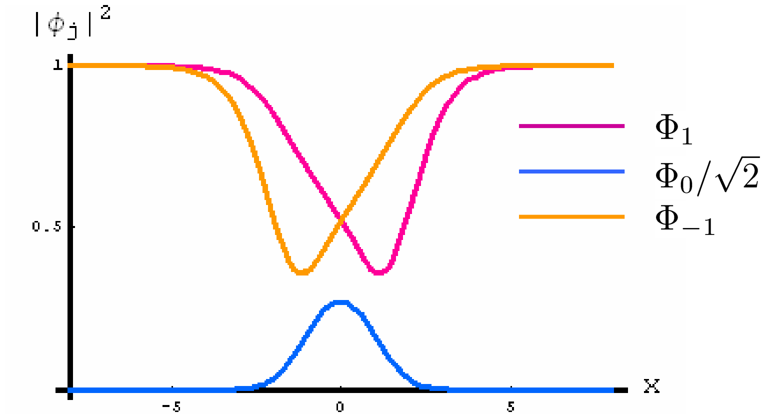
  |total spin| = total hole number

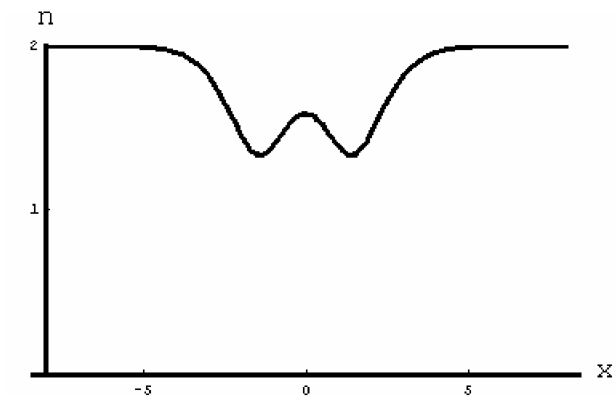


$$\lambda_0 = 1, k = 0, \varphi = 0.927, \Pi = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$$

Polar state

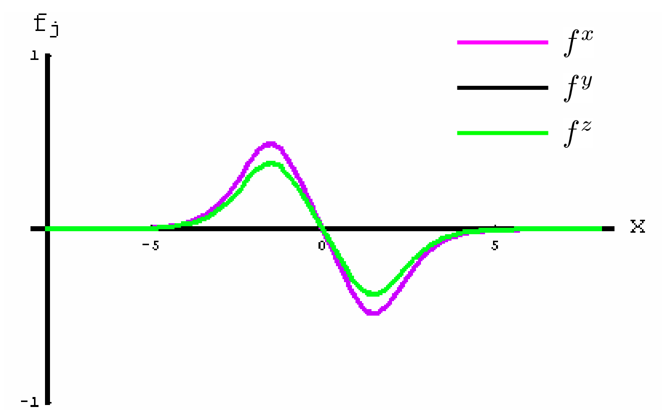
$$\det \Pi \neq 0$$





- boundary intensities of each wavefunction are same for $x \to \pm\infty$
- a twin-valley dark soliton for number density
- dipole-shape spin density and

  total spin = zero

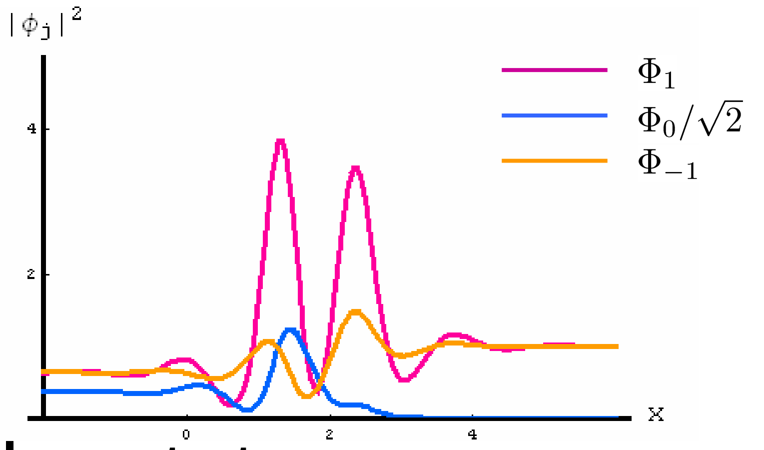


A degenerate 2-soliton behaving like a 1-soliton

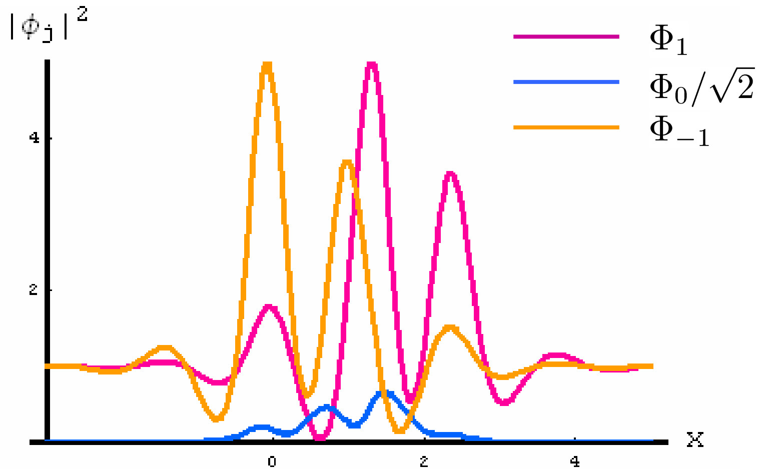
$$\lambda_0 = 1, k = 0, \varphi = 0.927, \Pi = \begin{pmatrix} 8 & 3.88 \\ 3.88 & 2 \end{pmatrix}$$

1-soliton (c<0: attractive & ferromagnetic)

- Ferromagnetic state



- Polar state



- bright solitons with same characters as c>0
- oscillating profiles because φ can be complex

2-soliton (c>0)

- 2-soliton solution

$$Q(x,t) = \lambda_0 e^{i\phi(x,t)} \frac{\mathfrak{B}}{\mathfrak{A}}$$

$\mathfrak{A},\ \mathfrak{B} = (\mathfrak{B}_{ij})_{i,j=1,2}$: complicated formulas involving $\varphi_i,\ \Pi_i\ (i=1,2)$

- As $t \to \pm\infty$, 2-soliton solution becomes two 1-solitons.

  Three combinations of 1-soliton states are possible.

  i.e. Polar & Polar

  Ferromagnetic & Polar

  Ferromagnetic & Ferromagnetic

✓ $$\mathfrak{A} =\Xi^2 \det\Pi_1 \det\Pi_2 \mathrm{e}^{2(\chi_1+\rho_1)+2(\chi_2+\rho_2)} + \Xi \det\Pi_1 \mathrm{tr}\Pi_2 \mathrm{e}^{2(\chi_1+\rho_1)+(\chi_2+\rho_2)} + \det\Pi_1 \mathrm{e}^{2(\chi_1+\rho_1)} + \Xi \mathrm{tr}\Pi_1 \det\Pi_2 \mathrm{e}^{(\chi_1+\rho_1)+2(\chi_2+\rho_2)}$$
$$+ \left[\mathrm{e}^{\rho_1+\rho_2}\mathrm{tr}\Pi_1\mathrm{tr}\Pi_2 - \sin^{-2}\left(\frac{1}{2}(\varphi_1+\varphi_2)\right)\mathrm{tr}\Pi_1\Pi_2\right]\mathrm{e}^{\chi_1+\chi_2} + \mathrm{tr}\Pi_1\mathrm{e}^{\chi_1+\rho_1} + \det\Pi_2\mathrm{e}^{2(\chi_2+\rho_2)} + \mathrm{tr}\Pi_2\mathrm{e}^{\chi_2+\rho_2} + 1,$$

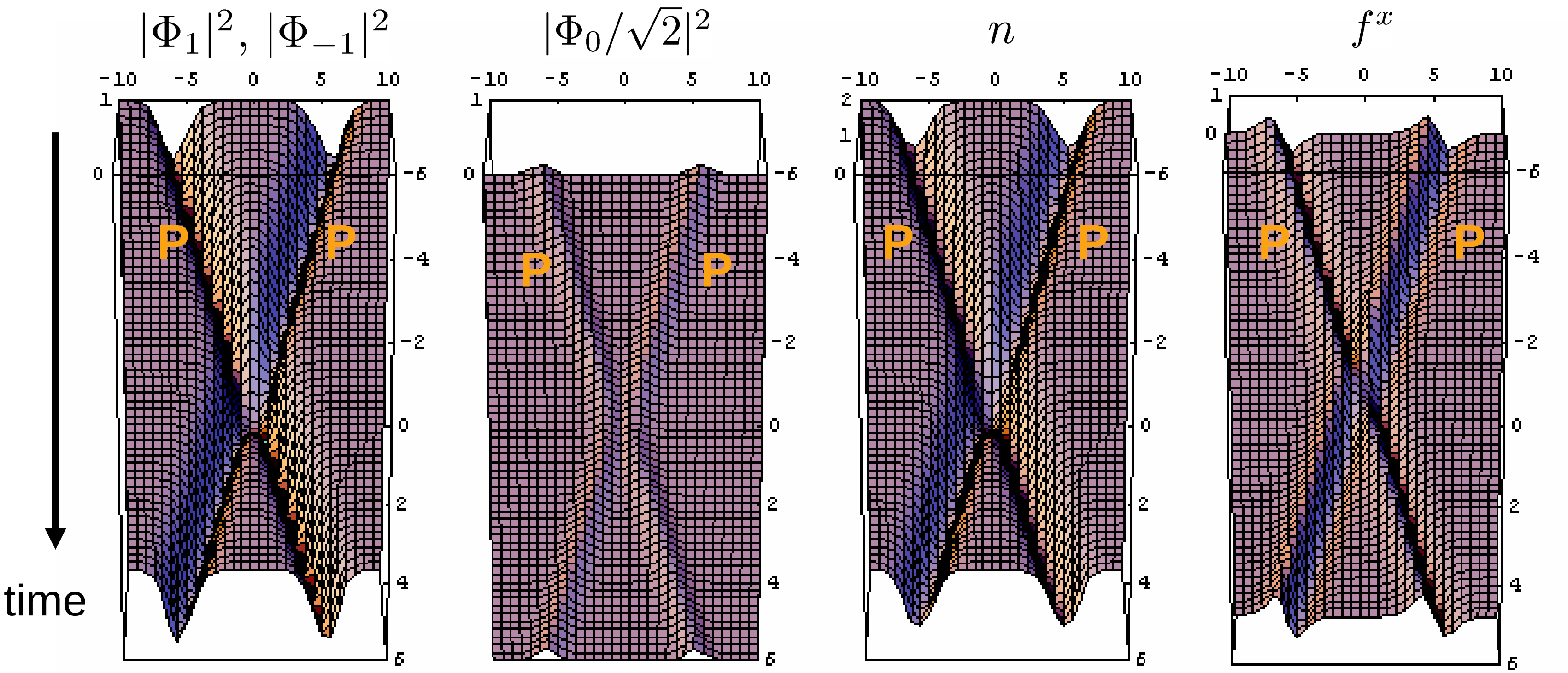
✓ $$\mathfrak{B}_{11} =\Xi^2 \mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2)} \det\Pi_1 \det\Pi_2 \mathrm{e}^{2(\chi_1+\rho_1)+2(\chi_2+\rho_2)}$$
$$+ \Xi \det\Pi_1 \left[\mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2)}(\Pi_2)_{11} + \mathrm{e}^{2\mathrm{i}\varphi_1}(\Pi_2)_{22}\right]\mathrm{e}^{2(\chi_1+\rho_1)+(\chi_2+\rho_2)} + \mathrm{e}^{2\mathrm{i}\varphi_1}\det\Pi_1\mathrm{e}^{2(\chi_1+\rho_1)}$$
$$+ \Xi \det\Pi_2 \left[\mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2)}(\Pi_1)_{11} + \mathrm{e}^{2\mathrm{i}\varphi_2}(\Pi_1)_{22}\right]\mathrm{e}^{(\chi_1+\rho_1)+2(\chi_2+\rho_2)}$$
$$+ \Big\{\Xi\mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2)}(\Pi_1)_{11}(\Pi_2)_{11} + \Xi(\Pi_1)_{22}(\Pi_2)_{22} + \mathrm{e}^{2\mathrm{i}\varphi_1}(\Pi_1)_{11}(\Pi_2)_{22} + \mathrm{e}^{2\mathrm{i}\varphi_2}(\Pi_1)_{22}(\Pi_2)_{11}$$
$$- \sin^{-2}\left(\frac{1}{2}(\varphi_1+\varphi_2)\right)\mathrm{e}^{-(\rho_1+\rho_2)+\mathrm{i}(\varphi_1+\varphi_2)}\left[(\Pi_1)_{12}(\Pi_2)_{21} + (\Pi_1)_{21}(\Pi_2)_{12}\right]\Big\}\mathrm{e}^{(\chi_1+\rho_1)+(\chi_2+\rho_2)}$$
$$+ \left[\mathrm{e}^{2\mathrm{i}\varphi_1}(\Pi_1)_{11} + (\Pi_1)_{22}\right]\mathrm{e}^{\chi_1+\rho_1} + \mathrm{e}^{2\mathrm{i}\varphi_2}\det\Pi_2\mathrm{e}^{2(\chi_2+\rho_2)} + \left[\mathrm{e}^{2\mathrm{i}\varphi_2}(\Pi_2)_{11} + (\Pi_2)_{22}\right]\mathrm{e}^{\chi_2+\rho_2} + 1$$

✓ $$\mathfrak{B}_{12} =2\mathrm{i}\Big\{\Xi\mathrm{e}^{2\rho_1+2\mathrm{i}\varphi_1+\mathrm{i}\varphi_2}\det\Pi_1(\Pi_2)_{12}\mathrm{e}^{2\chi_1+\chi_2} + \Xi\mathrm{e}^{2\rho_2+\mathrm{i}\varphi_1+2\mathrm{i}\varphi_2}(\Pi_1)_{12}\det\Pi_2\mathrm{e}^{\chi_1+2\chi_2}$$
$$+ \left[-\Xi^{1/2}\mathrm{e}^{\rho_2+\mathrm{i}\varphi_1+\mathrm{i}\varphi_2}(\Pi_1)_{12}\mathrm{tr}\Pi_2 + \Xi^{1/2}\mathrm{e}^{\rho_1+\mathrm{i}\varphi_1+\mathrm{i}\varphi_2}\mathrm{tr}\Pi_1(\Pi_2)_{12}\right]\mathrm{e}^{\chi_1+\chi_2} + \mathrm{e}^{\mathrm{i}\varphi_1}(\Pi_1)_{12}\mathrm{e}^{\chi_1} + \mathrm{e}^{\mathrm{i}\varphi_2}(\Pi_2)_{12}\mathrm{e}^{\chi_2}\Big\}$$

where $$\Xi = \frac{\sin^2\left(\frac{1}{2}(\varphi_2-\varphi_1)\right)}{\sin^2\left(\frac{1}{2}(\varphi_2+\varphi_1)\right)}.$$

$\mathfrak{B}_{21}$ (resp. $\mathfrak{B}_{22}$) is given by replacing the indices of Π_j with $1 \leftrightarrow 2$ in $\mathfrak{B}_{12}$ (resp. $\mathfrak{B}_{11}$).

2-soliton collision Polar v.s. Polar

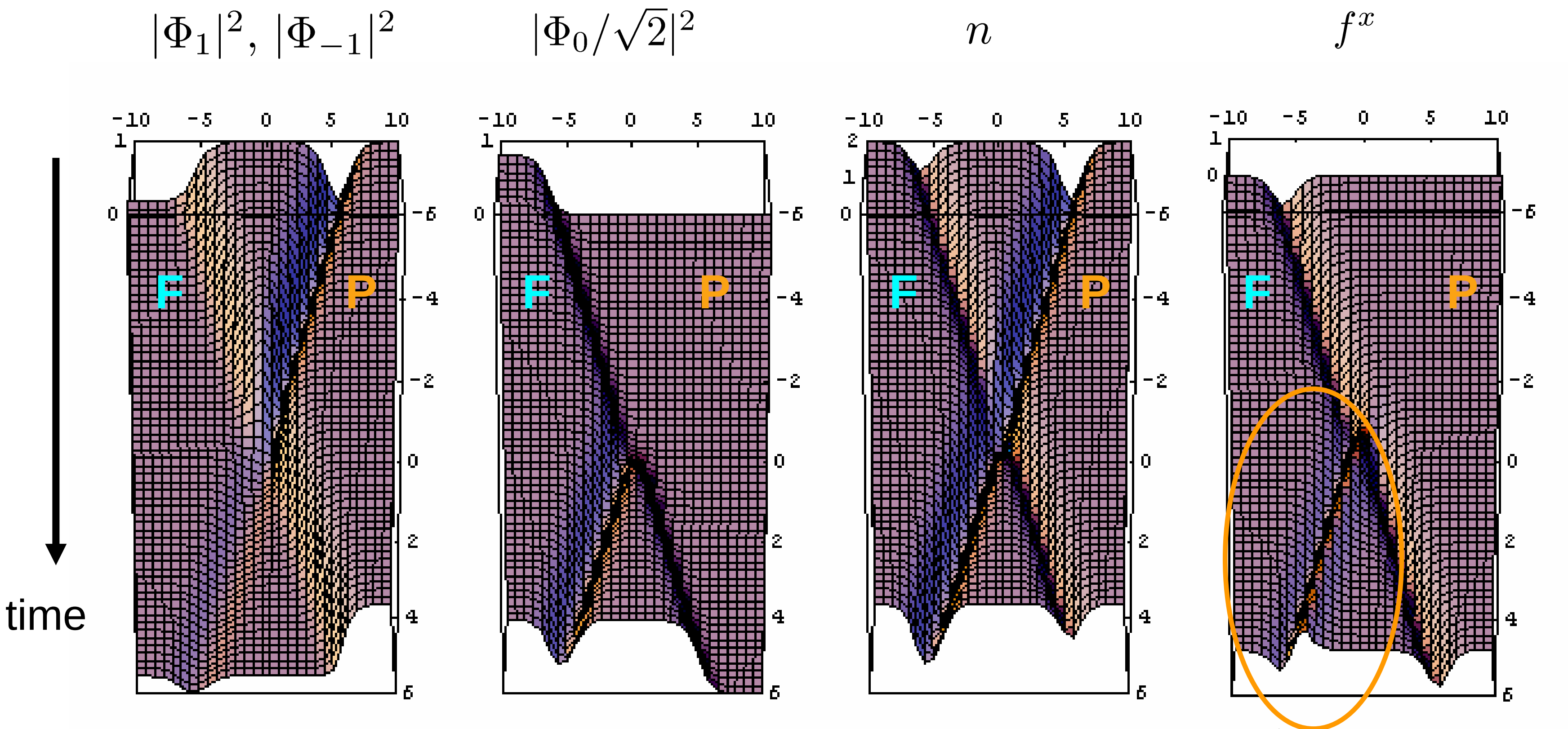


- no component-mixing occurs
- almost the same as collision for the one-component soliton

$$\Pi_1 = \Pi_2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

(twin-valleys look almost like single-valleys)

2-soliton collision Ferro v.s. Polar

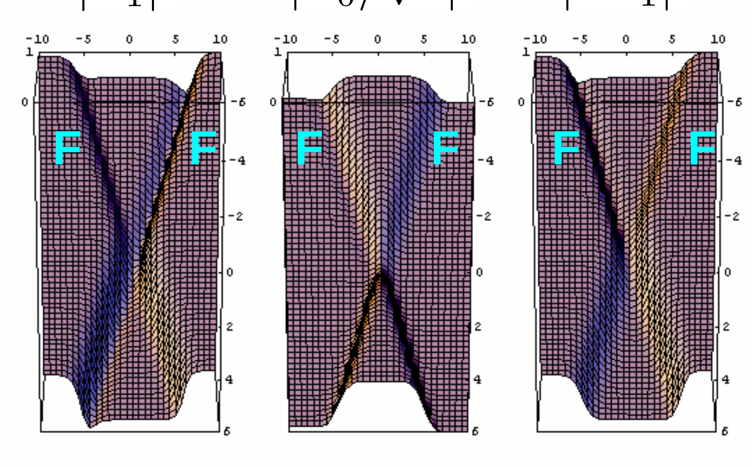


- component-mixing occurs for soliton **P**
- spin-switching phenomenon

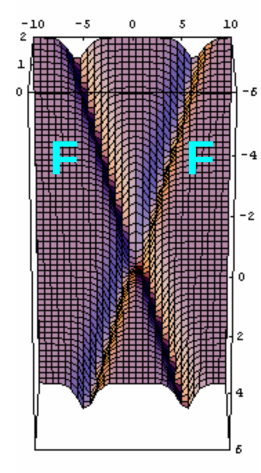
$$\Pi_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \Pi_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2-soliton collision Ferro v.s. Ferro

$|\Phi_1|^2$ $|\Phi_0/\sqrt{2}|^2$ $|\Phi_{-1}|^2$

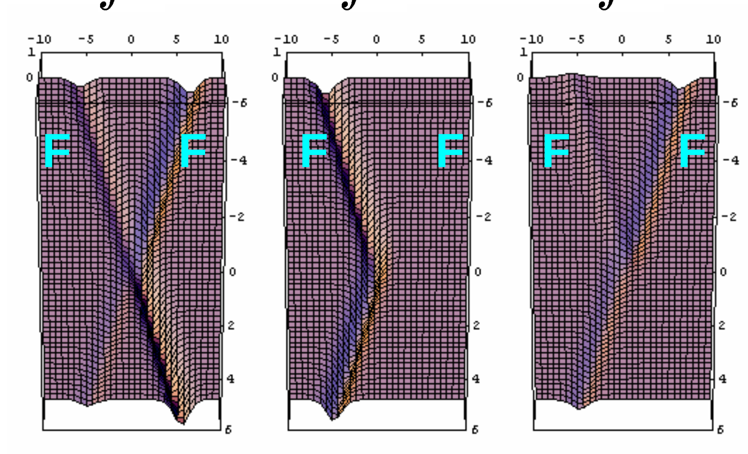


n



time

f^x f^y f^z



- component-mixing occurs for both solitons
- spin-precession (or spin-mixing) during the collision

Spin-mixing and Spin-switching phenomena

- a soliton in the ferromagnetic state has <u>nonzero</u> total spin

  → it makes the spin of another soliton rotate during the collision

  magnetically *operative*

- a soliton in the polar state has <u>zero</u> total spin

  → it cannot rotate the spin of another soliton during the collision

  magnetically *passive*

- Multi-soliton collision

  is remarkably factorized by successive two-soliton collisions.

Summary

- The multi-component Gross-Pitaevskii equation for the F=1 BEC is identified with the 2×2 matrix nonlinear Schrödinger equation at the integrable point, both for repulsive & anti-ferromagnetic and attractive & ferromagnetic cases.
- Under the nonvanishing boundary conditions, an explicit expression for 1-soliton is presented. Two states, ferromagnetic and polar, are found.
- Two-soliton collisions for every combination of states are clarified. The spin degrees of freedom give a novel phenomenon, spin-mixing for solitons.

References

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6. J. Ieda, M. Uchiyama and M. Wadati, *Inverse Scattering Method for the Multicomponent Nonlinear Schrödinger Equation under Nonvanishing Boundary Conditions,* J.Math.Phys. **48,** 013507—1~19 (2007).